Grothendieck groups

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Grothendieck group of definable sets

- From the family of definable sets in a structure, with the operation $\sqcup$ of disjoint union, we may naturally construct a group (even a ring) in the following way.
- First, we mod out by the equivalence relation saying that $X \sim Y$ when there is a definable bijection between X and Y.
- Then we want to induce a semigroup operation on the equivalence classes by setting $[X] + [Y] = [X \sqcup Y]$.
- We make the equivalence relation coarser in order to ensure the cancellation law: $[X] + [Z] = [Y] + [Z]$ implies $[X] = [Y]$.
- In order to make a group we artificially add inverses: for each nonempty X we add a new formal element $[-X]$. Then $[X_1] + [-Y_1] \simeq [X_2] + [-Y_2]$ when there are Z_1, Z_2 such that $[X_1 + Z_1] = [X_2 + Z_2]$ and $[Y_1 + Z_1] = [Y_2 + Z_2]$.
- Under this identification, the set of ($\simeq$-classes of) expressions of the form $[X] + [-Y]$ has the obvious group structure (extending the one given above).

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Grothendieck group of a general semigroup

- If S is an abelian semigroup satisfying the cancellation law, we can embed S into a group (namely, artificially adjoin a neutral element if necessary, along with inverses for elements that don't already have them).
- If S does not satisfy the cancellation law, we can mod out by an equivalence relation that forces this, and then embed in a group.
- If S is not abelian, then it may not embed in a group even if it is (two-sided) cancellative. (The universal theory of groups is not finitely axiomatisable.)
- However, there is a always a universal map from S to a group.
- This can be obtained either by considering a diagonal of all possible maps from S into groups of appropriately bounded size, or by considering the group with presentation $\langle \{g_s \mid s \in S\} \mid \{g_{s_1} g_{s_2} g_{s_1 s_2}^{-1} \mid s_1, s_2 \in S\} \rangle$.
- This group, along with the canonical homomorphism $S \to \mathbb{G}\mathrm{r}(S)$, is called the *Grothendieck group of S*.

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CLTS and Ellis groups

- Let S be a compact T_2 left topological semigroup (CLTS) (i.e. such that left multiplication $s \mapsto ss_0$ is continuous).
- Then S has minimal left ideals, which are all compact, principal and are disjoint unions of groups, called Ellis groups.
- Ellis groups are all isomorphic (as abstract groups), but are in general not topological nor closed in S.
- In general, if $\mathcal{M}$ is a minimal left ideal and $p \in \mathcal{M}$ is arbitrary, then $\mathcal{M} = Sp$ and $p\mathcal{M} = pSp$ is an Ellis group.
- By taking the identity u in an Ellis group, it follows that if $\varphi\colon S \to \mathrm{Gr}(S)$ is the canonical homomorphism, then $\varphi[uSu] = \varphi(u)\varphi[S]\varphi(u) = \varphi[S]$, so φ restricts to a (semigroup) homomorphism $u\mathcal{M} \to \mathrm{Gr}(S)$. Since $u\mathcal{M}$ is a group and $\varphi[S]$ generates $\mathrm{Gr}(S)$, it is easy to see that it is a surjective group homomorphism.

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- In particular, if S is a compact left topological semigroup (e.g. an Ellis group) and $\mathcal{M}$ is a minimal ideal in S, while $u \in \mathcal{M}$ is idempotent, then for every $s \in S$ we have $\varphi(usu) = \varphi(u)\varphi(s)\varphi(u) = e\varphi(s)e = \varphi(s)$. In particular, $\varphi[S] = \varphi[uSu] = \varphi[u\mathcal{M}]$.
- Thus, $u\mathcal{M}$ is a group contained in S, such that φ restricts to a semigroup homomorphism $u\mathcal{M} \to \mathrm{Gr}(S)$.
- Now, since the image of a group under a semigroup homomorphism is a group, and $\varphi[u\mathcal{M}] = \varphi[S]$, it follows that φ is onto and its restriction to $u\mathcal{M}$ is a group epimorphism.

Question (Kowalski)

Suppose S is the Ellis semigroup of a model-theoretic dynamical system.
Is the Ellis group the Grothendieck group (i.e. is the restriction of φ to $u\mathcal{M}$ injective)?

In general, no. There are examples when $\mathrm{Gr}(S)$ is trivial and $u\mathcal{M}$ is not.

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Grothendieck group is not the Ellis group

Example

- Let S_0 be a monoid with elements u, f, v such that u, fu, f^2u, f^3u are all distinct, $f^4 = 1$, f^2 is central, $u^2 = u$, $ufu = u$, and likewise, $v^2 = v$, $vfv = v$, and moreover $vu = f^3u$ and $uv = f^3v$.
- Then $S = \{u, fu, f^2u, f^3u, v, fv, f^2v, f^3v\}$ is a finite (hence compact, with discrete topology) subsemigroup.
  - e.g. $ufv = uff^4v = uf^2uv = f^2uv = f^5v = fv$.
- The minimal ideals in S are $\{u, fu, f^2u, f^3u\}$ and $\{v, fv, f^2v, f^3v\}$. The idempotents are u, fu, v, fv, and the Ellis groups are $\{u, f^2u\}, \{fu, f^3u\}, \{v, f^2v\}, \{fv, f^3v\}$
  - e.g. $f^3uf^3u = f^5(ufu) = fu = f(ufu)$ (since f^2 is central!)
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- Let S_0 be a monoid with elements u, f, v such that u, fu, f^2u, f^3u are all distinct, $f^4 = 1$, f^2 is central, $u^2 = u$, $ufu = u$, and likewise, $v^2 = v, vfv = v$, and moreover $vu = f^3u$ and $uv = f^3v$.
- Then $S = \{u, fu, f^2u, f^3u, v, fv, f^2v, f^3v\}$ is a finite (hence compact, with discrete topology) subsemigroup.
  - e.g. $ufv = uff^4v = uf^2uv = f^2uv = f^5v = fv$.
- The minimal ideals in S are $\{u, fu, f^2u, f^3u\}$ and $\{v, fv, f^2v, f^3v\}$. The idempotents are u, fu, v, fv, and the Ellis groups are $\{u, f^2u\}, \{fu, f^3u\}, \{v, f^2v\}, \{fv, f^3v\}$
  - e.g. $f^3uf^3u = f^5(ufu) = fu = f(ufu)$ (since f^2 is central!)
- The Grothendieck group is trivial, since $vu = f^3u$ maps to the identity, as does fu (because it is idempotent).

When Ellis group equals the Grothendieck group

Theorem

The following are equivalent:

1. $\mathrm{Gr}(S) = u\mathcal{M}$,
2. *the map* $S \to u\mathcal{M}$, $f \mapsto ufu$ *is a homomorphism,*
3. *for each idempotent* $u' \in \mathcal{M}$ *and minimal idempotent* $v \in S$ *equivalent to* u, $vu' = u$.
4. *The product of any two minimal idempotents is idempotent.*

They are implied by the following equivalent conditions:

5. *The map* $S \to \mathcal{M}$, $f \mapsto fu$ *is a homomorphism.*
6. *For each idempotent* $u' \in \mathcal{M}$ *and arbitrary* $f \in S$, $fu = fu'$.

(The last condition easily follows if $\mathcal{M}$ *is a group, 5. easily follows if* $\mathcal{M}$ *is the unique minimal ideal in* S*.)*

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Grothendieck kernel

- Let us write $\ker_{\mathrm{Gr}}(S)$ for the kernel of the epimorphism $u\mathcal{M} \to \mathrm{Gr}(S)$, where $u\mathcal{M} \subseteq S$ is an Ellis group.
  - $\ker_{\mathrm{Gr}}(S)$ does not depend on $u\mathcal{M}$: if $v\mathcal{N}$ is another Ellis group, then there is a φ-invariant isomorphism $u\mathcal{M} \to v\mathcal{N}$ (in particular, $u\mathcal{M}$ and $v\mathcal{N}$ are isomorphic as groups with a predicate for the kernel).
- General problem: understanding $\ker_{\mathrm{Gr}}(S)$, especially when S is some "naturally occuring" semigroup.
- We have a description of $\ker_{\mathrm{Gr}}(S)$ *as a normal subgroup of* $u\mathcal{M}$.
- In general, $\ker_{\mathrm{Gr}}(S)$ can be though of as an object "measuring" how badly the map $f \mapsto ufu$ fails to be a homomorphism, or how badly the product of two minimal idempotents can fail to be idempotent.

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Describing the kernel

Lemma

The kernel of $\varphi\restriction_{u\mathcal{M}}: u\mathcal{M} \to \mathrm{Gr}(S)$ is the normal subgroup generated by elements of the form $uf_1uf_2u(uf_1f_2u)^{-1}$ (inverse is in $u\mathcal{M}$).

Proof.

Write N for the normal subgroup. Then:

$$\varphi(uf_1uf_2u(uf_1f_2u)^{-1}) = \varphi(f_1)\varphi(f_2)\varphi(uf_1f_2u)^{-1} = \varphi(f_1)\varphi(f_2)(\varphi(f_1)\varphi(f_2))^{-1} = e_{\mathrm{Gr}(S)}.$$

Thus N is contained in the kernel, so we have an induced epimorphism $\varphi': u\mathcal{M}/N \to \mathrm{Gr}(S)$. In the other direction, consider the map $\psi: S \to u\mathcal{M}/N$ given by $f \mapsto ufuN$. It is easy to see that this is a semigroup homomorphism: $\psi(f_1)\psi(f_2) = uf_1uNuf_2uN = uf_1uf_2uN = uf_1f_2uN = \psi(f_1f_2)$. Thus, ψ factors through φ (by universality of $\mathrm{Gr}(S)$), so there is $\psi': \mathrm{Gr}(S) \to u\mathcal{M}/N$ such that $\psi = \psi' \circ \varphi$, and ψ' is inverse to φ', since $\varphi(f) = \varphi' \circ \psi(f)$ and $\psi(f) = \psi' \circ \varphi(f)$. □

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By manipulating idempotents, we get the following corollaries.

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The kernel is generated (as a normal subgroup of $u\mathcal{M}$) by the elements of the form $ufu(ufu')^{-1} = uf(ufu')^{-1} = ufu(fu')^{-1}$, where $f \in S$ and $u' \in \mathcal{M}$ is idempotent.

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Corollary

If $S_0 \subseteq S$ (both CLTS) is a subsemigroup containing all the minimal ideals of S, then $\mathrm{Gr}(S_0) = \mathrm{Gr}(S)$.
In particular, if S has finitely many minimal ideals, then for $S_0 = \bigcup\{\text{minimal ideals of } S\}$ we have $\mathrm{Gr}(S_0) = \mathrm{Gr}(S)$ (or, more generally, if this is closed in S).

Connection to Ludomir's talk

- Suppose G is a group definable in M and $N \succeq M$. Then we have a natural embedding $S_{\mathrm{fs},G}(\mathfrak{C}/M) \subseteq S_{\mathrm{fs},G}(\mathfrak{C}/N)$: a global type finitely satisfiable in M is finitely satisfiable in N.
- $S_{\mathrm{fs},G}(\mathfrak{C}/M)$ is an Ellis semigroup, as is $S_{\mathrm{fs},G}(\mathfrak{C}/N)$.
- Question: are these two groups related?
- For example, it may happen that $u\mathcal{M}$ naturally embeds into $v\mathcal{N}$, and in fact we have $u\mathcal{M} \overset{\cong}{\to} \mathfrak{Gr}(S_{\mathrm{fs},G}(\mathfrak{C}/M)) \hookrightarrow \mathfrak{Gr}(S_{\mathrm{fs},G}(\mathfrak{C}/N)) \overset{\cong}{\to} v\mathcal{N}$.
- I believe this actually does happen in the context of Theorem 2 in Ludomir's talk. Indeed, the minimal ideals of both semigroups are groups, so they are isomorphic to the Grothendieck groups, and the fact that the natural map $\mathfrak{Gr}(S_{\mathrm{fs},G}(\mathfrak{C}/M)) \to \mathfrak{Gr}(S_{\mathrm{fs},G}(\mathfrak{C}/N))$ is injective should follow from Ludomir's paper.
- Perhaps Theorem 1 could also be recovered, as under its assumptions, Ellis groups are still isomorphic to the Grothendieck group, and perhaps the restriction map induces an epimorphism from a subgroup of $\mathfrak{Gr}(S_{\mathrm{fs},G}(\mathfrak{C}/N))$ to $\mathfrak{Gr}(S_{\mathrm{fs},G}(\mathfrak{C}/M))$.

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- $S_{\mathrm{fs},G}(\mathfrak{C}/M)$ is an Ellis semigroup, as is $S_{\mathrm{fs},G}(\mathfrak{C}/N)$.
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- For example, it may happen that $u\mathcal{M}$ naturally embeds into $v\mathcal{N}$, and in fact we have $u\mathcal{M} \xrightarrow{\cong} \mathfrak{Gr}(S_{\mathrm{fs},G}(\mathfrak{C}/M)) \hookrightarrow \mathfrak{Gr}(S_{\mathrm{fs},G}(\mathfrak{C}/N)) \xrightarrow{\cong} v\mathcal{N}$.
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- Thus we have a sequence of homomorphisms $u\mathcal{M} \to \mathrm{Gr}(S_{\mathrm{fs},G}(\mathfrak{C}/M)) \to G^*/(G^*)^{000}_M$.
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- This leads to a new variant of Newelski's conjecture: (when?) is the natural map $\mathrm{Gr}(S_{\mathrm{fs},G}(\mathfrak{C}/M)) \to G^*/(G^*)^{000}_M$ an isomorphism?

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- The Ellis group of this flow does not depend on $\mathfrak{C}$ (Krupiński, Newelski, Simon).
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Topological aspects

- For topological semigroups, instead of homomorphisms into groups, we can consider only continuous homomorphisms into groups endowed with a topology.
- For instance, write $\mathrm{Gr}^{top}(S)$ for the universal continuous homomorphism from S to a Hausdorff topological group.
- Then, since $G^*/(G^*)^{00}_M$ is a Hausdorff topological group, we have again a sequence $u\mathcal{M} \to \mathrm{Gr}^{top}(S_{\mathrm{fs},G}(\mathfrak{C}/M)) \to G^*/(G^*)^{00}_M$ and it makes sense to ask if/when the maps appearing here are isomorphisms.
- For example, if all types are definable (e.g. G is definable in an o-minimal expansion of $\mathbb{R}$), then the latter function is an isomorphism.
- Even for the usual Grothendieck group, it might be interesting to consider what sort of topology is induced to the Grothendieck group from, say, $S_{\mathrm{fs},G}(\mathfrak{C}/M)$, or $u\mathcal{M}$ (with the τ-topology), in particular, (when) the group operations are continuous and when the topology is T_2 or T_1. (For example, it seems that the multiplication is always continuous on the left with respect to the topology induced from the semigroup.)

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