Finite-dimensional Groups

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Dimension

Definition

A theory T is *finite-dimensional* if there is a dimension function $\dim$ from the collection of all interpretable sets in models of T to $\{-\infty\} \cup \omega$, satisfying for a formula $\varphi(x, y)$ and interpretable sets X and Y:

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- *Algebraicity:* $\dim(\emptyset) = -\infty$, and $\dim X = 0$ iff X is finite.
- *Union:* $\dim(X \cup Y) = \max\{\dim(X), \dim(Y)\}$.
- *Fibration:* Let $f : X \to Y$ be an interpretable map.
  If $\dim(f^{-1}(y)) \geq d$ for all $y \in Y$, then $\dim(X) \geq \dim(Y) + d$;
  if $\dim(f^{-1}(y)) \leq d$ for all $y \in Y$, then $\dim(X) \leq \dim(Y) + d$.

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Examples of finite-dimensional theories include theories of finite Lascar rank, finite SU-rank, finite $U^{\flat}$-rank and *o*-minimal theories.

Finite-dimensional groups

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In a finite-dimensional theory, if $f : G \to H$ is a definable homomorphism of definable groups, then

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It follows that we have a descending and an ascending chain condition on definable subgroups up to finite index.

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In a finite-dimensional theory:

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The infinitesimals in a non-standard real closed field show that $\bigwedge$-definability is necessary in the last item.

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- *A definable field with infinitely many absolutely algebraic elements has no infinite type-definable family of definable automorphisms.*
- *A definable automorphism of a $\emptyset$-definable, additively or multiplicatively $\emptyset$-connected field is $\mathrm{acl}^{eq}(\emptyset)$-definable.*

Connected components

There is no *a priori* assumption on the existence of connected components (although we usually assume connectednes of *one* of the groups involved). This is a major complication, and much of the proofs is devoted to working around this.

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Theorem

Let G be a connected group acting on the infinite abelian group A in a finite-dimensional theory. Suppose that A has no infinite G-invariant definable subgroup of smaller dimension, and there is $g \in G$ with $(g-1)A$ infinite. Then A has a G-invariant G-minimal definable subgroup of finite index.

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Theorem

Let G be a finite-dimensional nilpotent connected group. Then G has a definable connected lower central series.

Proof.

Suppose $A^* \leq A$ is a G-invariant definable subgroup of finite index. Then $C_G(A/A^*)$ has finite index in G and must be the whole of G. So $(g-1)A \leq A^*$; it is contained in every G-invariant subgroup of finite index.

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Now consider finitely many $g_i \in G$ such that $B = \sum_i (g_i - 1)A$ has maximal dimension possible. As $(g-1)A$ is infinite for some $g \in G$, the group B is infinite; by maximality of $\dim B$ it must be commensurable with $gB \leq \sum_i [(gg_i - 1)A - (g-1)A]$ for any $g \in G$.

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By compactness the commensurability is uniform, and by Schlichting's Theorem there is a G-invariant definable subgroup B_0 commensurable with B.

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Then $\dim A = \dim B_0 = \dim B$, so there is a maximal sum $A_0 = \sum_j (g_j - 1)A$ of finite index in A. Then A_0 is G-invariant and G-minimal. □

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Soluble finite-dimensional groups

Theorem

Let M be a connected abelian group acting faithfully on the infinite abelian group A in a finite-dimensional theory. Suppose

- *A has no infinite M-invariant definable subgroup of smaller dimension.*
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Then A is M-minimal, and there is a definable field K such that $A \cong K^+$ and $M \hookrightarrow K^\times$.

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Theorem

Let G be a nilpotent connected group acting on an infinite abelian group A, in a finite-dimensional theory. Then either the action is nilpotent, or there is a definable centreless 2-soluble section which naturally interprets a field.

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By minimality of $\dim A$ the linear dimension is 1, and $A \cong K^+$; clearly $M \hookrightarrow R^\times = K^\times$. □

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Corollary

A connected soluble non-nilpotent finite-dimensional group interprets naturally a field.

Some **late** theorems

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Theorem

Let G be a connected group acting faithfully on the infinite abelian group A in a finite-dimensional theory. Suppose

- *A is G-minimal and $C_A(G) = \{0\}$.*
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Then there is a definable field K over which A is definably a vector space of finite linear dimension, such that the action of G is K-linear and the action of M scalar (so M is central in G).

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Corollary

A finite-dimensional connected soluble group has a nilpotent derived subgroup.

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If H_1 and H_2 are homogenies, their sum is defined pointwise, as

$$H_1 + H_2 = \{(a, b_1 + b_2) \in A \times B : (a, b_1) \in H \text{ and } (a, b_2) \in H_2\}.$$

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$$\pi_A(H_1 + H_2) = \pi_A H_1 \cap \pi_A H_2, \ \mathrm{coker}(H_1 + H_2) = \mathrm{coker} H_1 + \mathrm{coker} H_2.$$

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We shall call a pre-ring of endogenies *essentially* infinite/ unbounded, if it is infinite/unbounded modulo equivalence.

Endogenies

If $A = B$, we speak of *endogenies* rathen than homogenies.

Under addition and composition, the endogenies of A form a *prering*, where only left distributivity fails.

We call two homogenies H_1 and H_2 *equivalent* if $H_1 - H_2$ has finite image. Equivalence is preserved under sum and product, and left distributivity holds modulo equivalence. Hence the pre-ring of endogenies modulo equivalence is a ring.

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In our context A will be connected, and endogenies will be total. We shall use Greek letters to denote an endogeny of A.

Strong commutation

Definition

Two endogenies γ and δ of A *commute strongly* if

$$\mathrm{im}\,(\gamma\delta - \delta\gamma) = \mathrm{coker}\gamma + \mathrm{coker}\delta.$$

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Lemma

- *If γ and δ commute strongly, then* $\delta[\operatorname{coker}\gamma] \leq \operatorname{coker}\gamma + \operatorname{coker}\delta$.
- *$C^{\#}(\gamma)$ is a prering, i.e. is closed under $+$, $-$ and $\circ$.*

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Suppose γ and δ are commuting strongly.

- $\mathrm{coker}\gamma$ *is δ-invariant.*
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However,

- $\ker \gamma$ need not be δ-invariant, unless γ is a homomorphism.
- Even if δ is invertible, δ^{-1} need not commute strongly with γ.

The setting

- T is finite-dimensional.
- A is a definable, connected, abelian group.
- Γ and Δ are two invariant prerings of definable endogenies of A.
- Γ and Δ commute strongly.
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Theorem

If moreover every endogeny from Γ and Δ has a finite kernel, then there is a finite $F \leq A$ which is both Γ- and Δ-invariant, and such that on A/F, both Γ and Δ act by automorphisms.

Proof.

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Endomorphisms with trivial kernels of a finite-dimensional connected-by-finite group are automorphisms.

Linearisation

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- A *is* Γ*-minimal.*

Then there is a finite subgroup $F \leq A$ *which is* Γ*- and* Δ*-invariant,* Δ *is a (possibly skew) field,* A/F *is a finite-dimensional* Δ*-vector space, and* $\Gamma = \mathrm{End}_\Delta(A/F)$*.*

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Notice that these claims reduce to the first, i.e. proving that Δ is a definable skew field of automorphisms of A/F.

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However, this appears to be the natural set-up for model-theoretic linearisation.

At the moment, we have some proofs, but no theorem.

Thank you !