PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 33, Fasc. 1 (2013), pp. 149–174

# CHARACTERIZATIONS OF $\mathcal{F}$-STABLE AND $\mathcal{F}$-SEMISTABLE DISTRIBUTIONS

BY

NADJIB BOUZAR (INDIANAPOLIS)

*Abstract.* The notion of $\mathcal{F}$-stability of van Harn et al. [10] (see also Steutel and van Harn [20]) and the related concept of $\mathcal{F}$-semistability are intimately connected with continuous-time branching processes. $\mathcal{F}$-stable and $\mathcal{F}$-semistable distributions play also a significant role in the theory of integer-valued (semi-)self-similar processes and have arisen as stationary solutions of integer-valued autoregressive processes. The aim of this article is twofold. Firstly, we provide several new characterizations of univariate $\mathcal{F}$-stable and $\mathcal{F}$-semistable distributions. Secondly, we propose a systematic study of $\mathcal{F}$-stability and $\mathcal{F}$-semistability for distributions on the $d$-dimensional lattice $\mathbf{Z}_+^d$.

**2000 AMS Mathematics Subject Classification:** Primary: 60E07; Secondary: 62E10.

**Key words and phrases:** Semigroup, infinite divisibility, branching processes, probability generating functions, discrete multivariate distributions, the Lau–Rao theorem.

## 1. INTRODUCTION

In [10] van Harn et al. (see also Steutel and van Harn [20]) proposed a discrete analogue of the concept of stability for distributions on $\mathbf{Z}_+ := \{0, 1, 2, \ldots\}$. The authors' definition is based on the $\mathbf{Z}_+$-valued multiple $\alpha \odot_{\mathcal{F}} X$ of a $\mathbf{Z}_+$-valued random variable $X$ and $\alpha \in (0, 1)$ that they define as follows:

$$\alpha \odot_{\mathcal{F}} X = \sum_{k=1}^{X} Y_k(t) := Z_X(t) \quad (t = -\ln \alpha), \tag{1.1}$$

where $Y_1(\cdot), Y_2(\cdot), \ldots$ are independent copies of a continuous-time Markov branching process, independent of $X$, such that, for every $k \geqslant 1$, $P\big(Y_k(0) = 1\big) = 1$. The processes $\big(Y_k(\cdot), k \geqslant 1\big)$ are driven by a composition semigroup of probability generating functions (pgf's) $\mathcal{F} := (F_t, t \geqslant 0)$:

$$F_s \circ F_t(z) = F_{s+t}(z) \quad (|z| \leqslant 1;\ s, t \geqslant 0). \tag{1.2}$$

For every $k \geqslant 1$ and $t \geqslant 0$, $F_t(z)$ is the pgf of $Y_k(t)$, and the transition matrix $\{p_{ij}(t)\}$ of the Markov process $Y_k(\cdot)$ is determined by the equation

$$\sum_{j=0}^{\infty} p_{ij}(t) z^j = \{F_t(z)\}^i \quad (|z| \leqslant 1;\ i \geqslant 0). \tag{1.3}$$

Note that the process $Z_X(\cdot)$ of (1.1) is itself a Markov branching process driven by $\mathcal{F}$ and starting with $X$ individuals ($Z_X(0) = X$).

Let $P(z)$ be the pgf of $X$. Then the pgf $P_{\alpha \odot_{\mathcal{F}} X}(z)$ of $\alpha \odot_{\mathcal{F}} X$ is given by

$$P_{\alpha \odot_{\mathcal{F}} X}(z) = P\big(F_t(z)\big) \quad (\ t = -\ln \alpha;\ 0 \leqslant z \leqslant 1). \tag{1.4}$$

A distribution on $\mathbf{Z}_+$ with pgf $P(z)$ is said to be $\mathcal{F}$-*stable* if for any $t > 0$ there exists $\lambda > 0$ such that

$$P(z) = \big[P\big(F_t(z)\big)\big]^{\lambda} \quad (z \in [0, 1]). \tag{1.5}$$

If an $\mathcal{F}$-stable distribution exists, then $\lambda$ and $t$ in (1.5) satisfy the equation $\lambda = e^{\gamma t}$ for some constant $\gamma \in (0, 1]$ (see Steutel and van Harn [20], Chapter V, Sections 5 and 8). The number $\gamma$ is called the *exponent* of the $\mathcal{F}$-stable distribution.

In terms of random variables, the definition of $\mathcal{F}$-stability above can be restated as follows (see Steutel and van Harn [20], Chapter V, Section 5).

A $\mathbf{Z}_+$-valued random variable $X$ is said to be $\mathcal{F}$-*stable* if there exists $\gamma \in (0, 1]$ such that, for every $t > 0$,

$$X \stackrel{d}{=} e^{-t} \odot_{\mathcal{F}} X + (1 - e^{-\gamma t})^{1/\gamma} \odot_{\mathcal{F}} X', \tag{1.6}$$

where $X$ and $X'$ are independent and $X \stackrel{d}{=} X'$ (the symbol $\stackrel{d}{=}$ designates equality in distribution).

$\mathcal{F}$-stable distributions are infinitely divisible and are characterized by the following canonical form of their pgf's (Theorem 8.6, p. 306, in Steutel and van Harn [20]):

$$P(z) = \exp\{-cA(z)^{\gamma}\} \quad (0 \leqslant z \leqslant 1), \tag{1.7}$$

where $c > 0$, $0 < \gamma \leqslant 1$, and $A(z)$ is the $A$-function of $\mathcal{F}$ (see (1.13) below).

Adapting Lévy's definition of semistability (Lévy [13]) for continuous distributions to the discrete case, Krapavitskaite [11] introduced the notion of $\mathcal{F}$-semistability as follows.

A distribution on $\mathbf{Z}_+$ is said to be $\mathcal{F}$-*semistable* if its pgf $P(z)$ satisfies (1.5) for some $t > 0$ and $\lambda > 0$.

If an $\mathcal{F}$-semistable distribution exists, then $t$ and $\lambda$ of (1.5) must satisfy $\lambda = e^{\gamma t}$ for some $\gamma \in (0, 1]$ (Krapavitskaite [11]; see also Lemma 2.1 below). We will refer to $\gamma$ as the exponent of the distribution and $t$ its order.

A distribution on $\mathbf{Z}_+$ is $\mathcal{F}$-stable if and only if it is $\mathcal{F}$-semistable of all orders $t > 0$.

$\mathcal{F}$-semistable distributions are infinitely divisible and are characterized by the following representation of their pgf's (see Krapavitskaite [11] and Bouzar [5]):

$$P(z) = \exp\{-A(z)^\gamma g_{\gamma,t}(|\ln A(z)|)\} \quad (0 \leqslant z < 1), \tag{1.8}$$

where $g_{\gamma,t}(\cdot)$ is a continuous function from $\mathbf{R}_+$ to $\mathbf{R}_+$ that is periodic with period $t$.

The notions of $\mathcal{F}$-stability and $\mathcal{F}$-semistability have arisen in various contexts. Several authors studied the connection between $\mathcal{F}$-stable distributions and continuous-time branching processes. We cite Steutel et al. [21], Hansen [8], and Pakes [15]. $\mathcal{F}$-stable and $\mathcal{F}$-semistable distributions play an important role in the theory of integer-valued (semi-)self-similar processes (van Harn and Steutel [9] and Satheesh and Sandhya [17]). $\mathcal{F}$-stable distributions have also arisen as stationary solutions of integer-valued autoregressive processes (see Zhu and Joe [22] and the overview article by Aly [1]).

The aim of this article is twofold. Firstly, we provide several new characterizations of (univariate) $\mathcal{F}$-stable and $\mathcal{F}$-semistable distributions. That will be the object of Section 2. Our starting point is the functional equation

$$\ln P(z) = \sum_{i=1}^{m} \lambda_i \ln P(F_{t_i}(z)) \quad (0 \leqslant z \leqslant 1), \tag{1.9}$$

where $m \geqslant 1$, $\lambda_i$, $t_i > 0$ $(i = 1, \ldots, m)$, and $P(z)$ is a pgf such that $0 < P(0) < 1$.

We show that, depending on the commensurability of $t_1, \ldots, t_m$, or lack thereof, the solution to (1.9) is the pgf of either an $\mathcal{F}$-semistable or an $\mathcal{F}$-stable distribution. Several characterizations of $\mathcal{F}$-(semi)stability are then derived from this result. In particular, we identify necessary and sufficient conditions that will make an $\mathcal{F}$-semistable distribution an $\mathcal{F}$-stable one.

Secondly, we propose a systematic study of $\mathcal{F}$-stability and $\mathcal{F}$-semistability for distributions on the $d$-dimensional lattice $\mathbf{Z}_+^d := \mathbf{Z}_+ \times \ldots \times \mathbf{Z}_+$, where $d \geqslant 1$ is a natural number. We generalize the representations (1.7) and (1.8) to the multivariate case and show that the univariate characterizations obtained by Steutel and van Harn [20], as well as those obtained in Section 2, extend to the multivariate setting. We also establish a connection between classical (semi)stability for distributions on $\mathbf{R}_+^d := \mathbf{R}_+ \times \ldots \times \mathbf{R}_+$ and $\mathcal{F}$-(semi)stability. The treatment of the multivariate case is the object of Section 3. In Section 4, we show that $\mathcal{F}$-semistable and $\mathcal{F}$-stable distributions on $\mathbf{Z}_+^d$ arise as solutions to central limit problems. As a consequence, we obtain characterizations of $\mathcal{F}$-semistable (resp., stable) distributions on $\mathbf{Z}_+^d$ in terms of the Lévy measure of a semistable (resp., stable) distribution on $\mathbf{R}_+^d$.

In the remainder of this section we introduce some definitions and recall several basic facts about the semigroup $\mathcal{F} = (F_t, t \geqslant 0)$. For proofs and further details we refer to van Harn et al. [10] and Steutel and van Harn [20], and references therein.

As noted in Steutel and van Harn [20], Chapter V, Section 8, the multiplication $\odot_{\mathcal{F}}$ must satisfy some minimal conditions. We impose the following limit conditions on the composition semigroup $\mathcal{F}$:

$$\lim_{t\downarrow 0} F_t(z) = F_0(z) = z, \qquad \lim_{t\to\infty} F_t(z) = 1. \tag{1.10}$$

The first part of (1.10) implies the continuity of the semigroup $\mathcal{F}$ (by way of (1.2)) and the second part is equivalent to assuming that $E\big(Y_1(1)\big) = F_1'(1) \leqslant 1$, which implies the (sub)criticality of the continuous-time Markov branching process $Y_k(\cdot)$ in (1.1). We will restrict ourselves to the subcritical case ($F_1'(1) < 1$) and we will assume without loss of generality that $F_1'(1) = e^{-1}$ (see Remark 3.1 in [10]). In this case,

$$F_t'(1) = e^{-t} \qquad (t > 0). \tag{1.11}$$

The infinitesimal generator $U$ of the semigroup $\mathcal{F}$ is defined by

$$U(z) = \lim_{t\downarrow 0} \big(F_t(z) - z\big)/t \qquad (|z| \leqslant 1), \tag{1.12}$$

and satisfies $U(z) > 0$ for $0 \leqslant z < 1$.

The related $A$-function is defined by

$$A(z) = \exp\Big\{-\int_0^z \big(U(x)\big)^{-1}\,dx\Big\} \qquad (0 \leqslant z \leqslant 1). \tag{1.13}$$

The functions $U(z)$ and $A(z)$ satisfy, for any $t > 0$,

$$\frac{\partial}{\partial t} F_t(z) = U\big(F_t(z)\big) = U(z)F_t'(z) \qquad (|z| \leqslant 1) \tag{1.14a}$$

and

$$A\big(F_t(z)\big) = e^{-t}A(z) \qquad (0 \leqslant z < 1). \tag{1.14b}$$

Moreover, by (1.14b), the function $A(z)$ decreases from one to zero.

Finally, we will need the following theorem whose proof can be found in the monograph by Rao and Shanbhag [16].

THEOREM 1.1 (the Lau–Rao theorem). *Let $f$ be an $\mathbf{R}_+$-valued Borel measurable locally integrable function on $\mathbf{R}_+$, such that $l([f > 0]) \neq 0$, where $l$ is the Lebesgue measure. Let $\mu$ be a $\sigma$-finite measure on the Borel $\sigma$-field of $\mathbf{R}_+$ with $\mu(\{0\}) < 1$. Then*

$$f(x) = \int_{\mathbf{R}_+} f(x+y)\,\mu(dy), \tag{1.15}$$

*for almost all $x \in \mathbf{R}_+$ with respect to $l$, if and only if one of the following two conditions, with $\eta$ such that $\int_{\mathbf{R}_+} \exp\{\eta x\}\,\mu(dx) = 1$, holds:*

(i) *$\mu$ is arithmetic with some span $\kappa > 0$ and, for almost all $x \in \mathbf{R}_+$ with respect to $l$,*

$$f(x + n\kappa) = f(x) \exp\{n\kappa\eta\}, \quad n = 0, 1, \dots$$

(ii) *$\mu$ is non-arithmetic and, for some constant $c > 0$,*

$$f(x) = c \exp\{\eta x\}$$

*for almost all $x \in \mathbf{R}_+$ with respect to $l$.*

## 2. CHARACTERIZATIONS OF $\mathcal{F}$-SEMISTABLE DISTRIBUTIONS

The solution set of equation (1.9) is denoted by $\mathcal{S}_{\mathcal{F}}(m, \underline{\lambda}, \underline{t})$, where

$$\underline{\lambda} = (\lambda_1, \dots, \lambda_m) \quad \text{and} \quad \underline{t} = (t_1, \dots, t_m).$$

A necessary condition for $\mathcal{S}_{\mathcal{F}}(m, \underline{\lambda}, \underline{t})$ to be nonempty is identified first.

LEMMA 2.1. *Assume that $m \geqslant 1$, $t_i > 0$, $\lambda_i > 0$ $(i = 1, \dots, m)$.*
(i) *If $\mathcal{S}_{\mathcal{F}}(m, \underline{\lambda}, \underline{t}) \neq \emptyset$, then*

$$\sum_{i=1}^{m} \lambda_i e^{-t_i} \leqslant 1 < \sum_{i=1}^{m} \lambda_i. \tag{2.1}$$

*If, in addition, a solution distribution has finite mean, then*

$$\sum_{i=1}^{m} \lambda_i e^{-t_i} = 1. \tag{2.2}$$

(ii) *A nonempty $\mathcal{S}_{\mathcal{F}}(1, \lambda, t)$ (for some $\lambda > 0$ and $t > 0$) coincides with the set of discrete $\mathcal{F}$-semistable distributions with exponent $\gamma = (\ln \lambda)/t$ and order $t$. In this case, $\lambda e^{-t} \leqslant 1 < \lambda$.*

Proof. First, we note that the assumption $F_1'(1) = e^{-1}$ implies (see van Harn et al. [10])

$$F_t(z) > z \quad (t > 0;\ 0 \leqslant z < 1). \tag{2.3}$$

(i) Assume that $P(\cdot) \in \mathcal{S}_{\mathcal{F}}(m, \underline{\lambda}, \underline{t})$. We have, by (1.9) and (2.3),

$$\ln P(0) = \sum_{i=1}^{m} \lambda_i \ln P\big(F_{t_i}(0)\big) > \Big(\sum_{i=1}^{m} \lambda_i\Big) \ln P(0).$$

The second inequality in (2.1) thus holds (as $\ln P(0) < 0$). By differentiation,

$$\frac{P'(z)}{P(z)} = \sum_{i=1}^{m} \lambda_i F_{t_i}'(z) \frac{P'\big(F_{t_i}(z)\big)}{P\big(F_{t_i}(z)\big)}. \tag{2.4}$$

By (2.3) and the fact that $P'(z)$ is increasing over the interval $[0,1)$, we have

$$(2.5)\qquad \frac{1}{P(z)}=\sum_{i=1}^{m}\lambda_i F_{t_i}'(z)\frac{P'\big(F_{t_i}(z)\big)}{P'(z)}\frac{1}{P\big(F_{t_i}(z)\big)}\geqslant\sum_{i=1}^{m}\lambda_i F_{t_i}'(z)\frac{1}{P\big(F_{t_i}(z)\big)}.$$

By (1.11), $F_{t_i}'(1)=e^{-t_i}$, $i=1,\dots,m$, therefore the first inequality in (2.1) follows by letting $z\uparrow 1$ in (2.5). The additional assumption of finite mean is equivalent to $0<P'(1)<\infty$ (recall the distribution with pgf $P(z)$ is nondegenerate). Letting $z\uparrow 1$ in (2.4) yields (2.2).

To prove (ii), suppose $P(\cdot)\in\mathcal{S}_{\mathcal{F}}(1,\lambda,t)$, $\lambda>0$ and $t>0$. Then $\ln P(z)=\lambda\ln P\big(F_t(z)\big)$ for any $z\in[0,1]$. Note that, by part (i), $\lambda e^{-t}\leqslant 1<\lambda$. Letting $\gamma=(\ln\lambda)/t$, we have $0<\gamma\leqslant 1$. Therefore, $P(z)$ is discrete semistable with exponent $\gamma$ and order $t$. ∎

The real numbers $a_1,a_2,\dots,a_m$ are said to be *commensurable* if there exists a real number $a$ such that, for every $i\in\{1,\dots,m\}$, $a_i=r_ia$ for some integer $r_i$. The real number $a$ is called a *period* of the set $\{a_1,\dots,a_m\}$.

We now give a full description of $\mathcal{S}_{\mathcal{F}}(m,\underline{\lambda},\underline{t})$.

THEOREM 2.1. *Assume that* $m\geqslant 1$, $t_i>0$, $\lambda_i>0$ $(i=1,\dots,m)$ *satisfy* (2.1). *A pgf* $P(\cdot)$ *belongs to* $\mathcal{S}_{\mathcal{F}}(m,\underline{\lambda},\underline{t})$ *if and only if one of the following two conditions holds*, *with* $\gamma$ *being the unique solution to* $\sum_{i=1}^m\lambda_ie^{-\gamma t_i}=1$ *and* $\gamma$ *necessarily in* $(0,1]$:

(i) $(t_1,\dots,t_m)$ *are commensurable with some period* $t>0$ *and* $P(z)$ *is the pgf of an* $\mathcal{F}$*-semistable distribution with exponent* $\gamma$ *and order* $t$ (*and hence of orders* $t_1,\dots,t_m$).

(ii) $(t_1,\dots,t_m)$ *are noncommensurable and* $P(z)$ *is the pgf of an* $\mathcal{F}$*-stable distribution with exponent* $\gamma$.

Proof. To prove the “if” part assume that $\gamma\in(0,1]$ is solution to $\sum_{i=1}^m\lambda_ie^{-\gamma t_i}=1$. Under (i), if $(t_1,\dots,t_m)$ are commensurable with some period $t>0$ and $P(z)$ is the pgf of an $\mathcal{F}$-semistable distribution with exponent $\gamma$ and order $t$, then $P(z)$ admits the representation (1.8) where $g_{\gamma,t}(\cdot)$ is a nonnegative periodic function over $\mathbf{R}_+$ with period $t$. Since, for every $i\in\{1,\dots,m\}$, $t_i=rt$ for some positive integer $r_i$, it follows that $g_{\gamma,t}(\cdot)$ has periods $t_i$, $i=1,\dots,m$. Therefore,

$$(2.6)\qquad \begin{aligned}\sum_{i=1}^m\lambda_i\ln P\big(F_{t_i}(z)\big)&=-A(z)^{\gamma}\Big(\sum_{i=1}^m\lambda_ie^{-\gamma t_i}g_{\gamma,t}\big(|\ln A(z)|+t_i\big)\Big)\\&=\ln P(z),\end{aligned}$$

which implies $P(\cdot)\in\mathcal{S}_{\mathcal{F}}(m,\underline{\lambda},\underline{t})$. Under (ii), $P(z)$ is the pgf of an $\mathcal{F}$-stable distribution with exponent $\gamma$ (we note that the lack of commensurability of the $\ln\alpha_i$’s is not needed at this stage of the proof). Then, by (1.7), $\ln P(z) = -cA(z)^\gamma$ for some $c > 0$. It is easily shown that $P(z)$ satisfies (1.9). Therefore, $P(\cdot) \in \mathcal{S}_\mathcal{F}(m, \underline{\lambda}, \underline{t})$. This concludes the proof of the "if" part.

To prove the "only if" part, we will assume, without loss of generality, that $t_i \neq t_j$ for all $i, j \in \{1, \dots, m\}$, $i \neq j$. Let $P(\cdot) \in \mathcal{S}_\mathcal{F}(m, \underline{\lambda}, \underline{t})$ and define $f(x) = -\ln\left[P\left(A^{-1}(e^{-x})\right)\right]$, $x \geqslant 0$, where $A^{-1}(\cdot)$ is the inverse function of $A(\cdot)$ (note $A$ is one-to-one from $[0,1]$ onto $[0,1]$). Clearly, $f$ is nonnegative. By (1.14b), $F_t\left(A^{-1}(e^{-x})\right) = A^{-1}(e^{-t-x})$ for any $t > 0$ and $x \geqslant 0$. Therefore, by (1.9),

$$f(x) = -\sum_{i=1}^m \lambda_i \ln P\Big(F_{t_i}\left(A^{-1}(e^{-x})\right)\Big) = \sum_{i=1}^m \lambda_i f(x + t_i) \quad (x \geqslant 0). \tag{2.7}$$

For $a > 0$, let $\delta_a$ be the Dirac point-mass measure on the $\sigma$-field of the Borel sets of $\mathbf{R}_+$. Define

$$\mu(\cdot) = \sum_{i=1}^m \lambda_i \delta_{t_i}(\cdot). \tag{2.8}$$

The set function $\mu$ is a finite measure on the $\sigma$-field of the Borel sets of $\mathbf{R}_+$ with $\mu(\{0\}) = 0$. It is easily seen that equation (2.7) can be rewritten in the form of the integral equation (1.15) in Theorem 1.1 with $\mu$ of (2.8). The equation holds for every $x \geqslant 0$. By Theorem 1.1, there exists $\eta \in \mathbf{R}$, necessarily unique, such that $\sum_{i=1}^m \lambda_i e^{\eta t_i} = 1$. Setting $\gamma = -\eta$, we have $\sum_{i=1}^m \lambda_i e^{-\gamma t_i} = 1$. Suppose $\mu$ is arithmetic with some span $t$. We can assume, without loss of generality, that $t > 0$. Now the support of $\mu$ is $\{t_1, \dots, t_m\}$ ($\mu(\{t_i\}) = \lambda_i > 0$, $i = 1, \dots, m$). It follows that $(t_1, \dots, t_m)$ are commensurable with period $t$, or, for every $i \in \{1, \dots, m\}$, $t_i = r_i t$ for some positive integer $r_i$. By Theorem 1.1 (statement (i), $n = 1$),

$$f(x + t) = f(x) e^{\eta t} = e^{-\gamma t} f(x) \quad (x \geqslant 0)$$

or, equivalently, through the change of variable $z = A^{-1}(e^{-x})$,

$$\ln P\big(F_t(z)\big) = e^{-\gamma t} \ln P(z) \quad (0 \leqslant z < 1).$$

This implies that $P(z)$ satisfies (1.5) with $\lambda = e^{\gamma t}$, and hence $P(z)$ is the pgf of an $\mathcal{F}$-semistable distribution with exponent $\gamma$ and order $t$. The fact that $\gamma \in (0,1]$ follows from Lemma 2.1 (ii). Moreover, it is easy to see that the commensurability of the $t_i$'s implies that $P(z)$ is $\mathcal{F}$-semistable with exponent $\gamma$ and order $t_i$ for each $i = 1, \dots, m$. Assume now that $\mu$ is not arithmetic. Necessarily, $(t_1, \dots, t_m)$ are noncommensurable. By Theorem 1.1 (statement (ii)), there exists $c > 0$ such that $f(x) = ce^{\eta x} = ce^{-\gamma x}$, $x \geqslant 0$, or, equivalently, $\ln P(z) = -cA(z)^\gamma$, $0 \leqslant z < 1$. Thus $P(z)$ is the pgf of an $\mathcal{F}$-stable distribution with exponent $\gamma$, necessarily in $(0,1]$. ∎

We recall that a function $P(z)$ on $[0,1]$ is the pgf of an infinitely divisible discrete distribution if and only if it admits the representation (see Steutel and van Harn [20], Theorem 4.2, Chapter II)

$$\ln P(z) = -\int_z^1 R(x)\,dx \quad (0 \leqslant z \leqslant 1), \tag{2.9}$$

where $R(x) = \sum_{n=0}^{\infty} r_n x^n$ with $r_n \geqslant 0$ and, necessarily, $\sum_{n=0}^{\infty} r_n (n+1)^{-1} < \infty$. Following Steutel and van Harn [20], we will refer to $R(z)$ as the $R$-function of $P(z)$.

Since $\mathcal{F}$-stable and $\mathcal{F}$-semistable distributions are infinitely divisible, it is worthwhile to describe $\mathcal{S}_{\mathcal{F}}(m, \underline{\lambda}, \underline{t})$ in terms of $R$-functions.

THEOREM 2.2. *Assume that* $m \geqslant 1$, $t_i > 0$, $\lambda_i > 0$ $(i = 1, \ldots, m)$ *satisfy* (2.1) *and* $P(z)$ *is infinitely divisible. Then* $P(\cdot) \in \mathcal{S}_{\mathcal{F}}(m, \underline{\lambda}, \underline{t})$ *if and only if one of the following two conditions holds, with* $\gamma$ *being the solution to* $\sum_{i=1}^{m} \lambda_i e^{-\gamma t_i} = 1$ *and* $\gamma$ *necessarily in* $(0,1]$:

(i) $(t_1, \ldots, t_m)$ *are commensurable with some period* $t > 0$ *and* $P(z)$ *is the pgf of an* $\mathcal{F}$*-semistable distribution with exponent* $\gamma$, *order* $t$ (*and hence of orders* $t_1, \ldots, t_m$), *and an* $R$*-function with the representation*

$$R(z) = A(z)^{\gamma} r_{\gamma,t}\Big(\big|\ln\big(A(z)\big)\big|\Big)/U(z) \quad (0 \leqslant z < 1), \tag{2.10}$$

*where* $r_{\gamma,t}(\cdot)$ *is a nonnegative periodic function defined over* $[0,\infty)$, *with periods* $t$ *and* $t_i$, $i = 1, \ldots, m$.

(ii) $(t_1, \ldots, t_m)$ *are noncommensurable and* $P(z)$ *is the pgf of an* $\mathcal{F}$*-stable distribution with exponent* $\gamma$ *and an* $R$*-function of the form*

$$R(z) = kA(z)^{\gamma}/U(z) \quad (0 \leqslant z < 1) \tag{2.11}$$

*for some* $k > 0$.

Proof. By (1.9) and (2.9), $P(\cdot) \in \mathcal{S}_{\mathcal{F}}(m, \underline{\lambda}, \underline{t})$ if and only if its $R$-function is solution to the functional equation

$$R(z) = \sum_{i=1}^{m} \lambda_i F'_{t_i}(z) R\big(F_{t_i}(z)\big) \quad (0 \leqslant z < 1). \tag{2.12}$$

It is easy to see (by way of (1.14a) and (1.14b)) that $R(z)$ of (2.10) (resp., (2.11)), under condition (i) (resp., (ii)), satisfies (2.12). This establishes the "if" part. We now prove the "only if" part. Let $P(\cdot) \in \mathcal{S}_{\mathcal{F}}(m, \underline{\lambda}, \underline{t})$. We assume first that condition (i) in Theorem 2.1 holds. Noting that $R(z) = (d/dz)[\ln P(z)]$, we deduce the representation (2.10) from (1.8) and differentiation. In this case, $r_{\gamma,t}(x) = \gamma g_{\gamma,t}(x) - g'_{\gamma,t}(x)$, $x \geqslant 0$. We note that the differentiability of the function $g_{\gamma,t}(\cdot)$

on $(0,\infty)$ ensues from the proof of Theorem 3.1 in Bouzar [5]. The nonnegativity of $r_{\gamma,t}(x)$ follows from that of $R(x)$. Moreover, since $g_{\gamma,t}(x)$ is periodic with periods $t$ and $t_i$, $i=1,\dots,m$, it is easily shown that $r_{\gamma,t}(x)$ enjoys the same property. A similar argument, assuming this time that condition (ii) in Theorem 2.1 holds, leads to the representation (2.11). ∎

COROLLARY 2.1. *Let $P(z)$ be a pgf such that $0<P(0)<1$. Then*

$$P\big(F_{t_1}(z)\big)P\big(F_{t_2}(z)\big)=P\big(F_t(z)\big)\quad(0\leqslant z\leqslant 1)\tag{2.13}$$

*for some distinct $t_1>0$, $t_2>0$, $t\geqslant 0$ (that necessarily satisfy $t_i>t$, $i=1,2$) if and only if one of the following two conditions holds, with $\gamma\in(0,1]$ being the unique solution to $e^{-\gamma t_1}+e^{-\gamma t_2}=e^{-\gamma t}$:*

(i) *$t_1-t$ and $t_2-t$ are commensurable with period $s>0$ and $P(z)$ is the pgf of an $\mathcal{F}$-semistable distribution with exponent $\gamma$ and order $s>0$.*

(ii) *$t_1-t$ and $t_2-t$ are noncommensurable and $P(z)$ is the pgf of an $\mathcal{F}$-stable distribution with exponent $\gamma$.*

Proof. The proof of the "if" part is essentially the same as that of its counterpart in Theorem 2.1. The details are skipped. To prove the "only if" part, we first note that, by (2.13), $P\big(F_t(z)\big)<P\big(F_{t_i}(z)\big)$ for any $0\leqslant z<1$ and $i=1,2$. By (1.14a), $(\partial/\partial s)F_s(z)>0$ for any $0\leqslant z<1$, which implies that $F_s(z)$ is an increasing function of the variable $s$. Therefore, $t_i>t$, $i=1,2$. By (1.2) and (2.13),

$$P\big(F_t(z)\big)=P\Big(F_{t_1-t}\big(F_t(z)\big)\Big)P\Big(F_{t_2-t}\big(F_t(z)\big)\Big)\quad(0\leqslant z\leqslant 1).$$

A simple change of variable argument shows that $P(z)$ satisfies (1.9) for every $z\in[F_t(0),1]$, with $m=2$, $\underline{\lambda}=(1,1)$, and $\underline{t}=(t_1-t,t_2-t)$, and thus, by analytic continuation, for every $z\in[0,1]$. The conclusion follows by applying Theorem 2.1. ∎

We note that Corollary 2.1 extends a result obtained by Rao and Shanbhag [16], Theorem 6.4.6, p. 159, for $\mathcal{F}$-stable distributions, where $\mathcal{F}$ is the binomial thinning semigroup of Steutel and van Harn [20], Chapter V, Section 4,

$$F_t(z)=1-e^{-t}+e^{-t}z.\tag{2.14}$$

COROLLARY 2.2. *Let $P(z)$ be the pgf of a distribution on $\mathbf{Z}_+$ such that $0<P(0)<1$. The following assertions are equivalent:*

(i) *$P(z)$ is $\mathcal{F}$-stable.*

(ii) *$P(z)$ is $\mathcal{F}$-semistable and has two noncommensurable orders $t_1,t_2>0$.*

(iii) *There exist two noncommensurable numbers $t_1,t_2>0$ such that*

$$P\big(F_{t_1}(z)\big)P\big(F_{t_2}(z)\big)=P(z)\quad(z\in[0,1]).$$

P r o o f. To prove (i)⇔(ii) assume that $P(z)$ is the pgf of an $\mathcal{F}$-semistable distribution with two noncommensurable orders $t_1, t_2 > 0$. There exists $\lambda_i \geqslant 1$, $i = 1, 2$, such that $\ln P(z) = \lambda_i \ln P\big(F_{t_i}(z)\big)$, $z \in [0, 1]$, which implies

$$\ln P(z) = \frac{\lambda_1}{2} \ln P\big(F_{t_1}(z)\big) + \frac{\lambda_2}{2} \ln P\big(F_{t_2}(z)\big).$$

Therefore, $P(\cdot) \in \mathcal{S}_{\mathcal{F}}(2, \underline{\lambda}/2, \underline{t})$. Condition (2.1) holds by Lemma 2.1. Since $t_1$ and $t_2$ are noncommensurable, it follows by Theorem 2.1 that $P(z)$ is the pgf of an $\mathcal{F}$-stable distribution. The converse is trivially true. The equivalence (i)⇔(iii) follows straightforwardly from Theorem 2.1 with $m = 2$, $\lambda_1 = \lambda_2 = 1$. ■

COROLLARY 2.3. *Let $P(z)$ be the pgf of an $\mathcal{F}$-semistable distribution on $\mathbf{Z}_+$. Then there exists a unique $\gamma \in (0, 1]$ such that $\lambda = e^{\gamma t}$ for any $\lambda, t > 0$ satisfying* (1.5).

P r o o f. If $P(z)$ admits two noncommensurable orders, then the conclusion follows trivially from Corollary 2.2. Assume that $P(z)$ satisfies (1.5) for some $\lambda_1, t_1 > 0$ and $\lambda_2, t_2 > 0$ such that $t_1$ and $t_2$ are commensurable with period $t > 0$. By Lemma 2.1, $\lambda_i = e^{\gamma_i t_i}$, $\gamma_i \in (0, 1]$, $i = 1, 2$. The same argument used to prove Corollary 2.2 leads to $P(\cdot) \in \mathcal{S}_{\mathcal{F}}(2, \underline{\lambda}/2, \underline{t})$. By Theorem 2.1, part (ii), there exists $\gamma \in (0, 1]$ such that $P(z)$ is semistable with exponent $\gamma$ and of orders $t, t_1, t_2$. Therefore, $\gamma_1 = \gamma_2$. ■

Next, we state a characterization of $\mathcal{F}$-stability that is an extension of a result obtained by Gupta et al. [7], Theorem 5.2, for the binomial thinning semigroup (2.14).

THEOREM 2.3. *Let $(p_n, n \geqslant 0)$ be a sequence in $(0, 1)$ and $\xi$ a function from $(0, 1)$ to $(0, 1)$ such that*:

(i) $\liminf_n p_n = \liminf_n \xi(p_n) = 0$, $\limsup_n p_n = c$, *and* $\limsup_n \xi(p_n) = c'$, *where* $c, c' \in (0, 1)$.

(ii) *The smallest closed subgroup under addition of* $\mathbf{R}$ *generated by* $\big\{\ln(p_n/c'),\ \ln\big(\xi(p_n)/c\big) : n = 1, 2, \dots\big\}$ *is* $\mathbf{R}$ *itself.*

*Let $P_i(z)$, $i = 1, 2, 3$, be pgf's such that one of them is non-degenerate and, for $n \geqslant 1$, set $t_n = -\ln p_n$ and $s_n = -\ln \xi(p_n)$. Then*

$$P_1\big(F_{t_n}(z)\big) P_2\big(F_{s_n}(z)\big) = P_3(z) \quad (0 \leqslant z \leqslant 1;\ n \geqslant 1) \tag{2.15}$$

*if and only if $P_3(z) = P_1\big(F_{-\ln c'}(z)\big) = P_2\big(F_{-\ln c}(z)\big)$, $0 \leqslant z \leqslant 1$, and $P_3(z)$ is the pgf of an $\mathcal{F}$-stable distribution with some exponent $\gamma \in (0, 1]$ that satisfies $(p_n/c')^\gamma + \big(\xi(p_n)/c\big)^\gamma = 1$ for every $n \geqslant 1$.*

Proof. Set $t' = -\ln c'$ and $s' = -\ln c$. For the "if" part, we note that the condition $(p_n/c')^\gamma + \big(\xi(p_n)/c\big)^\gamma = 1$ implies $p_n < c'$ (or $t_n > t'$) and $\xi(p_n) < c$ (or $s_n > s'$). We have, by assumption and by (1.2),

$$P_1\big(F_{t_n}(z)\big) = P_3\big(F_{t_n - t'}(z)\big) \quad \text{and} \quad P_2\big(F_{s_n}(z)\big) = P_3\big(F_{s_n - s'}(z)\big).$$

Equation (2.15) follows then from (1.7). For the "only if" part, assume that (i), (ii), and (2.15) hold. By using the same argument as in Gupta and al. [7], the proof of Theorem 5.2, we arrive at $p_n < c'$, $\xi(p_n) < c$ for all $n \geqslant 1$, and

$$P_1\big(F_{t'}(z)\big) = P_3(z) \quad \text{and} \quad P_2\big(F_{t'}(z)\big) = P_3(z) \quad (z \in [0,1]).$$

Therefore, by (2.15),

$$P_3\big(F_{t_n - t'}(z)\big)P_3\big(F_{s_n - s'}(z)\big) = P_3(z) \quad (z \in [0,1];\ n \geqslant 1). \tag{2.16}$$

If $t_{n_0} - t'$ and $s_{n_0} - s'$ are noncommensurable for some $n_0 \geqslant 1$, then by Corollary 2.2, $P_3(z)$ is $\mathcal{F}$-stable with some exponent $\gamma \in (0,1]$ and, by (2.16), $(p_n/c')^\gamma + \big(\xi(p_n)/c\big)^\gamma = 1$ for every $n \geqslant 1$. Assume now that, for each $n \geqslant 1$, $t_n - t'$ and $s_n - s'$ are commensurable. It follows by Theorem 2.1 and Corollary 2.3 that $P_3(z)$ is $\mathcal{F}$-semistable with some exponent $\gamma \in (0,1]$ and of orders $\{t_n - t', s_n - s' : n = 1, 2, \ldots\}$, with $(p_n/c')^\gamma + \big(\xi(p_n)/c\big)^\gamma = 1$, $n \geqslant 1$ (recall $t_n - t' = -\ln(p_n/c')$ and $s_n - s' = -\ln\big(\xi(p_n)/c\big)$). The continuous function $g(\tau)$ in the representation (1.8) of $P_3(z)$ is periodic with periods $\big\{-\ln(p_n/c'), -\ln\big(\xi(p_n)/c\big) : n = 1, 2, \ldots\big\}$. Hence, by assumption (ii), $g(\tau)$ admits any positive number as its period, and thus it must be constant, making $P_3(z)$ $\mathcal{F}$-stable. ∎

REMARK 2.1. (i) The continuous version of Theorem 2.1 for distributions on $\mathbf{R}$ was obtained by Shimizu in [19]. Ben Alaya and Huillet [3] established the $\mathbf{R}_+$-version of the result. Various generalizations of Shimizu's result as well as related characterizations of stable laws both in the univariate and multivariate settings were proposed by several authors. We refer to Rao and Shanbhag [16] and the article by Gupta et al. [7], and the references therein, for more on the topic.

(ii) Theorems 2.1 and 2.2, as well as Corollary 2.2, were established by Bouzar [4] for the binomial thinning semigroup (2.14).

We conclude the section by discussing an example.

EXAMPLE 2.1. Let $m = 2$, $t > 0$, and $0 < \lambda < e^t(e^t - 1)$. Let also $\underline{\lambda} = (1, \lambda)$ and $\underline{t} = (t, 2t)$. Clearly, condition (2.1) holds and there is a (unique) $\gamma \in (0,1)$ such that $e^{-\gamma t} + \lambda e^{-2\gamma t} = 1$, specifically,

$$\gamma = -\ln\left[\frac{-1 + \sqrt{1 + 4\lambda}}{2\lambda}\right]/t.$$

Let $\psi_t(x)$ be a continuous bounded nonnegative and periodic function with period $t$. The function $P(z)$ defined by

$$P(z) = \exp\Big\{-\int_0^\infty (1 - e^{-A(z)x}) x^{-1-\gamma} \psi_t(\ln x)\, dx\Big\} \qquad (0 \leqslant z \leqslant 1)$$

is a pgf (see Bouzar [5]), where $A(z)$ is given by (1.13). It is easily verified that $P(z) \in \mathcal{S}_{\mathcal{F}}(2, \underline{\lambda}, \underline{t})$. By Theorem 2.1 (i) (or direct calculations), $P(z)$ is semistable with exponent $\gamma$ and order $t$.

### 3. MULTIVARIATE $\mathcal{F}$-SEMISTABILITY

We start out by extending the multiplication $\odot_{\mathcal{F}}$ of (1.1) to the multivariate setting.

Let $d \geqslant 1$ be a natural number, $(X_1, \dots, X_d)$ be a $\mathbf{Z}_+^d$-valued random vector, and $\alpha \in (0, 1)$. Then

$$\alpha \odot_{\mathcal{F}} (X_1, \dots, X_d) = (\alpha \odot_{\mathcal{F}} X_1, \dots, \alpha \odot_{\mathcal{F}} X_d). \tag{3.1}$$

The multiplications $\alpha \odot_{\mathcal{F}} X_j$ in (3.1) are performed independently for each $j$. More precisely, we suppose the existence of $d$ independent sequences $\big(Y_k^{(j)}(t), t \geqslant 0, k \geqslant 1\big)$, $j = 1, 2, \dots, d$, of iid continuous-time Markov branching processes driven by the semigroup $\mathcal{F}$ (see (1.1)), independent of $(X_1, \dots, X_d)$, such that

$$\alpha \odot_{\mathcal{F}} X_j = \sum_{k=1}^{X_j} Y_k^{(j)}(t) \qquad (t = -\ln \alpha). \tag{3.2}$$

By convention, and in compatibility with (1.10), we set

$$0 \odot_{\mathcal{F}} (X_1, \dots, X_d) = (0, \dots, 0) \quad \text{and} \quad 1 \odot \mathcal{F}(X_1, \dots, X_d) = (X_1, \dots, X_d).$$

We recall that the pgf $P(z_1, \dots, z_d)$ of a distribution $(p_{n_1,\dots,n_d}, n_1, \dots, n_d \in \mathbf{Z}_+)$ on $\mathbf{Z}_+^d$ is defined by

$$P(z_1, \dots, z_d) = \sum_{n_1,\dots,n_d} p_{n_1,\dots,n_d} z_1^{n_1} \dots z_d^{n_d} \qquad (|z_i| \leqslant 1;\ i = 1, \dots, d).$$

From the assumptions and a conditioning argument, it is easily shown that the pgf of $\alpha \odot_{\mathcal{F}} (X_1, \dots, X_d)$ is (for $|z_j| \leqslant 1$, $j = 1, \dots, d$)

$$P_{\alpha \odot_{\mathcal{F}} (X_1,\dots,X_d)}(z_1, \dots, z_d) = P\big(F_t(z_1), \dots, F_t(z_d)\big), \tag{3.3}$$

where $P(z_1, \dots, z_d)$ is the pgf of $(X_1, \dots, X_d)$ and $t = -\ln \alpha$.

For $\alpha, \beta \in (0, 1)$, we have, by (1.2) and (3.3),

$$\alpha \odot_{\mathcal{F}} \big(\beta \odot_{\mathcal{F}} (X_1, \dots, X_d)\big) \stackrel{d}{=} (\alpha\beta) \odot_{\mathcal{F}} (X_1, \dots, X_d). \tag{3.4a}$$

Moreover, if $(X_1, \ldots, X_d)$ and $(Y_1, \ldots, Y_d)$ are independent $\mathbf{Z}_+^d$-valued random vectors and $\alpha \in (0,1)$, then

$$\text{(3.4b)} \quad \alpha \odot_{\mathcal{F}} \big((X_1, \ldots, X_d) + (Y_1, \ldots, Y_d)\big) \stackrel{d}{=} \stackrel{d}{=} \alpha \odot_{\mathcal{F}} (X_1, \ldots, X_d) + \alpha \odot_{\mathcal{F}} (Y_1, \ldots, Y_d).$$

A distribution on $\mathbf{Z}_+^d$ is said to be $\mathcal{F}$*-semistable* if its pgf $P(z_1, \ldots, z_d)$ satisfies $0 < P(0, \ldots, 0) < 1$ and

$$\text{(3.5)} \quad P(z_1, \ldots, z_d) = \big[P\big(F_t(z_1), \ldots, F_t(z_d)\big)\big]^{\lambda} \quad (z_1, \ldots, z_d \in [0,1])$$

for some $t > 0$ and $\lambda > 0$.

A distribution on $\mathbf{Z}_+^d$ (or its pgf) with pgf $P(z_1, \ldots, z_d)$ such that $0 < P(0, \ldots, 0) < 1$ is said to be $\mathcal{F}$*-stable* if, for every $t > 0$, there exists $\lambda > 0$ such that (3.5) holds.

As a direct consequence of the definition, the marginal distributions of an $\mathcal{F}$-semistable (resp., $\mathcal{F}$-stable) distribution on $\mathbf{Z}_+^d$ are univariate $\mathcal{F}$-semistable (resp., $\mathcal{F}$-stable). Therefore, by Lemma 2.1 (ii), $\lambda$ and $t$ in (3.5) satisfy the equation $\lambda = e^{\gamma t}$ for some $\gamma \in (0,1]$. As in the univariate case, we will refer to $\gamma$ and $t$ as the exponent and order of an $\mathcal{F}$-semistable distribution, respectively.

By definition, a distribution on $\mathbf{Z}_+^d$ is $\mathcal{F}$-stable if and only if it is $\mathcal{F}$-semistable of all orders $t > 0$.

PROPOSITION 3.1. *Any $\mathcal{F}$-semistable, and thus any $\mathcal{F}$-stable, distribution on $\mathbf{Z}_+^d$ is infinitely divisible.*

Proof. Let $P(z_1, \ldots, z_d)$ be the pgf of an $\mathcal{F}$-semistable distribution with exponent $\gamma \in (0,1]$ and order $t > 0$. By (1.2), (3.5), and induction, we have, for any $n \geqslant 0$ and $z_1, \ldots, z_d \in [0,1]$,

$$\text{(3.6)} \qquad [P(z_1, \ldots, z_d)]^{e^{-n\gamma t}} = P\big(F_{nt}(z_1), \ldots, F_{nt}(z_d)\big).$$

Let $P_n(z_1, \ldots, z_d) = \exp\big\{e^{n\gamma t}\big([P(z_1, \ldots, z_d)]^{e^{-n\gamma t}} - 1\big)\big\}$ for $n \geqslant 0$. By (3.6), $P_n(z_1, \ldots, z_d)$ is the pgf of a compound Poisson distribution on $\mathbf{Z}_+^d$ and is therefore infinitely divisible. Moreover, we have

$$\lim_{n \to \infty} P_n(z_1, \ldots, z_d) = P(z_1, \ldots, z_d).$$

Hence, any $\mathcal{F}$-semistable distribution is the weak limit of a sequence of infinitely divisible distributions and is therefore infinitely divisible. ∎

We state a useful lemma.

LEMMA 3.1. *Let* $P(z_1,\dots,z_d)$ *be the pgf of an* $\mathcal{F}$*-semistable distribution on* $\mathbf{Z}_+^d$*. Then there exists a unique* $\gamma\in(0,1]$ *such that* $\lambda=e^{\gamma t}$ *for any* $\lambda,t>0$ *satisfying* (3.5).

Proof. The marginal pgf $Q_1(z)=P(z,1,\dots,1)$ of $P(z_1,\dots,z_d)$ is (univariate) $\mathcal{F}$-semistable. The conclusion follows by applying Corollary 2.3 to the marginal pgf $Q_1(z)$. ■

Next, we give characterization results for $\mathcal{F}$-semistability and $\mathcal{F}$-stability for distributions on $\mathbf{Z}_+^d$.

THEOREM 3.1. *Let* $t>0$ *and* $0<\gamma\leqslant 1$*. A distribution on* $\mathbf{Z}_+^d$ *with pgf* $P(z_1,\dots,z_d)$ *is* $\mathcal{F}$*-semistable with exponent* $\gamma$ *and order* $t$ *if and only if, for any* $z_1,\dots,z_d\in[0,1)$,

$$\ln P(z_1,\dots,z_d)=-\Big(\prod_{i=1}^d A(z_i)\Big)^{\gamma/d} g_{\gamma,t}\big(|\ln A(z_1)|,\dots,|\ln A(z_d)|\big), \tag{3.7}$$

*where* $A(z)$ *is the A-function of* $\mathcal{F}$ (see (1.13)) *and* $g_{\gamma,t}(\tau_1,\dots,\tau_d)$ *is a continuous function from* $\mathbf{R}_+^d$ *to* $\mathbf{R}_+$ *such that*

$$g_{\gamma,t}(\tau_1+t,\dots,\tau_d+t)=g_{\gamma,t}(\tau_1,\dots,\tau_d)\qquad(\tau_i\geqslant 0;\ i=1,\dots,d). \tag{3.8}$$

Proof. Let $P(z_1,\dots,z_d)$ be the pgf of an $\mathcal{F}$-semistable distribution on $\mathbf{Z}_+^d$ with exponent $\gamma$ and order $t$. Define, for $z_1,\dots,z_d\in[0,1)$,

$$f_{\gamma,t}(z_1,\dots,z_d)=-\Big(\prod_{i=1}^d A(z_i)\Big)^{-\gamma/d}\ln P(z_1,\dots,z_d).$$

By (3.5) and (1.14b), $f_{\gamma,t}\big(F_t(z_1),\dots,F_t(z_d)\big)=f_{\gamma,t}(z_1,\dots,z_d)$. Therefore,

$$\ln P(z_1,\dots,z_d)=-\Big(\prod_{i=1}^d A(z_i)\Big)^{\gamma/d} f_{\gamma,t}(z_1,\dots,z_d). \tag{3.9}$$

For $\tau_i\geqslant 0$, $i=1,\dots,d$, define

$$g_{\gamma,t}(\tau_1,\dots,\tau_d)=f_{\gamma,t}\big(A^{-1}(e^{-\tau_1}),\dots,A^{-1}(e^{-\tau_d})\big).$$

We clearly have, for any $z_1,\dots,z_d\in[0,1)$,

$$g_{\gamma,t}\big(-\ln A(z_1),\dots,-\ln A(z_d)\big)=f_{\gamma,t}(z_1,\dots,z_d). \tag{3.10}$$

Combining (3.9) and (3.10) leads to (3.7). Now $A\big[F_t\big(A^{-1}(e^{-\tau})\big)\big]=e^{-\tau-t}$ for any $\tau\geqslant 0$. Therefore,

$$g_{\gamma,t}(\tau_1+t,\dots,\tau_d+t)=f_{\gamma,t}\Big(F_t\big(A^{-1}(e^{-\tau_1})\big),\dots,F_t\big(A^{-1}(e^{-\tau_d})\big)\Big),$$

which, along with (3.10), implies (3.8). Conversely, a pgf of a distribution on $\mathbf{Z}_+^d$ that admits the representation (3.7) must satisfy (3.5) for any $z_1, \ldots, z_d \in [0,1)$, and thus, by the principle of analytic continuation, for any $z_1, \ldots, z_d$ such that $|z_i| \leqslant 1$. ∎

THEOREM 3.2. *Let $0 < \gamma \leqslant 1$. A distribution on $\mathbf{Z}_+^d$ with pgf $P(z_1, \ldots, z_d)$ is $\mathcal{F}$-stable with exponent $\gamma$ if and only if, for any $z_1, \ldots, z_d \in [0,1)$,*

(3.11)
$$\ln P(z_1, \ldots, z_d) = \\ = -\Big(\prod_{i=1}^{d} A(z_i)\Big)^{\gamma/d} Q_\gamma\big(\ln[A(z_1)/A(z_2)], \ldots, \ln[A(z_1)/A(z_d)]\big),$$

*where $A(z)$ is the $A$-function of $\mathcal{F}$* (see (1.13)) *and $Q_\gamma(x_1, \ldots, x_{d-1})$ is a nonnegative function on $\mathbf{R}^{d-1}$ if $d \geqslant 2$, and a constant if $d = 1$.*

Proof. If $P(z_1, \ldots, z_d)$ satisfies (3.11), it is easily shown that it also satisfies (3.5) for every $t > 0$ (with $\lambda = e^{\gamma t}$). Therefore, $P(z_1, \ldots, z_d)$ is $\mathcal{F}$-stable. Conversely, if $P(z_1, \ldots, z_d)$ is $\mathcal{F}$-stable, then it is $\mathcal{F}$-semistable for every $t > 0$ and some exponent $\gamma \in (0,1]$. By Lemma 3.1, the exponent $\gamma$ is independent of $t$. Therefore, the nonnegative function $g_{\gamma,t}(\tau_1, \ldots, \tau_d)$ in the representation (3.7) of $P(z_1, \ldots, z_d)$ is also independent of $t$. We denote it by $g(\tau_1, \ldots, \tau_d)$. Note that $g(\tau_1, \ldots, \tau_d)$ is defined on $\mathbf{R}_+^d$ and satisfies (3.8) for every $t > 0$. In the case $d = 1$, this implies that $g(\tau_1, \ldots, \tau_d)$ is constant. Assume $d \geqslant 2$. We define the subsets $B_j, j = 0, 1, \ldots, d-1$, of $\mathbf{R}^{d-1}$ as follows:

$$B_0 = \{(x_1, \ldots, x_{d-1}) \mid x_i \geqslant 0, i = 1, \ldots, d-1\},$$

$$B_i = \{(x_1, \ldots, x_{d-1}) \mid \\ x_i < 0, x_i < x_1, \ldots, x_i < x_{i-1}, x_i \leqslant x_{i+1}, \ldots, x_i \leqslant x_{d-1}\}$$

for $1 \leqslant i \leqslant d-2$, and

$$B_{d-1} = \{(x_1, \ldots, x_{d-1}) \mid x_{d-1} < 0, x_{d-1} < x_1, \ldots, x_{d-1} < x_{d-2}\}.$$

$\{B_i : i = 0, 1, \ldots, d-1\}$ forms a partition of $\mathbf{R}^{d-1}$. We introduce a function $Q_\gamma$ on $\mathbf{R}^{d-1}$ as follows. For $(x_1, \ldots, x_{d-1})$ in $\mathbf{R}^{d-1}$, set $x_0 = 0$ and let $x_{ij} = x_j - x_i$ for $i, j \in \{0, 1, \ldots, d-1\}$. Then $Q_\gamma$ is defined by

$$Q_\gamma(x_1, \ldots, x_{d-1}) = g(x_{i0}, x_{i1}, \ldots, x_{i,d-1}) \qquad ((x_1, \ldots, x_{d-1}) \in B_i)$$

for each $i \in \{0, 1, \ldots, d-1\}$. It is easily checked that for $\tau_i \geqslant 0,\ i = 1, \ldots, d$,

(3.12)
$$g(\tau_1, \ldots, \tau_d) = Q_\gamma(\tau_2 - \tau_1, \tau_3 - \tau_1, \ldots, \tau_d - \tau_1).$$

The representation (3.11) follows from (3.7) and (3.12). ∎

Let $(X_1, \ldots, X_d)$ be a $\mathbf{Z}_+^d$-valued random vector and $\alpha_1, \ldots, \alpha_d \in (0,1]$. Using the operator $\odot_{\mathcal{F}}$ of (1.1), we define a linear combination of the $X_i$'s as follows:

$$Y = \sum_{j=1}^{d} \alpha_j \odot_{\mathcal{F}} X_j \quad (\text{with } 1 \odot_{\mathcal{F}} X = X). \tag{3.13}$$

The multiplications $\alpha_j \odot_{\mathcal{F}} X_j$ in (3.13) are performed independently for each $j$ (see (3.2) and the discussion preceding it). From the assumptions and a conditioning argument, the pgf $P_Y(z)$ of the linear combination (3.13) is given by

$$P_Y(z) = P\big(F_{s_1}(z), \ldots, F_{s_d}(z)\big) \quad (s_j = -\ln \alpha_j;\ j = 1, \ldots, d), \tag{3.14}$$

where $P(z_1, \ldots, z_d)$ is the pgf of $(X_1, \ldots, X_d)$.

THEOREM 3.3. *A $\mathbf{Z}_+^d$-valued random vector $(X_1, \ldots, X_d)$ has an $\mathcal{F}$-semistable (resp., $\mathcal{F}$-stable) distribution with exponent $\gamma \in (0,1]$ and order $t > 0$ (resp., exponent $\gamma$) if and only if for every $(\alpha_1, \ldots, \alpha_d) \in (0,1]^d$ the linear combination* (3.13) *is univariate $\mathcal{F}$-semistable (resp., $\mathcal{F}$-stable) with exponent $\gamma$ and order $t$ (resp., exponent $\gamma$).*

Proof. It suffices to establish the result for semistability. The "only if" part follows easily from (3.5), (3.14), and the semigroup property (1.2). Assume that the linear combination (3.13) is univariate $\mathcal{F}$-semistable with exponent $\gamma \in (0,1]$ and order $t > 0$ for every $\alpha_1, \ldots, \alpha_d \in (0,1]$. Let $P(z_1, \ldots, z_d)$ be the pgf of $(X_1, \ldots, X_d)$ and $\lambda = e^{\gamma t}$. We have, by (1.5) and (3.14),

$$P\big(F_{s_1}(z), \ldots, F_{s_d}(z)\big) = \big[P\big(F_{s_1+t}(z), \ldots, F_{s_d+t}(z)\big)\big]^{\lambda} \tag{3.15}$$

for any $s_1, \ldots, s_d > 0$ and $z \in [0,1]$. Choose $z_1, \ldots, z_d$ arbitrarily in $[0,1)$. By (1.13) and (1.14b), the function $\varphi(t) = F_t(0)$ is one-to-one from $[0,\infty)$ onto $[0,1)$. Its inverse is $\varphi^{-1}(z) = \int_0^z \big(1/U(x)\big)\,dx$, $z \in [0,1)$. Thus, there exist $s_1, \ldots, s_d \in [0,\infty)$ such that $z_j = F_{s_j}(0)$, $j = 1, \ldots, d$. By setting $z = 0$ in (3.15), we have shown that (3.5) holds for any $z_1, \ldots, z_d \in [0,1)$, and thus, by the principle of analytic continuation, for any $z_1, \ldots, z_d$ such that $|z_i| \leqslant 1$. ∎

Next, we extend Theorem 2.1 to the multivariate setting by considering the functional equation

$$\ln P(z_1, \ldots, z_d) = \sum_{i=1}^{m} \lambda_i \ln P\big(F_{t_i}(z_1), \ldots, F_{t_i}(z_d)\big) \tag{3.16}$$

for $z_1, \ldots, z_d \in [0,1]$, where $m \geqslant 1$, $\lambda_i,\ t_i > 0$ $(i = 1, \ldots, m)$, and $P(z_1, \ldots, z_d)$ belongs to the set of pgf's of nondegenerate distributions on $\mathbf{Z}_+^d$ such that $0 < P(0, \ldots, 0) < 1$.

LEMMA 3.2. *A pgf $P(z_1, \ldots, z_d)$ is a solution to equation* (3.16) *if and only if the univariate pgf $Q_{s_1,\ldots,s_d}(z) = P\big(F_{s_1}(z), \ldots, F_{s_d}(z)\big)$* (see (3.14)) *is a solution to equation* (1.9) *with the same $\lambda_i$'s, $t_i$'s, and $m$ for every $s_1, \ldots, s_d \geqslant 0$.*

P r o o f. The "only if" part is straightforward. The same argument used in the second part of the proof of Theorem 3.3 shows that the converse holds. The details are omitted. ■

THEOREM 3.4. *Assume that $m \geqslant 1$, $t_i > 0$, $\lambda_i > 0$ $(i = 1, \ldots, m)$ satisfy the condition* (2.1). *A pgf $P(z_1, \ldots, z_d)$ is a solution to equation* (3.16) *if and only if one of the following two conditions holds, with $\gamma$ being the unique solution to $\sum_{i=1}^{m} \lambda_i e^{-\gamma t_i} = 1$ and $\gamma$ necessarily in $(0, 1]$:*

(i) *$(t_1, \ldots, t_m)$ are commensurable with some period $t > 0$ and $P(z_1, \ldots, z_d)$ is the pgf of an $\mathcal{F}$-semistable distribution on $\mathbf{Z}_+^d$ with exponent $\gamma$ and order $t$ (and hence of orders $t_1, \ldots, t_m$).*

(ii) *$(t_1, \ldots, t_m)$ are noncommensurable and $P(z_1, \ldots, z_d)$ is the pgf of an $\mathcal{F}$-stable distribution on $\mathbf{Z}_+^d$ with exponent $\gamma$.*

P r o o f. The proof of the "if" part is essentially the same as the one given in the univariate case (Theorem 2.1). The representations (3.7) (resp., (3.11)) clearly satisfy (3.16). Noting that in the univariate case, the exponent $\gamma \in (0, 1]$ and order $t > 0$ depend only on the $\lambda_i$'s, the $t_i$'s, and $m$ (see Theorem 2.1 and its proof), the "only if" part follows readily from Lemma 3.2 and Theorem 3.3. ■

The following result is the multivariate extension of Corollary 2.1 (and is proved along the same lines). The $\mathcal{F}$-stability part was first established by Rao and Shanbhag [16] for the binomial thinning semigroup (2.14).

COROLLARY 3.1. *Let $P(z)$ be a pgf of a distribution on $\mathbf{Z}_+^d$ such that $0 < P(0, \ldots, 0) < 1$. Then, there exist distinct numbers $t_1 > 0$, $t_2 > 0$, $t \geqslant 0$ (that necessarily satisfy $t_i > t$, $i = 1, 2$) such that*

$$(3.17)\quad P\big(F_{t_1}(z_1), \ldots, F_{t_1}(z_d)\big)P\big(F_{t_2}(z_1), \ldots, F_{t_2}(z_d)\big) = P\big(F_t(z_1), \ldots, F_t(z_d)\big)$$

*for every $z_1, \ldots, z_d \in [0, 1]$ if only if one of the following two conditions holds, with $\gamma \in (0, 1]$ being the unique solution to $e^{-\gamma t_1} + e^{-\gamma t_2} = e^{-\gamma t}$:*

(i) *$t_1 - t$ and $t_2 - t$ are commensurable with period $s > 0$ and $P(z_1, \ldots, z_d)$ is the pgf of an $\mathcal{F}$-semistable distribution with exponent $\gamma$ and order $s > 0$.*

(ii) *$t_1 - t$ and $t_2 - t$ are noncommensurable and $P(z_1, \ldots, z_d)$ is the pgf of an $\mathcal{F}$-stable distribution with exponent $\gamma$.*

We gather several characterizations of multivariate $\mathcal{F}$-stability that extend some results in the univariate case found in Section 2 and in Steutel and van Harn [20], Chapter V, Section 8.

COROLLARY 3.2. *Let $P(z_1,\dots,z_d)$ be the pgf of a distribution on $\mathbf{Z}_+^d$ such that $0 < P(0,\dots,0) < 1$. The following assertions are equivalent*:

(i) *$P(z_1,\dots,z_d)$ is $\mathcal{F}$-stable.*

(ii) *$P(z_1,\dots,z_d)$ is $\mathcal{F}$-semistable and admits two noncommensurable orders $t_1 > 0$ and $t_2 > 0$.*

(iii) *There exists $\gamma \in (0,1)$ such that*

$$P(z_1,\dots,z_d) = P\big(F_s(z_1),\dots,F_s(z_d)\big)P\big(F_t(z_1),\dots,F_t(z_d)\big) \tag{3.18}$$

*for every $s,t > 0$ that satisfy $e^{-\gamma s} + e^{-\gamma t} = 1$ and $z_1,\dots,z_d \in [0,1]$.*

(iv) *For every $n \geqslant 1$, there exists $t_n \geqslant 0$ such that*

$$P(z_1,\dots,z_d) = \big[P\big(F_{t_n}(z_1),\dots,F_{t_n}(z_d)\big)\big]^n \quad (z_1,\dots,z_d \in [0,1]). \tag{3.19}$$

*In this case, there exists $\gamma \in (0,1]$ such that $t_n = n^{1/\gamma}$ for every $n \geqslant 1$.*

(v) *There exists two noncommensurable numbers $t_1, t_2 > 0$ such that*

$$P\big(F_{t_1}(z_1),\dots,F_{t_1}(z_d)\big)P\big(F_{t_2}(z_1),\dots,F_{t_2}(z_d)\big) = P(z_1,\dots,z_d) \tag{3.20}$$

*for $z_1,\dots,z_d \in [0,1]$.*

Proof. For (i)⇔(ii) the proof is the same as that of Corollary 2.2.

To prove (i)⇔(iii) observe that (i)⇒(iii) is straightforward, via (3.11), and for (iii)⇒(i), we choose $s,t > 0$, noncommensurable and such that $e^{-\gamma s} + e^{-\gamma t} = 1$; then (i) follows from Theorem 3.4 with $m = 2$, $t_1 = s$, $t_2 = t$, $\lambda_1 = \lambda_2 = 1$.

For (i)⇔(iv), it is easy to see that $P(z_1,\dots,z_d)$ of (3.11) satisfies (3.19) by choosing $t_n = n^{1/\gamma}$, and thus (i)⇒(iv). Conversely, by (3.19), $P(z_1,\dots,z_d)$ is $\mathcal{F}$-semistable and satisfies (3.5) for every $n \geqslant 1$ ($\lambda = n$ and $t = t_n$). By Lemma 3.1, there exists $\gamma \in (0,1]$ such that $n = e^{\gamma t_n}$, or $t_n = n^{1/\gamma}$, for every $n \geqslant 1$. Choose $n$ and $m$ such that $t_n$ and $t_m$ are noncommensurable. The conclusion follows from (ii)⇒(i).

Finally, (i)⇔(v) is a direct consequence of Theorem 3.4 by taking $m = 2$, $\lambda_1 = \lambda_2 = 1$. ■

It is worthwhile translating Corollary 3.2 in terms of random vectors. The translation is based on equations (3.1), (3.3), and (3.4b).

COROLLARY 3.3. *Let $(X_1,\dots,X_d)$ be a $\mathbf{Z}_+^d$-valued random vector. The following assertions are equivalent*:

(i) *$(X_1,\dots,X_d)$ has an $\mathcal{F}$-stable distribution.*

(ii) *There exists $\gamma \in (0,1]$ such that, for every $\alpha \in (0,1)$,*

$$(X_1,\dots,X_d) \stackrel{d}{=} \alpha \odot_{\mathcal{F}} (X_1,\dots,X_d) + (1-\alpha^\gamma)^{1/\gamma} \odot_{\mathcal{F}} (X_1',\dots,X_d'),$$

*where $(X_1,\dots,X_d)$ and $(X_1',\dots,X_d')$ are independent and $(X_1,\dots,X_d) \stackrel{d}{=} (X_1',\dots,X_d')$.*

(iii) *For every* $n \geqslant 1$, *there exists* $\alpha_n \in (0,1)$ *such that*

$$(X_1, \ldots, X_d) \stackrel{d}{=} \alpha_n \odot_{\mathcal{F}} \sum_{i=1}^{n} (X_1^{(i)}, \ldots, X_d^{(i)}),$$

*where* $\big((X_1^{(i)}, \ldots, X_d^{(i)}), i \geqslant 1\big)$ *is iid, independent of* $(X_1, \ldots, X_d)$, *and* $(X_1, \ldots, X_d) \stackrel{d}{=} (X_1^{(i)}, \ldots, X_d^{(i)})$. *In this case, there exists* $\gamma \in (0,1]$ *such that* $\alpha_n = n^{-1/\gamma}$ *for every* $n \geqslant 1$.

We conclude the section by discussing the relationship between the notions of semistability on $\mathbf{Z}_+^d$ and $\mathbf{R}_+^d$.

We recall (see Sato [18]) that a distribution on $\mathbf{R}_+^d$ with Laplace–Stieltjes transform (LST) $\phi(u_1, \ldots, u_d)$ is said to be *stable* if for any $\alpha \in (0,1)$ there exists $\lambda > 0$ such that

$$\phi(u_1, \ldots, u_d) = [\phi(\alpha u_1, \ldots, \alpha u_d)]^{\lambda} \quad (u_1, \ldots, u_d \geqslant 0). \tag{3.21}$$

It is said to be *semistable* if (3.21) is satisfied for some $\lambda > 0$ and $\alpha \in (0,1)$.

The real numbers $\alpha$ and $\lambda$ in (3.21) satisfy the equation $\lambda = \alpha^{-\gamma}$ for some $\gamma \in (0,1]$. As above, we refer to $\gamma$ as the exponent of the distribution, and in the case of semistability, $t$ as its order.

LEMMA 3.3. *Let* $\phi(u_1, \ldots, u_d)$ *be the LST of a distribution on* $\mathbf{R}_+^d$ *and* $A(z)$ *the A-function of* $\mathcal{F}$ (see (1.13)). *Then, for any* $\theta > 0$,

$$P_\theta(z_1, \ldots, z_d) = \phi\big(\theta A(z_1), \ldots, \theta A(z_d)\big) \quad (z_1, \ldots, z_d \in [0,1]) \tag{3.22}$$

*is the pgf of a distribution on* $\mathbf{Z}_+^d$.

Proof. We use an argument due to Barndorff-Nielsen et al. [2] to construct a multiparameter $\mathbf{Z}_+^d$-valued stochastic process. Let us assume $\theta > 0$. Let $\{X_1(t)\}, \{X_2(t)\}, \ldots, \{X_d(t)\}$ be independent Lévy processes such that, for every $i = 1, \ldots, d$, the pgf of $X_i(1)$ is $\mathcal{F}$-discrete stable with pgf $Q_\theta(z) = e^{-\theta A(z)}$. Let $(T_1, \ldots, T_d)$ be an $\mathbf{R}_+^d$-valued random vector independent of the process $\{\big(X_1(t), \ldots, X_d(t)\big)\}$. Define the multiparameter process $X(s_1, \ldots, s_d)$ by

$$X(s_1, \ldots, s_d) = \big(X_1(s_1), \ldots, X_d(s_d)\big) \quad (s_1, \ldots, s_d \geqslant 0).$$

The conditional pgf of $X(T_1, \ldots, T_d)$ given $(T_1, \ldots, T_d)$ is

$$E\big(\prod_{i=1}^{d} z_i^{X_i(s_i)} | T_1 = s_1, \ldots, T_d = s_d\big) = \prod_{i=1}^{d} E(z_i^{X_i(s_i)}) = \prod_{i=1}^{d} \big(Q_\theta(z_i)\big)^{s_i}.$$

Letting $\phi(u_1, \ldots, u_d)$ be the LST of $(T_1, \ldots, T_d)$, we conclude that the (unconditional) pgf of $X(T_1, \ldots, T_d)$ is

$$P(z_1, \ldots, z_d) = E\Big(\exp\big(-\sum_{i=1}^{d} \theta A(z_i) T_i\big)\Big) = \phi\big(\theta A(z_1), \ldots, \theta A(z_d)\big)$$

for $z_1, \ldots, z_d \in [0,1]$. ∎

We now show that $\mathcal{F}$-semistable and $\mathcal{F}$-stable distributions on $\mathbf{Z}_+^d$ can be obtained from their continuous counterparts with support on $\mathbf{R}_+^d$.

THEOREM 3.5. *A function $\phi(u_1, \ldots, u_d)$ defined on $\mathbf{R}_+^d$ is the LST of a stable (resp., semistable) distribution on $\mathbf{R}_+^d$ with exponent $\gamma \in (0,1]$ (resp., exponent $\gamma$ and order $\alpha \in (0,1)$) if and only if, for every $\theta > 0$, $P_\theta(z_1, \ldots, z_d)$ of* (3.22) *is the pgf of an $\mathcal{F}$-stable (resp., $\mathcal{F}$-semistable) distribution on $\mathbf{Z}_+^d$ with exponent $\gamma$ (resp., exponent $\gamma$ and order $t = -\ln\alpha$).*

P r o o f. It suffices to prove the result for the semistable case. Let $\theta > 0$ and assume that $\phi(u_1, \ldots, u_d)$ satisfies (3.21) for $\alpha \in (0,1)$, $\lambda = \alpha^{-\gamma}$, and $\gamma \in (0,1]$. Setting $t = -\ln\alpha$, we have, by (3.21), (3.22), and (1.14b),

$$P_\theta(z_1, \ldots, z_d) = \big[\phi\big(e^{-t}\theta A(z_1), \ldots, e^{-t}\theta A(z_d)\big)\big]^\lambda = \big[P_\theta\big(F_t(z_1), \ldots, F_t(z_d)\big)\big]^\lambda,$$

which proves the "only if" part. Conversely, assume that $\phi(u_1, \ldots, u_d)$ is an LST with the property that there exist $\alpha \in (0,1)$, $\lambda = \alpha^{-\gamma}$ for some $\gamma \in (0,1]$, such that, for any $\theta > 0$, $P_\theta(z_1, \ldots, z_d)$ of (3.22) satisfies (3.5) with $t = -\ln\alpha$. Select $u_1, \ldots, u_d \geqslant 0$ and choose $\theta > \max_{1 \leqslant i \leqslant d} u_i$. Define $z_i = A^{-1}(u_i/\theta)$ for $i = 1, \ldots, d$. Then, by (3.5), (3.22), and (1.14b),

$$P_\theta(z_1, \ldots, z_d) = \big[\phi\big(\alpha\theta A(z_1), \ldots, \alpha\theta A(z_d)\big)\big]^\lambda = [\phi(\alpha u_1, \ldots, \alpha u_d)]^\lambda,$$

which implies (3.21), since $\phi(u_1, \ldots, u_d) = P_\theta(z_1, \ldots, z_d)$. ■

REMARK 3.1. (i) Theorem 2.3 extends to distributions on $\mathbf{Z}_+^d$ (a result also derived by Gupta and al. [7] for the binomial thinning semigroup (2.14)). The proof reduces to the univariate case via Theorem 3.3. The details are omitted.

(ii) Corollary 3.2 [(i)⇔(ii)] is the discrete analogue of a result in the continuous case (see, e.g., Maejima [14], Theorem 1.4).

(iii) Corollary 3.2 [(i)⇔(v)] was proved by Gupta et al. [7] for the binomial thinning semigroup (2.14).

## 4. LIMIT THEOREMS

In this section we show that $\mathcal{F}$-semistable and $\mathcal{F}$-stable distributions on $\mathbf{Z}_+^d$ arise as solutions to central limit problems. We will need the following function:

(4.1) $$V(x) = 1 - F_{\ln x}(0) \quad (x \geqslant 1).$$

First, we state the multivariate version of Theorem 8.4 (part (i)) due to van Harn et al. [10]. Their proof extends almost verbatim to the multivariate setting. The details are omitted.

THEOREM 4.1. *Let* $\big((S_1^{(n)},\dots,S_d^{(n)}),n\geqslant 1\big)$ *be a sequence of* $\mathbf{Z}_+^d$*-valued random vectors with pgf's* $P_n(z_1,\dots,z_d)$, $n\geqslant 1$. *Then there exist* $c_n\to\infty$ *and a pgf* $P(z_1,\dots,z_d)$ *such that*

$$\lim_{n\to\infty} P_n\big(F_{\ln c_n}(z_1),\dots,F_{\ln c_n}(z_d)\big)=P(z_1,\dots,z_d)$$

*if and only if there exist* $a_n\to\infty$ *such that* $a_n^{-1}(S_1^{(n)},\dots,S_d^{(n)})$ *converges weakly to a distribution on* $\mathbf{R}_+^d$ *with LST* $\phi(u_1,\dots,u_d)$ (*as* $n\to\infty$). *In this case*,

$$\lim_{n\to\infty} a_n V(c_n)=\theta \quad (\textit{for some } \theta>0) \tag{4.2}$$

*and*

$$P(z_1,\dots,z_d)=\phi\big(\theta A(z_1),\dots,\theta A(z_d)\big). \tag{4.3}$$

The next result identifies $\mathcal{F}$-semistable distributions as weak limits of subsequences of weighted sums of $\mathbf{Z}_+^d$-valued iid random vectors.

THEOREM 4.2. *Let* $\big((X_1^{(n)},\dots,X_d^{(n)}),n\geqslant 1\big)$ *be a sequence of iid* $\mathbf{Z}_+^d$*-valued random vectors and* $0<\alpha<1$. *Let* $(c_n,n\geqslant 1)$ *be an increasing sequence of real numbers such that* $c_n\geqslant 1$ *and* $c_n\uparrow\infty$ *and let* $(k_n,n\geqslant 1)$ *be a sequence in* $\mathbf{Z}_+$ *such that* $k_n\uparrow\infty$. *Furthermore, assume that*

(i) $c_n^{-1}\odot_{\mathcal{F}}\sum_{i=1}^{k_n}(X_1^{(i)},\dots,X_d^{(i)})$ *converges weakly to a* $\mathbf{Z}_+^d$*-valued random vector* $(X_1,\dots,X_d)$;

(ii) $\lim_{n\to\infty} c_n/c_{n+1}=\alpha$.

*Then* $(X_1,\dots,X_d)$ *has an* $\mathcal{F}$*-semistable distribution with some exponent* $0<\gamma\leqslant 1$ *and order* $t=-\ln\alpha$. *Moreover, the pgf of* $(X_1,\dots,X_d)$ *admits a representation of the type* (3.22) (*for some* $\theta>0$), *where* $\phi(u_1,\dots,u_d)$ *is the LST of a semistable distribution on* $\mathbf{R}_+^d$ *with exponent* $\gamma$ *and order* $\alpha$. *The sequence* $(k_n,n\geqslant 1)$ *necessarily satisfies*

$$\lim_{n\to\infty}\frac{k_n}{k_{n+1}}=\alpha^\gamma. \tag{4.4}$$

*Conversely, if a* $\mathbf{Z}_+^d$*-valued random vector* $(X_1,\dots,X_d)$ *has an* $\mathcal{F}$*-semistable distribution with exponent* $0<\gamma\leqslant 1$ *and order* $t>0$, *then there exist sequences* $\big((X_1^{(n)},\dots,X_d^{(n)}),n\geqslant 1\big)$, $(c_n,n\geqslant 1)$, *and* $(k_n,n\geqslant 1)$, *as defined above, that satisfy* (i)–(ii) *and* (4.4), *with* $\alpha=e^{-t}$.

P r o o f. Let $P(z_1,\dots,z_d)$ be the pgf of $(X_1,\dots,X_d)$ and $Q_i(z_1,\dots,z_d)$, $i\geqslant 1$, be that of $(X_1^{(i)},\dots,X_d^{(i)})$. By the assumption (i) and Theorem 4.1, there exists a sequence $(a_n,n\geqslant 0)$, $\lim_{n\to\infty}a_n=\infty$, such that the $\mathbf{R}_+^d$-valued sequence $\big(a_n^{-1}\sum_{i=1}^{k_n}(X_1^{(i)},\dots,X_d^{(i)}),n\geqslant 1\big)$ converges weakly to an $\mathbf{R}_+^d$-valued random vector $(Y_1, \dots, Y_d)$ and $P(z_1, \dots, z_d)$ admits the representation (3.22), where $\phi(u_1, \dots, u_d)$ is the LST of $(Y_1, \dots, Y_d)$ and $\theta > 0$ is given by (4.2). By van Harn et al. [10], $V(x) = x^{-1}L(x)$ $(x \geqslant 1)$ for some function $L$ that varies slowly at infinity. Therefore, by condition (ii) and (4.2),

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{a_n V(c_n)}{a_{n+1}V(c_{n+1})} \frac{c_{n+1}^{-1}L(c_{n+1})}{c_n^{-1}L(c_n)} = \alpha. \tag{4.5}$$

It follows by Theorem 2.1 (i) in Maejima [14] (see also Kruglov [12]) that $(Y_1, \dots, Y_d)$ has a semistable distribution on $\mathbf{R}_+^d$ with order $\alpha$ and exponent $\gamma \in (0, 1]$ that satisfies (4.4). We conclude by Theorem 3.5 that $P(z_1, \dots, z_d)$ is the pgf of an $\mathcal{F}$-semistable distribution on $\mathbf{Z}_+^d$ with exponent $\gamma$ and order $\alpha$.

Conversely, assume that $(X_1, \dots, X_d)$ has an $\mathcal{F}$-semistable distribution with exponent $0 < \gamma \leqslant 1$ and order $t > 0$. Let $\alpha = e^{-t}$. For $n \geqslant 1$, let $k_n = [\alpha^{-n\gamma}]$ (where $[x]$ denotes the largest integer less than or equal to $x$) and $c_n = \alpha^{-n}$. Clearly, $\{c_n\}$ and $\{k_n\}$ satisfy (ii) and (4.4), respectively. Define a sequence of iid $\mathbf{Z}_+^d$-valued random vectors $\{(X_1^{(n)}, \dots, X_d^{(n)})\}$ such that $(X_1^{(n)}, \dots, X_d^{(n)}) \stackrel{d}{=} (X_1, \dots, X_d)$. Let us denote by $P_n(z_1, \dots, z_d)$ and $P(z_1, \dots, z_d)$ the pgf's of $c_n^{-1} \odot_{\mathcal{F}} \sum_{i=1}^{k_n} (X_1^{(i)}, \dots, X_d^{(i)})$ and $(X_1, \dots, X_d)$, respectively. Noting that for every $n \geqslant 1$, $P(z_1, \dots, z_d) = \big[P\big(F_{\ln c_n}(z_1), \dots, F_{\ln c_n}(z_d)\big)\big]^{\alpha^{-n\gamma}}$ (by (3.5) and (3.6)), $P_n(z_1, \dots, z_d) = \big[P\big(F_{\ln c_n}(z_1), \dots, F_{\ln c_n}(z_d)\big)\big]^{k_n}$, and $\alpha^{-n\gamma} = k_n + \xi_n$ for some $0 \leqslant \xi_n < 1$, it follows by (1.10) that

$$\begin{aligned}\lim_{n\to\infty} &|P_n(z_1, \dots, z_d) - P(z_1, \dots, z_d)| \\ &\leqslant \lim_{n\to\infty} \big|1 - P^{\xi_n}\big(F_{\ln c_n}(z_1), \dots, F_{\ln c_n}(z_d)\big)\big| = 0. \quad \blacksquare\end{aligned}$$

The following corollary is a direct consequence of Theorems 3.5 and 4.2.

COROLLARY 4.1. *A $\mathbf{Z}_+^d$-valued random vector $(X_1, \dots, X_d)$ has an $\mathcal{F}$-semistable distribution with exponent $0<\gamma\leqslant 1$ and order $t > 0$ if and only if its pgf admits the representation* (3.22) (*for some $\theta > 0$*), *where $\phi(u_1, \dots, u_d)$ is the LST of a semistable distribution on $\mathbf{R}_+^d$ with exponent $\gamma$ and order $\alpha = e^{-t}$.*

Theorem 4.2 extends to $\mathcal{F}$-stable distributions on $\mathbf{Z}_+^d$ as follows.

THEOREM 4.3. *Let $\big((X_1^{(n)}, \dots, X_d^{(n)}), n\geqslant 1\big)$ be a sequence of iid $\mathbf{Z}_+^d$-valued random vectors and $0 < \alpha < 1$. Let $(c_n, n \geqslant 1)$ be an increasing sequence of real numbers such that $c_n \geqslant 1$ and $c_n \uparrow \infty$. Assume $c_n^{-1} \odot_{\mathcal{F}} \sum_{i=1}^{n} (X_1^{(i)}, \dots, X_d^{(i)})$ converges weakly to a $\mathbf{Z}_+^d$-valued random vector $(X_1, \dots, X_d)$. Then $(X_1, \dots, X_d)$ has an $\mathcal{F}$-stable distribution with some exponent $0 < \gamma \leqslant 1$. Moreover, the pgf of $(X_1, \dots, X_d)$ admits a representation of the type* (3.22) (*for some $\theta > 0$*), *where $\phi(u_1, \dots, u_d)$ is the LST of a stable distribution on $\mathbf{R}_+^d$ with exponent $\gamma$.*

*Conversely, if a* $\mathbf{Z}_+^d$*-valued random vector* $(X_1, \ldots, X_d)$ *has an* $\mathcal{F}$*-stable distribution with exponent* $0 < \gamma \leqslant 1$*, then there exist sequences* $(c_n, n \geqslant 1)$ *and* $\big((X_1^{(n)}, \ldots, X_d^{(n)}), n \geqslant 1\big)$ *such that* $c_n^{-1} \odot_{\mathcal{F}} \sum_{i=1}^n (X_1^{(i)}, \ldots, X_d^{(i)})$ *converges weakly to* $(X_1, \ldots, X_d)$.

P r o o f. The first part is proven along the same lines as the semistable case. We rely on Theorem 15.7 in Sato [18] instead of Theorem 2.1 in Maejima [14]. The converse follows from Corollary 3.3 (iii), with $c_n^{-1} = n^{-1/\gamma}$. ■

COROLLARY 4.2. *A* $\mathbf{Z}_+^d$*-valued random vector* $(X_1, \ldots, X_d)$ *has an* $\mathcal{F}$*-stable distribution with exponent* $0 < \gamma \leqslant 1$ *if and only if its pgf admits the representation* (3.22) (*for some* $\theta > 0$), *where* $\phi(u_1, \ldots, u_d)$ *is the LST of a stable distribution on* $\mathbf{R}_+^d$ *with exponent* $\gamma$.

We derive some new representation results for $\mathcal{F}$-semistable and $\mathcal{F}$-stable distributions on $\mathbf{Z}_+^d$.

Let $S$ be the unit sphere on $\mathbf{R^d}$. We define $S_+ = S \cap \mathbf{R}_+^d$ and denote by $\mathcal{B}(\mathbf{R}_+)$ and $\mathcal{B}(S_+)$ the $\sigma$-algebra of Borel sets in $\mathbf{R}_+$ and $S_+$, respectively. For $E \subset \mathbf{R}_+$ and $B \subset S_+$, let $EB = \{(ux_1, \ldots, ux_d) : u \in E, (x_1, \ldots, x_d) \in B\}$.

We recall (see Choi [6]) that a distribution on $\mathbf{R}_+^d$ is semistable with exponent $\gamma \in (0, 1]$ and $\alpha \in (0, 1)$ if and only if it is infinitely divisible and its Lévy measure $\Lambda$ satisfies, for any $B \in \mathcal{B}(S_+)$ and $E \in \mathcal{B}(\mathbf{R}_+)$,

$$\Lambda(EB) = \int_B \cdots \int \mu(dx_1, \ldots, dx_d) \int_E d\big(-H(x_1, \ldots, x_d; u) u^{-\gamma}\big), \tag{4.6}$$

where $\mu$ is a finite measure on $\mathcal{B}(S_+)$, $H(x_1, \ldots, x_d; u)$ is nonnegative and right-continuous in $u$, and Borel measurable in $(x_1, \ldots, x_d)$, $H(x_1, \ldots, x_d; u)u^{-\gamma}$ is nonincreasing in $u$, $H(x_1, \ldots, x_d; 1) = 1$, and, finally, $H(x_1, \ldots, x_d; \alpha u) = H(x_1, \ldots, x_d; u)$.

COROLLARY 4.3. *A distribution on* $\mathbf{Z}_+^d$ *is* $\mathcal{F}$*-semistable with exponent* $\gamma \in (0, 1]$ *and order* $t > 0$ *if and only if its pgf* $P(z_1, \ldots, z_d)$ *admits the representation* (3.7) *with the function* $g_{\gamma,t}(\tau_1, \ldots, \tau_d)$ *given by*

$$\begin{aligned} g_{\gamma,t}(\tau_1, \ldots, \tau_d) = {} & \exp\left(\frac{\gamma}{d} \sum_{i=1}^d \tau_i\right) \\ & \times \int_0^\infty \cdots \int_0^\infty \Big(1 - \exp\big(-\theta \sum_{i=1}^d e^{-\tau_i} x_i\big)\Big) \Lambda(dx_1, \ldots, dx_d), \end{aligned} \tag{4.7}$$

*where* $\Lambda$ *is the Lévy measure* (*given by* (4.6)) *of a semistable distribution on* $\mathbf{R}_+^d$ *with exponent* $\gamma$ *and order* $\alpha = e^{-t}$, *and* $\theta > 0$.

P r o o f. The "only if" part follows from Corollary 4.1, (3.7), (3.22), and the Lévy–Khintchine representation of the LST of a semistable distribution (Proposition 2.3 in Choi [6]) with Lévy measures given by (4.6). The "if" part can be easily checked. ■

In the univariate case ($d = 1$), the function $g_{\gamma,t}$ of (4.7) reduces to

$$g_{\gamma,t}(\tau) = e^{\gamma\tau} \int_0^\infty \big(1 - \exp(-\theta e^{-\tau} x)\big)\Lambda(dx) \quad (\tau \geqslant 0) \tag{4.8}$$

for some $\theta > 0$. The measure $\Lambda$ is given by

$$\Lambda(E) = \int_E d\big( - H(u)u^{-\gamma}\big) \quad (E \in \mathcal{B}(\mathbf{R}_+)), \tag{4.9}$$

where $H(u)$ is nonnegative and right-continuous in $u$, $H(u)u^{-\gamma}$ is nonincreasing, $H(1) = 1$, and $H(\alpha u) = H(u)$ with $\alpha = e^{-t}$.

A similar characterization of the function $r_{\gamma,t}$ in the representation (2.10) of the $R$-function of an $\mathcal{F}$-semistable distribution on $\mathbf{Z}_+$ can be obtained.

COROLLARY 4.4. *An infinitely divisible distribution on $\mathbf{Z}_+$ is $\mathcal{F}$-semistable with exponent $\gamma \in (0, 1]$ and order $t > 0$ if and only if its $R$-function $R(z)$ admits the representation* (2.10) *with the function $r_{\gamma,t}(\tau)$ given by*

$$r_{\gamma,t}(\tau) = \theta e^{(\gamma-1)\tau} \int_0^\infty x \exp(-\theta e^{-\tau} x)\Lambda(dx), \tag{4.10}$$

*where $\Lambda$ is the Lévy measure* (*given by* (4.9)) *of a semistable distribution on $\mathbf{R}_+$ with exponent $\gamma$ and order $\alpha = e^{-t}$, and $\theta > 0$.*

P r o o f. If $R(z)$ is the $R$-function of an $\mathcal{F}$-semistable distribution with exponent $\gamma \in (0, 1]$ and order $t > 0$, then, by Corollary 4.1,

$$R(z) = \theta A'(z) \frac{\phi'\big(\theta A(z)\big)}{\phi\big(\theta A(z)\big)}, \tag{4.11}$$

where $\phi(\tau)$ is the LST of a semistable distribution on $\mathbf{R}_+$ with exponent $\gamma$ and order $\alpha = e^{-t}$, and $\theta > 0$. By the Lévy–Khintchine representation of $\phi(\tau)$, we have

$$\frac{\phi'(\tau)}{\phi(\tau)} = - \int_0^\infty x e^{-\tau x}\Lambda(dx) \quad (\tau > 0), \tag{4.12}$$

where $\Lambda$ is the Lévy measure (described by (4.9)) of $\phi(\tau)$. It follows by (2.10), (4.11), (4.12), and the fact that $1/U(z) = -A'(z)/A(z)$ (see (1.13)) that

$$r_{\gamma,t}\big( - \ln A(z)\big) = \theta A(z)^{1-\gamma} \int_0^\infty x e^{-\theta A(z) x}\Lambda(dx) \quad (z \in [0, 1)). \tag{4.13}$$

Letting $\tau = - \ln A(z)$ yields (4.10). The converse is easily shown to hold. ■

By using Corollary 4.2, and noting that the Lévy measure of a stable distributions on $\mathbf{R}_+^d$ takes the form (see Theorem 14.3 in Sato [18])

$$\Lambda(B) = \int_{S_+} \dots \int \mu(dx_1, \dots, dx_d) \int_0^\infty I_B(ux_1, \dots, ux_d) u^{-1-\gamma}\, du, \tag{4.14}$$

where $\mu$ is a finite measure on $\mathcal{B}(S_+)$ and $B \in \mathcal{B}(S_+)$, we obtain the following characterization of $\mathcal{F}$-stable distributions on $\mathbf{Z}_+^d$.

COROLLARY 4.5. *Let* $0 < \gamma \leqslant 1$ *and* $d > 1$. *A distribution on* $\mathbf{Z}_+^d$ *is* $\mathcal{F}$*-stable with exponent* $\gamma$ *if and only if its pgf* $P(z_1, \dots, z_d)$ *admits the representation* (3.11) *and the function* $Q_\gamma(u_1, \dots, u_d)$ *satisfies* (*for any* $\tau_1, \dots, \tau_d \geqslant 0$)

$$Q_\gamma(\tau_2 - \tau_1, \dots, \tau_d - \tau_1) = \exp\left(\frac{\gamma}{d} \sum_{i=1}^d \tau_i\right) \times \int_0^\infty \dots \int_0^\infty \Big(1 - \exp\big(-\theta \sum_{i=1}^d e^{-\tau_i} x_i\big)\Big) \Lambda(dx_1, \dots, dx_d), \tag{4.15}$$

*where* $\Lambda$ *is the Lévy measure* (*given by* (4.14)) *of a stable distribution on* $\mathbf{R}_+^d$ *with exponent* $\gamma$ *and* $\theta > 0$.

**Acknowledgments.** The author is indebted to two referees for very valuable comments and suggestions.

### REFERENCES


[1] E.-E. A. A. Aly, *Models for* $\mathbf{Z}_+$*-valued time series based on thinning*, in: *International Encyclopedia of Statistical Science*, Part 13, Springer, New York 2011, pp. 844–847.

[2] O. E. Barndorff-Nielsen, J. Pedersen, and K. Sato, *Multivariate subordination, self-decomposability and stability*, Adv. in Appl. Probab. 33 (2001), pp. 160–187.

[3] M. Ben Alaya and T. Huillet, *On a functional equation generalizing the class of semistable distributions*, Ann. Inst. Statist. Math. 57 (2005), pp. 817–831.

[4] N. Bouzar, *A functional equation that leads to semistability*, Probab. Math. Statist. 30 (2010), pp. 87–102.

[5] N. Bouzar, *Semi-self-decomposability induced by semigroups*, Electron. J. Probab. 16 (2011), pp. 1117–1133.

[6] G. S. Choi, *Criteria for recurrence and transience of semistable processes*, Nagoya Math. J. 134 (1994), pp. 91–106.

[7] A. K. Gupta, K. Jagannathan, T. T. Nguyen, and D. N. Shanbhag, *Characterizations of stable laws via functional equations*, Math. Nachr. 279 (2006), pp. 571–580.

[8] B. G. Hansen, *Self-decomposable distributions and branching processes*, Memoir COSOR 89-06, Eindhoven Univ. of Technology, Eindhoven, Netherlands, 1989.

[9] K. van Harn and F. W. Steutel, *Integer-valued self-similar processes*, Comm. Statist. Stochastic Models 1 (1985), pp. 191–208.

[10] K. van Harn, F. W. Steutel, and W. Vervaat, *Self-decomposable discrete distributions and branching processes*, Z. Wahrsch. Verw. Gebiete 61 (1982), pp. 97–118.

[11] D. Krapavitskaite, *Discrete semistable probability distributions*, J. Soviet Math. 38 (1987), pp. 2309–2319.
[12] V. M. Kruglov, *On the extension of the class of stable distributions*, Theory Probab. Appl. 17 (1972), pp. 685–694.
[13] P. Lévy, *Théorie de l'addition des variables aléatoires*, Gauthier-Villars, Paris 1937.
[14] M. Maejima, *Semistable distributions*, in: *Lévy Processes*, O. E. Barndorff-Nielsen, T. Mikosch, and S. I. Resnick (Eds.), Birkhäuser, Boston, MA, 2001, pp. 169–183.
[15] A. G. Pakes, *Characterization of discrete laws via mixed sums and Markov branching processes*, Stochastic Process. Appl. 55 (1995), pp. 285–300.
[16] C. R. Rao and D. N. Shanbhag, *Choquet–Deny Type Functional Equations with Applications to Stochastic Models*, Wiley, Chichester 1994.
[17] S. Satheesh and E. Sandhya, *Semi-selfsimilar processes on $R$ and $I_0$*, Int. Math. Forum 3 (2008), pp. 1781–1784.
[18] K. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge University Press, Cambridge, U.K, 1999.
[19] R. Shimizu, *Solution to a functional equation and its application to some characterization problems*, Sankhya A 40 (1978), pp. 319–332.
[20] F. W. Steutel and K. van Harn, *Infinite Divisibility of Probability Distributions on the Real Line*, Marcel Dekker, Inc., New York–Basel 2004.
[21] F. W. Steutel, W. Vervaat, and S. J. Wolfe, *Integer-valued branching processes with immigration*, Adv. in Appl. Probab. 15 (1983), pp. 713–725.
[22] R. Zhu and H. Joe, *Count data time series models based on expectation thinning*, Stoch. Models 26 (2010), pp. 431–462.

University of Indianapolis
Department of Mathematics and Computer Science
1400 E. Hanna Ave.
Indianapolis, IN 46227, USA
*E-mail*: nbouzar@uindy.edu

*Received on 11.1.2012;*
*revised version on 17.4.2013*