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$\mathcal{J}$-REGULARITY OF MULTIVARIATE STATIONARY SEQUENCES FOR SOME FAMILIES $\mathcal{J}$

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Abstract. Let $n \in \mathbb{N}$, S be a nonempty finite subset of the set of integers, S^{c} be its complement, and $\mathcal{J}$ be the family of translations of S^{c} by ln, $l \in \mathbb{Z}$. For such a family, $\mathcal{J}$-regularity of a q-variate stationary sequence over $\mathbb{Z}$ is studied. If S contains exactly n elements, a description of a $\mathcal{J}$-regular sequence in terms of its spectral density is obtained. Some examples are given for the case where S contains more than n elements.

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1. INTRODUCTION

The notion of $\mathcal{J}$-regularity, where $\mathcal{J}$ is a family of nonempty subsets of the parameter set of a stationary process, plays an important role in the prediction theory; for its definition see [7], Definition 2.10 (ii), or [6], Definition 3.7 (a). It is usually assumed that the family $\mathcal{J}$ is closed under translations. We mention the papers [1] and [6]–[8] which deal with families occurring in the interpolation problem and have close connections with our study.

Here we are concerned with a multivariate stationary sequence over $\mathbb{Z}$ and a family $\mathcal{J}$ consisting, for some $n \in \mathbb{N}$, of all translations of the complement of a nonempty finite subset S of $\mathbb{Z}$ by ln, $l \in \mathbb{Z}$. Such families might be of interest in prediction on the basis of periodic observations. Note that for $n > 1$ the family $\mathcal{J}$ is not closed under translations. It is not hard to see that if S does not contain representatives of all residue classes modulo n, the only $\mathcal{J}$-regular multivariate stationary sequence is the sequence identically zero, cf. Theorem 3.2. If S contains representatives of all residue classes modulo n and the cardinality of S is equal to n, a description of a $\mathcal{J}$-regular sequence in terms of its spectral density can be given, cf. Theorem 4.2. To obtain it we apply an idea which was used in the theory of periodically correlated sequences, cf. [2]–[5]. If the cardinality of S is greater than n, it seems to be more difficult to obtain such a description. To point out some problems occurring in this case we conclude our paper with some examples.

2. DEFINITIONS AND BASIC FACTS

For $q \in \mathbb{N}$, let $\mathcal{M}_q$ be the algebra of $q \times q$ matrices with complex entries and $\mathcal{M}_q^{\geqslant}$ be the cone of nonnegative Hermitian $q \times q$ matrices. For a matrix A denote by $\mathcal{R}(A)$, $\mathcal{N}(A)$, A^* and A^+ its range, null space, adjoint and Moore–Penrose inverse, resp. Let E_q denote the unit matrix of $\mathcal{M}_q$ and let $\operatorname{diag}[A_1, \ldots, A_n]$ be the block diagonal matrix of $\mathcal{M}_{nq}$ with matrices $A_j \in \mathcal{M}_q$, $j \in \{1, \ldots, n\}$, $n \in \mathbb{N}$, in the prescribed order on the principal diagonal.

Let $\mathcal{H}$ be a Hilbert space over the field of complex numbers and $\mathcal{H}^q$ be the Cartesian product of q copies of $\mathcal{H}$.

DEFINITION 2.1. A *q-variate stationary sequence* $\mathbf{X} := (X_k)_{k \in \mathbb{Z}}$ is a function on $\mathbb{Z}$ such that $X_k \in \mathcal{H}^q$ and the Gramian matrix $\langle X_k, X_l \rangle$ depends only on the difference $k - l$: $\langle X_k, X_l \rangle =: \Gamma_X(k - l)$, $k, l \in \mathbb{Z}$.

To avoid trivialities we exclude the case where $\mathbf{X}$ is identically zero. The correlation function Γ_X is positive definite and by Herglotz's theorem admits an integral representation $\Gamma_X(k) = \langle X_k, X_0 \rangle = \int_I \mathrm{e}^{\mathrm{i}k\gamma} F_X(\mathrm{d}\gamma)$, where F_X is a unique $\mathcal{M}_q^{\geqslant}$-valued Borel measure on the interval $I := [0, 2\pi)$, the so-called *spectral measure* of $\mathbf{X}$.

For a nonempty subset K of $\mathbb{Z}$ denote by $\mathfrak{M}_X(K)$ the closed $\mathcal{M}_q$-submodule of $\mathcal{H}^q$ spanned by X_k, $k \in K$, with coefficients from $\mathcal{M}_q$. By $\mathfrak{M}_X := \mathfrak{M}_X(\mathbb{Z})$ denote the time domain of $\mathbf{X}$. Recall that $\mathfrak{M}_X(K)$ has the form $\mathcal{H}_X(K)^q$, where $\mathcal{H}_X(K)$ is the closed linear hull of $x_{k,r}$, $k \in K$, $r \in \{1, \ldots, q\}$. Here $x_{k,r}$ denotes the r-th component of X_k.

By $L_2(F_X)$ denote the Hilbert $\mathcal{M}_q$-module of (equivalence classes of) Borel measurable $\mathcal{M}_q$-valued functions Φ on I such that the integral $\int_I \Phi \mathrm{d}F_X \Phi^*$ exists. The mapping

$$V_X \colon X_k \mapsto \mathrm{e}^{\mathrm{i}k\cdot} E_q, \qquad k \in \mathbb{Z},$$

induces an $\mathcal{M}_q$-linear isomorphism between $\mathfrak{M}_X$ and $L_2(F_X)$, the so-called *Kolmogorov isomorphism*.

DEFINITION 2.2. Let $\mathcal{J}$ be a family of nonempty subsets of $\mathbb{Z}$. A q-variate stationary sequence $\mathbf{X}$ is called *$\mathcal{J}$-regular* if $\bigcap_{K \in \mathcal{J}} \mathfrak{M}_X(K) = \{0\}$.

3. FORMULATION OF THE PROBLEM

Let $n, m \in \mathbb{N}$, $S := \{k_1, \ldots, k_m\}$ be a set of integers and $S^{\mathrm{c}} := \mathbb{Z} \setminus S$. Let $\mathcal{J}(S^{\mathrm{c}}; n)$ denote the family of all translations of S^{c} by ln, i.e., $\mathcal{J}(S^{\mathrm{c}}; n)$ is equal to $\mathcal{J}(S^{\mathrm{c}}; n) := \{\tau_{ln} S^{\mathrm{c}} \colon l \in \mathbb{Z}\}$, where $\tau_{ln} S^{\mathrm{c}}$ is the translation $\tau_{ln} S^{\mathrm{c}} := \{k + ln \colon k \in S^{\mathrm{c}}\} = \mathbb{Z} \setminus \{k_1 + ln, \ldots, k_m + ln\}$, $l \in \mathbb{Z}$.

PROBLEM. Describe the spectral measure of a $\mathcal{J}(S^{\mathrm{c}}; n)$-regular q-variate stationary sequence $\mathbf{X}$.

Denote by λ the Lebesgue measure on I. It is well known that, for an arbitrary nonempty finite set S and any $n \in \mathbb{N}$, a sequence $\mathbf{X}$ is not $\mathcal{J}(S^{\mathrm{c}}; n)$-regular if its spectral measure F_X has a nonzero singular part (with respect to λ). Therefore, we can and shall assume to the end of the paper that F_X is absolutely continuous, i.e., $\mathrm{d}F_X = f_X \mathrm{d}\lambda$, where f_X denotes the corresponding Radon–Nikodym derivative and is called the *spectral density* of $\mathbf{X}$.

Among the known results on $\mathcal{J}(S^{\mathrm{c}}; 1)$-regularity we only mention a special case of Theorem 5.3 of [6], which proves to be useful for us.

THEOREM 3.1. *A q-variate stationary sequence $\mathbf{X}$ is $\mathcal{J}(\{0\}^{\mathrm{c}}; 1)$-regular if and only if $\mathcal{R}(f_X)$ is constant λ-a.e. and f_X^{+} is integrable with respect to λ.*

The following simple result shows that we can assume that the set S contains representatives of all residue classes modulo n.

THEOREM 3.2. *If S does not contain representatives of all residue classes modulo n, then there does not exist a $\mathcal{J}(S^{\mathrm{c}}; n)$-regular q-variate stationary sequence different from the sequence identically zero.*

Proof. Let $j_0 \in \{0, \ldots, n-1\}$ be such that $k_t \not\equiv j_0 \pmod{n}$ for all $t \in \{1, \ldots, m\}$. It follows that $j_0 \in \tau_{ln} S^{\mathrm{c}}$ for $l \in \mathbb{Z}$, and hence

$$X_{j_0} \in \bigcap_{l \in \mathbb{Z}} \mathfrak{M}_X(\tau_{ln} S^{\mathrm{c}}). \quad \blacksquare$$

REMARK 3.1. Note that the preceding theorem is true for an infinite set S as well.

To the end of the paper we shall assume that S contains representatives of all residue classes modulo n. Thus $m \geqslant n$, and the cases $m = n$ and $m > n$ will be discussed separately in Sections 4 and 5, respectively.

4. $\mathcal{J}(S^{\mathrm{c}}; n)$-REGULARITY IF $m = n$

If $m = n$, the elements of S can be numbered in such a way that one has $k_t \equiv t - 1 \pmod{n}$. Let $\mathbf{X} = (X_k)_{k \in \mathbb{Z}}$ be a q-variate stationary sequence with spectral density f_X. Define an nq-variate stationary sequence $\mathbf{Y} := (Y_k)_{k \in \mathbb{Z}}$ by

$$Y_k := \begin{bmatrix} X_{kn+k_1} \\ \vdots \\ X_{kn+k_n} \end{bmatrix}, \qquad k \in \mathbb{Z}.$$

Let $\xi := \mathrm{e}^{2\pi \mathrm{i}/n}$.

LEMMA 4.1. *The spectral density of $\mathbf{Y}$ is equal to an $\mathcal{M}_{nq}^{\geqslant}$-valued Borel measurable function f_Y which for $\gamma \in I$ is defined by*

$$f_Y(\gamma) := V(\gamma) f_\xi(\gamma) V(\gamma)^*,$$

where

$$f_\xi(\gamma) := \frac{1}{n}\left[\sum_{t=0}^{n-1} \xi^{(j-l)t} f_X\left(\frac{\gamma}{n} + \frac{2\pi}{n}t\right)\right]_{j,l=1,\dots,n}$$

and

$$V(\gamma) := \mathrm{diag}[\mathrm{e}^{\mathrm{i}k_1\gamma/n}E_q, \dots, \mathrm{e}^{\mathrm{i}k_n\gamma/n}E_q].$$

P r o o f. For $j, l \in \{1, \dots, n\}$, the $q \times q$ matrix block of $\Gamma_Y(k)$ at place (j, l) is equal to

$$\begin{aligned}
\text{(4.1)}\qquad \langle X_{kn+k_j}, X_{k_l}\rangle &= \int_I \mathrm{e}^{\mathrm{i}(kn+k_j)\gamma}\mathrm{e}^{-\mathrm{i}k_l\gamma} f_X(\gamma)\lambda(\mathrm{d}\gamma)\\
&= \int_I \mathrm{e}^{\mathrm{i}kn\gamma}\mathrm{e}^{\mathrm{i}(k_j-k_l)\gamma} f_X(\gamma)\lambda(\mathrm{d}\gamma)\\
&= \sum_{t=0}^{n-1}\int_{I_t} \mathrm{e}^{\mathrm{i}kn\gamma}\mathrm{e}^{\mathrm{i}(k_j-k_l)\gamma} f_X(\gamma)\lambda(\mathrm{d}\gamma),
\end{aligned}$$

where $I_t := \big[2\pi t/n, 2\pi(t+1)/n\big)$, $t \in \{0, \dots, n-1\}$. Since $k_j = nd_j + j - 1$ and $k_l = nd_l + l - 1$ for some $d_j, d_l \in \mathbb{Z}$, it follows that

$$\begin{aligned}
&\int_{I_t} \mathrm{e}^{\mathrm{i}kn\gamma}\mathrm{e}^{\mathrm{i}(k_j-k_l)\gamma} f_X(\gamma)\lambda(\mathrm{d}\gamma)\\
&\qquad= \int_{I_0} \mathrm{e}^{\mathrm{i}kn(\gamma+2\pi t/n)}\mathrm{e}^{\mathrm{i}(k_j-k_l)(\gamma+2\pi t/n)} f_X\left(\gamma + \frac{2\pi}{n}t\right)\lambda(\mathrm{d}\gamma)\\
&\qquad= \int_{I_0} \mathrm{e}^{\mathrm{i}kn\gamma}\mathrm{e}^{\mathrm{i}(k_j-k_l)\gamma}\mathrm{e}^{\mathrm{i}[n(d_j-d_l)+j-l]2\pi t/n} f_X\left(\gamma + \frac{2\pi}{n}t\right)\lambda(\mathrm{d}\gamma)\\
&\qquad= \int_{I_0} \mathrm{e}^{\mathrm{i}kn\gamma}\mathrm{e}^{\mathrm{i}(k_j-k_l)\gamma}\xi^{(j-l)t} f_X\left(\gamma + \frac{2\pi}{n}t\right)\lambda(\mathrm{d}\gamma)\\
&\qquad= \frac{1}{n}\int_{I} \mathrm{e}^{\mathrm{i}k\gamma}\mathrm{e}^{\mathrm{i}(k_j-k_l)\gamma/n}\xi^{(j-l)t} f_X\left(\frac{\gamma}{n} + \frac{2\pi}{n}t\right)\lambda(\mathrm{d}\gamma)
\end{aligned}$$

for all $t \in \{0, \dots, n-1\}$. Taking into account (4.1), we get the result. ∎

LEMMA 4.2. *For $l \in \mathbb{Z}$, $\mathcal{H}_X(\tau_{ln}S^{\mathrm{c}}) = \mathcal{H}_Y(\mathbb{Z} \setminus \{l\})$.*

P r o o f. Since the numbers $k_1, \dots, k_n$ belong to pairwise different residue classes modulo n, any integer k can be written as $k = dn + k_t$ for unique $d \in \mathbb{Z}$ and $t \in \{1, \dots, n\}$. If $x_{k,r} \in \mathcal{H}_X(\tau_{ln}S^{\mathrm{c}})$, then $k \in \mathbb{Z} \setminus \{ln + k_t\colon t = 1, \dots, n\}$, $r \in \{1, \dots, q\}$; hence $d \neq l$. It follows that $x_{k,r}$ is not a component of Y_l, which means that $x_{k,r} \in \mathcal{H}_Y(\mathbb{Z} \setminus \{l\})$. On the other hand, a component $y_{k,s}$, $s \in \{1, \dots, nq\}$, of Y_k is equal to $x_{kn+k_t,r}$ for some $t \in \{1, \dots, n\}$ and $r \in \{1, \dots, q\}$. If $y_{k,s} \in \mathcal{H}_Y(\mathbb{Z} \setminus \{l\})$, then $k \neq l$, which yields $kn + k_t \in \mathbb{Z} \setminus \{ln + k_j\colon j = 1, \dots, n\}$. Therefore $y_{k,s} \in \mathcal{H}_X(\tau_{ln}S^{\mathrm{c}})$. ∎

From Theorem 3.1 and Lemmas 4.1 and 4.2 we immediately obtain the following result.

THEOREM 4.1. *Let $n \in \mathbb{N}$ and k_t, $t \in \{1, \dots, n\}$, be integers belonging to pairwise different residue classes modulo n. A q-variate stationary sequence $\mathbf{X}$ with spectral density f_X is $\mathcal{J}(S^{\mathrm{c}}; n)$-regular if and only if the function f_Y defined in Lemma* 4.1 *has constant range λ-a.e. and f_Y^{+} is integrable with respect to λ.*

Now we describe those functions f_X for which f_Y has the properties requested in the preceding theorem.

Let $t \in \{0, \dots, n-1\}$, $A_t \in \mathcal{M}_q^{\geqslant}$, $\mathcal{E}_t$ be the set of pairwise different eigenvalues of A_t, i.e., without counting multiplicities, and $\mathcal{N}_t(\mu) := \mathcal{N}(A_t - \mu E_q)$, $\mu \in \mathbb{C}$. Thus, if $\mu \in \mathcal{E}_t$, then $\mathcal{N}_t(\mu)$ is the corresponding eigenspace. Define an Hermitian Toeplitz matrix $\mathbb{A} \in \mathcal{M}_{nq}$ by

$$\mathbb{A} := \frac{1}{n}\Big[\sum_{t=0}^{n-1} \xi^{(j-l)t} A_t\Big]_{j,l=1,\dots,n},$$

and let $\mathcal{E}$ be the set of pairwise different eigenvalues of $\mathbb{A}$ and $\mathcal{N}(\mu) := \mathcal{N}(\mathbb{A} - \mu E_{nq})$, $\mu \in \mathbb{C}$.

Define an isometry $U_t \colon \mathbb{C}^q \to \mathbb{C}^{nq}$ by

$$U_t u := \frac{1}{\sqrt{n}} \begin{bmatrix} u \\ \xi^t u \\ \vdots \\ \xi^{(n-1)t} u \end{bmatrix}, \quad u \in \mathbb{C}^q.$$

LEMMA 4.3. *If $r, t \in \{0, \dots, n-1\}$ and $r \neq t$, then $\mathcal{R}(U_r)$ is orthogonal to $\mathcal{R}(U_t)$.*

Proof. Let $(\cdot,\cdot)_{\mathbb{C}^q}$ denote the inner product of $\mathbb{C}^q$. For $u, v \in \mathbb{C}^q$ and $r, t \in \{0, \dots, n-1\}, r \neq t$, one has $(U_r u, U_t v)_{\mathbb{C}^{nq}} = \frac{1}{n}\sum_{s=0}^{n-1} \xi^{(r-t)s}(u, v)_{\mathbb{C}^q} = 0$ since $\sum_{s=0}^{n-1} \xi^{(r-t)s} = 0$. ∎

LEMMA 4.4. *If $\mu_1, \mu_2 \in \mathcal{E}_t$ and $\mu_1 \neq \mu_2$, then $U_t \mathcal{N}_t(\mu_1)$ is orthogonal to $U_t \mathcal{N}_t(\mu_2)$, $t \in \{0, \dots, n-1\}$.*

Proof. The assertion is trivial since U_t is an isometry and $\mathcal{N}_t(\mu_1)$ is orthogonal to $\mathcal{N}_t(\mu_2)$ if $\mu_1 \neq \mu_2$. ∎

LEMMA 4.5. *If $\mu \in \mathcal{E}_t$ and $u \in \mathcal{N}_t(\mu)$, then $\mu \in \mathcal{E}$ and $U_t u \in \mathcal{N}(\mu)$, $t \in \{0, \dots, n-1\}$.*

Proof. For $r \in \{1, \dots, n\}$, the r-th vector component of $\mathbb{A} U_t u$ is equal to $\frac{1}{n\sqrt{n}} \sum_{l=1}^{n} \sum_{s=0}^{n-1} \xi^{(r-l)s} A_s(\xi^{t(l-1)} u) = \frac{1}{n\sqrt{n}} \sum_{s=0}^{n-1} A_s\big(\sum_{l=1}^{n} \xi^{(r-l)s} \xi^{(l-1)t} u\big)$.

The equality $s=t$ implies $\sum_{l=1}^{n}\xi^{(r-l)s}\xi^{(l-1)t}=\xi^{(r-1)t}\sum_{l=1}^{n}1=n\xi^{(r-1)t}$. If $s\neq t$, then $\sum_{l=1}^{n}\xi^{(r-l)s}\xi^{(l-1)t}=\xi^{s(r-1)}\sum_{l=1}^{n}(\xi^{t-s})^{l-1}=0$. It follows that the r-th vector component of $\mathbb{A}U_t u$ is equal to

$$\frac{1}{n\sqrt{n}}A_t(n\xi^{(r-1)t}u)=\frac{1}{\sqrt{n}}\mu\xi^{(r-1)t}u,$$

which gives $\mathbb{A}U_t u=\mu U_t u$. ∎

The following relations are immediate consequences of the preceding three lemmas:

$$\mathcal{E}=\bigcup_{t=0}^{n-1}\mathcal{E}_t, \tag{4.2}$$

$$\mathcal{N}(\mu)=\bigoplus_{t=0}^{n-1}U_t\mathcal{N}_t(\mu),\qquad \mu\in\mathbb{C},$$

$$\mathcal{N}(\mathbb{A})=\bigoplus_{t=0}^{n-1}U_t\mathcal{N}(A_t),$$

$$\mathcal{R}(\mathbb{A})=\bigoplus_{t=0}^{n-1}U_t\mathcal{R}(A_t). \tag{4.3}$$

LEMMA 4.6. *The function f_Y^+ is integrable if and only if the function f_X^+ is integrable.*

Proof. Let $\mu_Y(\gamma)$, $\mu_X(\gamma)$, and $\mu_X^{(t)}(\gamma)$ denote the smallest positive eigenvalue of $f_Y(\gamma)$, $f_X(\gamma)$, and $f_X(\gamma/n+2\pi t/n)$, resp., $\gamma\in I$, $t\in\{0,\dots,n-1\}$. By (4.2) and Lemma 4.1, $\mu_Y(\gamma)^{-1}=\max\{\mu_X^{(t)}(\gamma)^{-1}\colon t=0,\dots,n-1\}$. It follows that f_Y^+ is integrable if and only if the function $\sum_{t=0}^{n-1}\mu_X^{(t)}(\cdot)^{-1}$ is integrable. Since $\int_I\sum_{t=0}^{n-1}\mu_X^{(t)}(\gamma)^{-1}\lambda(\mathrm{d}\gamma)=n\sum_{t=0}^{n-1}\int_{I_t}\mu_X(\gamma)^{-1}\lambda(\mathrm{d}\gamma)$, we get the result. ∎

LEMMA 4.7. *The range function $\mathcal{R}(f_Y)$ is constant λ-a.e. if and only if the range function $\mathcal{R}(f_X)$ is constant λ-a.e.*

Proof. For $t\in\{0,\dots,n-1\}$ and $\gamma\in I$, let $P_t(\gamma)$ be the orthoprojection in $\mathbb{C}^q$ onto $\mathcal{R}\big(f_X(\gamma/n+2\pi t/n)\big)$, $Q_\xi(\gamma)$ be the orthoprojection in $\mathbb{C}^{nq}$ onto $\mathcal{R}\big(f_\xi(\gamma)\big)$, and $Q_Y(\gamma)$ be the orthoprojection in $\mathbb{C}^{nq}$ onto $\mathcal{R}\big(f_Y(\gamma)\big)$. Then $U_tP_t(\gamma)U_t^*$ is the orthoprojection in $\mathbb{C}^{nq}$ onto $U_t\mathcal{R}\big(f_X(\gamma/n+2\pi t/n)\big)$, and from (4.3) it follows that

$$Q_\xi(\gamma)=\sum_{t=0}^{n-1}U_tP_t(\gamma)U_t^*=\frac{1}{n}\Big[\sum_{t=0}^{n-1}\xi^{(j-l)t}P_t(\gamma)\Big]_{j,l=1,\dots,n}.$$

Therefore, Lemma 4.1 yields

$$Q_Y(\gamma) = V(\gamma)Q_\xi(\gamma)V(\gamma)^* = \frac{1}{n}\Big[\mathrm{e}^{\mathrm{i}(k_j-k_l)\gamma/n}\sum_{t=0}^{n-1}\xi^{(j-l)t}P_t(\gamma)\Big]_{j,l=1,\ldots,n}. \tag{4.4}$$

If $\mathcal{R}(f_X)$ is constant λ-a.e., there exists an orthoprojection P such that $P = P_t(\gamma)$ for $t \in \{0,\ldots,n-1\}$ and λ-a.a. $\gamma \in I$. Therefore, (4.4) gives $Q_Y = \operatorname{diag}(P,\ldots,P)$ λ-a.e., which implies that $\mathcal{R}(f_Y)$ is constant λ-a.e.

Now assume that $\mathcal{R}(f_Y)$ is constant λ-a.e. To simplify the presentation let us assume that $k_j < k_{j+1}$, $j \in \{1,\ldots,n-1\}$. Then there exist matrices $C_{jl} \in \mathcal{M}_q$, $j,l \in \{1,\ldots,n\}$, such that $Q_Y(\gamma) = [C_{jl}]_{j,l=1,\ldots,n}$ for λ-a.a. $\gamma \in I$. Setting $D_m(\gamma) := \frac{1}{n}\sum_{t=0}^{n-1}\xi^{mt}P_t(\gamma)$, $m \in \{-(n-1),\ldots,n-1\}$, we have $C_{jl} = \mathrm{e}^{\mathrm{i}(k_j-k_l)\gamma/n}D_{j-l}(\gamma)$ for all $j,l \in \{1,\ldots,n\}$ and λ-a.a. $\gamma \in I$. Note that, for $l \in \{1,\ldots,n-1\}$, $D_{n-l} = D_{-l}$; hence

$$\begin{aligned} C_{nl} &= \mathrm{e}^{\mathrm{i}(k_n-k_l)\gamma/n}D_{-l}(\gamma) = \mathrm{e}^{\mathrm{i}[(k_n-k_l)-(k_1-k_{l+1})]\gamma/n}\mathrm{e}^{\mathrm{i}(k_1-k_{l+1})\gamma/n}D_{1-(l+1)}(\gamma) \\ &= \mathrm{e}^{\mathrm{i}[(k_n-k_l)+(k_{l+1}-k_1)]\gamma/n}C_{1,l+1} \end{aligned}$$

for λ-a.a. $\gamma \in I$. Since $k_l < k_n$ and $k_1 < k_{l+1}$, we get $C_{1,l+1} = 0$, from which it follows that $D_{-l}(\gamma) = 0$ for λ-a.a. $\gamma \in I$. Thus, for $l \in \{1,\ldots,n-1\}$ and λ-a.a. $\gamma \in I$ the matrix equalities

$$\sum_{t=0}^{n-1} P_t(\gamma) = nC_{11}, \qquad \sum_{t=0}^{n-1}\xi^{-lt}P_t(\gamma) = 0 \tag{4.5}$$

are satisfied. Multiplying the $(l+1)$-st equality of the system (4.5) by ξ^{ls} and adding we arrive at the equality $nP_s(\gamma) = nC_{11}$ for $s \in \{0,\ldots,n-1\}$ and λ-a.a. $\gamma \in I$, which implies that $\mathcal{R}(f_X)$ is constant λ-a.e. ∎

Now from Theorem 4.1 and Lemmas 4.6 and 4.7 the following generalization of Theorem 3.1 can be derived immediately.

THEOREM 4.2. *Let $n \in \mathbb{N}$, let k_t, $t \in \{1,\ldots,n\}$, be integers belonging to pairwise different residue classes modulo n, and $S := \{k_1,\ldots,k_n\}$. A q-variate stationary sequence $\mathbf{X}$ with spectral density f_X is $\mathcal{J}(S^{\mathrm{c}};n)$-regular if and only if $\mathcal{R}(f_X)$ is constant λ-a.e. and f_X^+ is integrable with respect to λ.*

5. $\mathcal{J}(S^{\mathrm{c}};n)$-REGULARITY IF $m > n$

Now let the cardinality of the set S be greater than n. Let $\tilde{S}$ be a subset of cardinality n containing representatives of all residue classes modulo n. Since a q-variate stationary sequence is $\mathcal{J}(S^{\mathrm{c}};n)$-regular if it is $\mathcal{J}(\tilde{S}^{\mathrm{c}};n)$-regular, from Theorems 3.1 and 4.2 the following assertion can be derived.

THEOREM 5.1. *Let S have more than n elements and contain representatives of all residue classes modulo n. If the spectral density f_X of a q-variate stationary sequence $\mathbf{X}$ has constant range λ-a.e. and f_X^+ is integrable with respect to λ, then $\mathbf{X}$ is $\mathcal{J}(S^c; n)$-regular.*

We do not know a characterization of a $\mathcal{J}(S^c; n)$-regular sequence in terms of its spectral density. The following examples show that neither the existence of the integral $\int_I f_X(\gamma)^+ \lambda(\mathrm{d}\gamma)$ nor the condition that f_X has constant range λ-a.e. is necessary for $\mathcal{J}(S^c; n)$-regularity of $\mathbf{X}$.

EXAMPLE 5.1. Let $\mathbf{X}$ be a univariate stationary sequence with spectral density f_X defined by $f_X(\gamma) := |1 - \mathrm{e}^{\mathrm{i}\gamma}|^2$, $\gamma \in I$, and let $S := \{0, \ldots, n\}$. For an integer $k \in \{0, \ldots, n-1\}$ define a function g_k so that $g_k(\gamma) := \mathrm{e}^{\mathrm{i}k\gamma}(1 - \mathrm{e}^{\mathrm{i}\gamma})$, $\gamma \in I$. Then $g_k/f_X \in L_2(f_X \mathrm{d}\lambda)$, and for $j \in \mathbb{Z} \setminus \{0, \ldots, n\}$ one has

$$\int_I \mathrm{e}^{-\mathrm{i}j\gamma} \frac{g_k(\gamma)}{f_X(\gamma)} f_X(\gamma) \lambda(\mathrm{d}\gamma) = 0.$$

It follows that if a function h belongs to the image of $\bigcap_{K \in \mathcal{J}(S^c; n)} \mathfrak{M}_X(K)$ under the Kolmogorov isomorphism, then

$$\int_I \mathrm{e}^{iln\gamma} \frac{\mathrm{e}^{\mathrm{i}k\gamma}(1 - \mathrm{e}^{\mathrm{i}\gamma})}{f_X(\gamma)} h(\gamma)^* f_X(\gamma) \lambda(\mathrm{d}\gamma) = 0, \qquad l \in \mathbb{Z},\ k \in \{0, \ldots, n-1\},$$

or, equivalently,

$$\int_I \mathrm{e}^{\mathrm{i}r\gamma}(1 - \mathrm{e}^{\mathrm{i}\gamma}) h(\gamma)^* \lambda(\mathrm{d}\gamma) = 0, \qquad r \in \mathbb{Z}.$$

Therefore, $h = 0$ λ-a.e., and the Kolmogorov isomorphism theorem implies that $\mathbf{X}$ is $\mathcal{J}(S^c; n)$-regular. Note, however, that by Theorems 3.2 and 4.2 the sequence $\mathbf{X}$ is not $\mathcal{J}(S_1^c; n)$-regular if S_1 is an arbitrary set of n integers.

EXAMPLE 5.2. Let $\mathbf{X}$ be a bivariate stationary sequence with spectral density f_X defined by

$$f_X(\gamma) := \begin{bmatrix} 1 & \mathrm{e}^{\mathrm{i}\gamma} \\ \mathrm{e}^{-\mathrm{i}\gamma} & 1 \end{bmatrix}, \qquad \gamma \in I.$$

Let $n := 2$ and $S := \{0, 1, 2\}$. Then $f_X^+ = f_X/4$, and for the matrix polynomial P defined by

$$P(\mathrm{e}^{\mathrm{i}\gamma}) := \begin{bmatrix} 1 & \mathrm{e}^{\mathrm{i}\gamma} \\ \mathrm{e}^{\mathrm{i}\gamma} & \mathrm{e}^{2\mathrm{i}\gamma} \end{bmatrix}, \qquad \gamma \in I,$$

one has $P(\mathrm{e}^{\mathrm{i}\gamma}) f_X^+(\gamma) = P(\mathrm{e}^{\mathrm{i}\gamma})/2$. Therefore, $\int_I P(\mathrm{e}^{\mathrm{i}\gamma}) f_X^+(\gamma) f_X(\gamma) \mathrm{e}^{-il\gamma} \lambda(\mathrm{d}\gamma) = \int_I P(\mathrm{e}^{i\gamma}) \mathrm{e}^{-il\gamma} \lambda(\mathrm{d}\gamma) = 0$ if $l \in S^c$. For a matrix function

$$H =: \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}$$

of the set $V_X\big(\bigcap_{K\in\mathcal{J}(S^{\mathrm{c}};2)}\mathfrak{M}_X(K)\big)$ it follows that

$$\int\limits_I \mathrm{e}^{2l\mathrm{i}\gamma}P(\mathrm{e}^{\mathrm{i}\gamma})f_X^+(\gamma)f_X(\gamma)H(\gamma)^*\lambda(\mathrm{d}\gamma)$$
$$=\int\limits_I \mathrm{e}^{2l\mathrm{i}\gamma}\begin{bmatrix} h_{11}(\gamma)^*+\mathrm{e}^{\mathrm{i}\gamma}h_{12}(\gamma)^* & h_{21}(\gamma)^*+\mathrm{e}^{i\gamma}h_{22}(\gamma)^* \\ \mathrm{e}^{i\gamma}h_{11}(\gamma)^*+\mathrm{e}^{2i\gamma}h_{12}(\gamma)^* & \mathrm{e}^{\mathrm{i}\gamma}h_{21}(\gamma)^*+\mathrm{e}^{2\mathrm{i}\gamma}h_{22}(\gamma)^*\end{bmatrix}\lambda(\mathrm{d}\gamma)=0$$

for $l\in\mathbb{Z}$. Thus

$$H(\gamma)=\begin{bmatrix} h_{11}(\gamma) & 0 \\ 0 & h_{22}(\gamma)\end{bmatrix}\begin{bmatrix} 1 & -\mathrm{e}^{\mathrm{i}\gamma} \\ -\mathrm{e}^{-\mathrm{i}\gamma} & 1\end{bmatrix} \quad \text{for } \lambda\text{-a.a. } \gamma\in I,$$

which implies that $H=0$ in $L_2(f_X\mathrm{d}\lambda)$. According to the Kolmogorov isomorphism theorem the sequence $\mathbf{X}$ is $\mathcal{J}(S^{\mathrm{c}};2)$-regular. Note, however, that by Theorem 4.2 it is not $\mathcal{J}(S_1^{\mathrm{c}};2)$-regular if S_1 is an arbitrary set of two integers.

EXAMPLE 5.3. Let $(X_k)_{k\in\mathbb{Z}}$ be a univariate sequence according to Example 5.1 and $(W_k)_{k\in\mathbb{Z}}$ be a bivariate sequence according to Example 5.2, and let X_k be orthogonal to W_l, $k,l\in\mathbb{Z}$. Then the three-variate sequence $\big(\big[\begin{smallmatrix}X_k\\W_k\end{smallmatrix}\big]\big)_{k\in\mathbb{Z}}$ is $\mathcal{J}(\{0,1,2\}^{\mathrm{c}};2)$-regular and has a spectral density which neither has constant range λ-a.e. nor an integrable Moore–Penrose inverse.

From Theorem 4.2 it can be immediately concluded that if a sequence is $\mathcal{J}(S^{\mathrm{c}};n)$-regular for some set of cardinality n, then it is $\mathcal{J}(S^{\mathrm{c}};n)$-regular for any set S of cardinality n. The following example shows that such a result does not remain true if the cardinality of S is greater than n.

EXAMPLE 5.4. Let $\mathbf{X}$ be a univariate stationary sequence with spectral density f_X defined by $f_X(\gamma):=|1-\mathrm{e}^{2\mathrm{i}\gamma}|^2$, $\gamma\in I$, and S be a finite subset of $\mathbb{Z}$. It is well known that $\mathbf{X}$ is $\mathcal{J}(S^{\mathrm{c}};1)$-regular if and only if there exists a trigonometric polynomial p of the form $p(\mathrm{e}^{\mathrm{i}\gamma})=\sum_{j\in S}a_j\mathrm{e}^{\mathrm{i}j\gamma}$, $\gamma\in I$, $a_j\in\mathbb{C}$, and such that the inequalities $0<\int_I|p(\mathrm{e}^{\mathrm{i}\gamma})|^2/f_X(\gamma)\lambda(\mathrm{d}\gamma)<\infty$ are satisfied; cf. [6], Theorem 4.1, and [8], Lemma 4.7 (b), for a much more general result. Therefore, the sequence $\mathbf{X}$ is $\mathcal{J}(\{0,2\}^{\mathrm{c}};1)$-regular, but not $\mathcal{J}(\{0,1\}^{\mathrm{c}};1)$-regular.

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