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# WEAK CONVERGENCE OF RANDOM SUMS OF INFIMA OF INDEPENDENT RANDOM VARIABLES

BY

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*Abstract.* Let $\{Y_n, n \geqslant 1\}$ be a sequence of independent positive random variables, defined on a probability space $(\Omega, \mathscr{A}, P)$, with a common distribution function $F$. Put

$$Y_m^* = \inf(Y_1, Y_2, \ldots, Y_m),\ m \geqslant 1 \text{ and } S_n = \sum_{m=1}^{n} Y_m^*,\ n \geqslant 1.$$

In this paper mixing limit theorem for the sums $S_n$, $n \geqslant 1$, is given and the random central limit theorem is proved.

**1. Introduction and results.** Let $\{Y_n, n \geqslant 1\}$ be a sequence of independent positive random variables with a common distribution function $F$. Let us put

$$Y_m^* = \inf(Y_1, Y_2, \ldots, Y_m),\ m \geqslant 1, \quad \text{and} \quad S_n = \sum_{m=1}^{n} Y_m^*,\ n \geqslant 1.$$

The three convergences: in probability, almost sure and in law were established in [4]–[7] for sums $S_n$ of infima of independent random variables uniformly distributed on [0, 1]. The almost sure invariance principle was investigated in [8].

Now, let $\{Y_n, n \geqslant 1\}$ be a sequence of independent positive random variables with a common distribution function $F$ such that

$$(1) \qquad \int_0^1 \left|F(x) - \frac{x}{b}\right| x^{-2}\, dx < \infty \quad \text{for } 0 < b < \infty.$$

T. Höglund proved in [9] the following central limit theorem:

THEOREM 0. *Under assumption* (1)

$$\lim_{n\to\infty} \mathrm{P}(Z_n < x) = \Phi(x),$$

*where*

$$Z_n = \frac{S_n - b\log n}{b\sqrt{2\log n}}, \qquad n > 1,$$

(2)

$$S_n = \sum_{k=1}^{n} Y_k^*, \quad Y_k^* = \inf(Y_1, Y_2, \ldots, Y_k), \; k \geqslant 1, n \geqslant 1,$$

*and* $\Phi$ *is the standard normal distribution function.*

In this paper we give a mixing limit theorem and a random central limit theorem for $\{Z_n, n > 1\}$.

THEOREM 1. (i) *Under the assumptions of Theorem* 0 *the sequence* $\{Z_n, n > 1\}$ *is mixing, i.e.*

$$\lim_{n\to\infty} \mathrm{P}(Z_n < x | B) = \Phi(x)$$

*for any event* $B \in \mathcal{A}$ *such that* $\mathrm{P}(B) > 0$.

(ii) *Let* $\{N_n, n \geqslant 1\}$ *be a sequence of positive integer-valued random variables such that*

(3) $$N_n/a_n \xrightarrow{\mathrm{P}} \lambda \quad \text{as } n \to \infty,$$

*where* $\lambda$ *is a positive random variable dependent only on finitely many* $Y_n$, $n \geqslant 1$, *and* $\{a_n, n \geqslant 1\}$ *is a sequence of positive numbers tending to* $+\infty$. *Then*

(4) $$\lim_{n\to\infty} \mathrm{P}(Z_{N_n} < x) = \Phi(x).$$

**2. Proofs of results.** In the proof of Theorem 1 we apply some lemmas given by Deheúvels [5] and Höglund [9]. For the sake of completeness we present them in Section 3.

Proof of Theorem 1. (i) Let $\{Z_n, n > 1\}$ be defined by (2) and let $Y_{m,n}^* = \inf(Y_{m+1}, \ldots, Y_n)$ for $n > m$. Denote by $A_k$ the event $\{Z_k < x\}$ for $k \geqslant n_0$, where $n_0$ is such that $P(A_k) > 0$ for all $k \geqslant n_0$. We prove that the sequence $\{Z_n, n > 1\}$ is mixing.

By Theorem 1 ([10], p. 406) it is sufficient to show that

(5) $$\lim_{n\to\infty} \mathrm{P}(A_n | A_k) = \Phi(x), \qquad k \geqslant n_0,$$

as, by Theorem 0, $\lim_{n\to\infty} \mathrm{P}(A_n|\Omega) = \Phi(x)$. Since

$$Z_n = \frac{S_k}{b\sqrt{2\log n}} + \frac{\sum_{l=k+1}^{n} (Y_l^* - Y_{k,l}^*)}{b\sqrt{2\log n}} + \frac{\sum_{l=k+1}^{n} Y_{k,l}^* - b\log n}{b\sqrt{2\log n}},$$

we have $S_k/b\sqrt{2\log n}\to 0$ a.s. as $n\to\infty$, and, by Lemma 3.4,

$$\sum_{l=k+1}^{n}(Y_l^*-Y_{k,l}^*)/b\sqrt{2\log n}\to 0 \text{ a.s.} \quad \text{as } n\to\infty.$$

The random variables $\sum Y_{k,l}^*$ are independent of $S_k$ for every $k\geqslant n_0$, so, by Theorem 0, we immediately obtain (5) and the proof of (i) is completed.

(ii) To prove that $\mathrm{P}(Z_{N_n}<x)\to\Phi(x)$ as $n\to\infty$ for every $\{N_n, n\geqslant 1\}$ satisfying (3), it is sufficient to note that the sequence $\{Z_n, n>1\}$ satisfies assumptions of Theorem 3 in [3].

By (i) and since the random variable $\lambda$ depends only on finitely many $Y_n$, $n\geqslant 1$, we have

$$\lim_{n\to\infty}\mathrm{P}(Z_n<x|A)=\Phi(x) \tag{6}$$

for all $A\in\mathscr{F}_\lambda$, where $\mathscr{F}_\lambda$ is the $\sigma$-field generated by the random variable $\lambda$.

Now we show that $\{Z_n, n>1\}$ satisfies the generalized Anscombe's condition with the norming sequence $\{k_n=n, n\geqslant 1\}$, i.e. that for every $\varepsilon>0$ there exists a $\delta>0$ such that

$$\limsup_{n\to\infty}\mathrm{P}_A\Big(\max_{(1-\delta)n\leqslant i<(1+\delta)n}|Z_n-Z_i|\geqslant\varepsilon\Big)\leqslant\varepsilon\mathrm{P}(A) \tag{7}$$

holds for every $A\in\mathscr{F}_\lambda$, where $\mathrm{P}_A(B)=\mathrm{P}(A\cap B)$.

If we write $D_n(\delta)=\{i\colon (1-\delta)n\leqslant i<(1+\delta)n\}$, then by a simple estimation we obtain

$$\max_{i\in D_n(\delta)}|Z_n-Z_i| = \max_{i\in D_n(\delta)}\left|\frac{S_n-b\log n}{b\sqrt{2\log n}}-\frac{S_i-b\log i}{b\sqrt{2\log i}}\right| \tag{8}$$

$$\leqslant \max_{i\in D_n(\delta)}\left|\frac{S_n}{b\sqrt{2\log n}}-\frac{S_i}{b\sqrt{2\log i}}\right|+\max_{i\in D_n(\delta)}\left|\frac{\log n}{\sqrt{2\log n}}-\frac{\log i}{\sqrt{2\log i}}\right|$$

$$\leqslant \max_{i\in D_n(\delta)}\max\left(\frac{S_n}{b\sqrt{2\log n}}-\frac{S_i}{b\sqrt{2\log i}},\ \frac{S_i}{b\sqrt{2\log i}}-\frac{S_n}{b\sqrt{2\log n}}\right)+$$

$$+\frac{1}{\sqrt{2}}\max_{i\in D_n(\delta)}\max(\sqrt{\log n}-\sqrt{\log i},\ \sqrt{\log i}-\sqrt{\log n})$$

$$\leqslant \max\left(\frac{S_n}{b\sqrt{2\log n}}-\frac{S_{[n(1-\delta)]}}{b\sqrt{2\log n(1+\delta)}},\ \frac{S_{[n(1+\delta)]}}{b\sqrt{2\log n(1-\delta)}}-\frac{S_n}{b\sqrt{2\log n}}\right)+$$

$$+\frac{1}{\sqrt{2}}\max(\sqrt{\log n}-\sqrt{\log n(1-\delta)},\ \sqrt{\log n(1+\delta)}-\sqrt{\log n})$$

$$\leqslant \max\left(S_{[n(1-\delta)]}\left(\frac{1}{b\sqrt{2\log n}}-\frac{1}{b\sqrt{2\log n(1+\delta)}}\right)+\frac{\sum_{k=[n(1-\delta)]+1}^{n} Y_k^*}{b\sqrt{2\log n}},\right.$$

$$\left. S_n\left(\frac{1}{b\sqrt{2\log n(1-\delta)}}-\frac{1}{b\sqrt{2\log n}}\right)+\frac{\sum_{k=n+1}^{[n(1+\delta)]} Y_k^*}{b\sqrt{2\log n(1-\delta)}}\right)+$$

$$+\left(\sqrt{\log n(1+\delta)}-\sqrt{\log n(1-\delta)}\right)/\sqrt{2}$$

$$\leqslant \frac{\sum_{k=[n(1-\delta)]+1}^{[n(1+\delta)]} Y_k^*}{b\sqrt{2\log n(1-\delta)}}+\max\left(\frac{S_{[n(1-\delta)]}}{b\log n(1-\delta)}b_n, \frac{S_n}{b\log n}b_n'\right)+c_n,$$

where

$$b_n = \log n(1-\delta)\left[\frac{1}{\sqrt{2\log n}}-\frac{1}{\sqrt{2\log n(1+\delta)}}\right],$$

$$b_n' = \log n\left[\frac{1}{\sqrt{2\log n(1-\delta)}}-\frac{1}{\sqrt{2\log n}}\right],$$

$$c_n = \frac{1}{\sqrt{2}}\left(\sqrt{\log n(1+\delta)}-\sqrt{\log n(1-\delta)}\right).$$

It is easy to see that $b_n \to 0$, $b_n' \to 0$ and $c_n \to 0$ as $n \to \infty$.

Now let $\{X_n, n \geqslant 1\}$ be a sequence of independent random variables uniformly distributed on [0, 1].

Put $G(t) = \inf\{x \geqslant 0\colon F(x) \geqslant t\}$. Then, by [6], the sequences $\{G(X_n), n \geqslant 1\}$ and $\{Y_n, n \geqslant 1\}$ are the same in law.

Furthermore, the sequence $S_n = \sum_{k=1}^{n} Y_k^*$ may be represented as $\bar{S}_n = \sum_{k=1}^{n} G(X_k^*)$, where $X_k^* = \inf(X_1, X_2, \ldots, X_k)$, $k \geqslant 1$.

On the other hand, Höglund [9] proved that

$$\frac{\sum_{k=1}^{n} G(X_k^*)-b\log n}{b\sqrt{2\log n}} = \frac{\sum_{k=1}^{n} X_k^*-\log n}{\sqrt{2\log n}}+r_n$$

holds in law, where $r_n \xrightarrow{P} 0$ as $n \to \infty$. Therefore, by Lemma 3.1,

(9) $$\frac{\bar{S}_{[n(1-\delta)]}}{b\log n(1-\delta)}b_n = \frac{\bar{\bar{S}}_{[n(1-\delta)]}}{\log n(1-\delta)}b_n+r_nb_n \to 0, \text{ a.s. } \quad \text{as } n \to \infty$$

and

(10) $$\frac{\bar{S}_n}{b\log n}b'_n = \frac{\tilde{S}_n}{\log n}b'_n + r_n b'_n \to 0 \text{ a.s.} \quad \text{as } n\to\infty,$$

where $\tilde{S}_n = \sum_{k=1}^{n} X_k^*$, $n \geqslant 1$. So, by (8)–(10) we get

(11) $$[\max_{i\in D_n(\delta)} |Z_n - Z_i| \geqslant \varepsilon] \subset \left[\frac{\sum_{k=[n(1-\delta)]+1}^{[n(1+\delta)]} Y_k^*}{b\sqrt{2\log n(1-\delta)}} \geqslant \frac{\varepsilon}{2}\right]$$

for any $\varepsilon > 0$ and sufficiently large $n$.

Observe that

$$\sum_{k=[n(1-\delta)]+1}^{[n(1+\delta)]} Y_k^* = \sum_{k=[n(1-\delta)]+1}^{[n(1+\delta)]} (Y_k^* - Y_{[n(1-\delta)],k}^*) + \sum_{k=[n(1-\delta)]+1}^{[n(1+\delta)]} Y_{[n(1-\delta)],k}^*.$$

By Lemma 3.4 and the fact that the random variables $\lambda$ and

$$\sum_{k=[n(1-\delta)]+1}^{[n(1+\delta)]} Y_{[n(1-\delta)],k}^*$$

are independent for sufficiently large $n$, one can check that condition (7) is a consequence of the following well-known Anscombe condition:

(12) $$\limsup_{n\to\infty} \mathrm{P}(\max_{i\in D_n(\delta)} |Z_n - Z_i| \geqslant \delta) \leqslant \varepsilon.$$

By (11), Lemma 3.3, the Markoff inequality and Lemma 3.2 we obtain

$$\mathrm{P}[\max_{i\in D_n(\delta)} |Z_n - Z_i| \geqslant \varepsilon] \leqslant \mathrm{P}\left[\frac{\sum_{k=[n(1-\delta)]+1}^{[n(1+\delta)]} Y_k^*}{b\sqrt{2\log n(1-\delta)}} \geqslant \frac{\varepsilon}{2}\right]$$

$$\leqslant \mathrm{P}\left[\frac{\sum_{k=[n(1-\delta)]+1}^{[n(1+\delta)]} X_k^*}{\sqrt{2\log n(1-\delta)}} \geqslant \frac{\varepsilon}{3}\right] \leqslant 3\frac{E(\sum_{k=[n(1-\delta)]+1}^{[n(1+\delta)]} X_k^*)}{\varepsilon\sqrt{2\log n(1-\delta)}}$$

$$= \frac{O(1)}{\sqrt{2\log n(1-\delta)}} \to 0 \quad \text{as } n\to\infty.$$

Hence, from Theorem 3 of [3], we immediately obtain (4) for every $\{N_n, n \geqslant 1\}$ satisfying (3). This completes the proof of Theorem 1.

**3. Lemmas.** In this section we present some lemmas we needed in the proofs of Theorem 1.

LEMMA 3.1. *Let $\{X_n, n \geqslant 1\}$ be a sequence of independent random variables uniformly distributed on* [0, 1]. *Then $\tilde{S}_n/\log n \to 1$ a.s. as $n \to \infty$, where $\tilde{S}_n = \sum_{k=1}^{n} X_k^*$, and $X_k^* = \inf(X_1, X_2, \ldots, X_k)$, $k \geqslant 1$, $n \geqslant 1$.*

LEMMA 3.2. $EX_k^* = (k+1)^{-1}$ $(k \geqslant 1)$, $E\tilde{S}_n - \log n = O(1)$.

LEMMA 3.3. *Under the assumptions of Theorem* 0

$$\frac{\sum_{k=1}^{n} G(X_k^*) - b\log n}{b\sqrt{2\log n}} = \frac{\sum_{k=1}^{n} X_k^* - \log n}{\sqrt{2\log n}} + r_n \text{ in law,}$$

*where $r_n \xrightarrow{P} 0$ as $n \to \infty$, and*

$$\frac{\sum_{k=1}^{n} y_k |G(X_k^*) - bX_k^*|}{\sqrt{\log n}} \xrightarrow{P} 0 \quad \text{as } n \to \infty,$$

*where, for $0 < \delta < 1$, $y_k = 1$ if $X_k^* \leqslant \delta$ and $y_k = 0$ if $X_k^* > \delta$, and $G(t) = \inf\{x \geqslant 0\colon F(x) \geqslant t\}$.*

LEMMA 3.4. *Let $\{Y_n, n \geqslant 1\}$ be a sequence of positive independent random variables with the common distribution function $F$ such that $F(x) = 0$ for $x \leqslant 0$, $F(x) > 0$ for $x > 0$. Let us put $Y_n^* = \inf(Y_1, \ldots, Y_n)$, $Y_{m,n}^* = \inf(Y_{m+1}, \ldots, Y_n)$, $n > m$, $n \geqslant 1$.*

*Then the sum $\sum_{n=m+1}^{\infty} (Y_{m,n}^* - Y_n^*)$ converges almost surely.*

Proof. We observe that

$$0 \leqslant Y_{m,n}^* - Y_n^* \leqslant \begin{cases} 0 & \text{if } Y_{m,n}^* \leqslant Y_m^*, \\ Y_{m,n}^* & \text{if } Y_{m,n}^* > Y_m^*. \end{cases}$$

Then

$$\sum_{n=m+1}^{\infty} (Y_{m,n}^* - Y_n^*) \leqslant \sum_{n=m+1}^{\infty} Y_{m,n}^* I_{[Y_{m,n}^* > Y_m]}.$$

Now, it is sufficient to show that

$$\lim_{K\to\infty} P\Big(\sum_{n=m+1}^{\infty} Y_{m,n}^* I_{[Y_{m,n}^* > Y_m]} \geqslant K\Big) = 0.$$

Indeed,

$$\lim_{K\to\infty} \mathrm{P}\Big(\sum_{n=m+1}^{\infty} Y_{m,n}^* I_{[Y_{m,n}^* > Y_m^*]} \geqslant K\Big)$$

$$= \int \lim_{K\to\infty} \mathrm{P}\Big(\sum_{n=m+1}^{\infty} Y_{m,n}^* I_{[Y_{m,n}^* > C]} \geqslant K\Big) \mathrm{P}_{Y_m^*}(dC) = 0$$

by

$$\lim_{K\to\infty} P\Big(\sum_{n=m+1}^{\infty} Y^*_{m,n} I_{[Y^*_{m,n}>C]} \geqslant K\Big) = 0 \quad \text{for every } C > 0,$$

and $P(Y_m = C) = 0$ for $C = 0$.

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