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ASYMPTOTIC MULTIVARIATE NORMALITY FOR THE SUBSERIES VALUES OF A GENERAL STATISTIC FROM A STATIONARY SEQUENCE WITH APPLICATIONS TO NONPARAMETRIC CONFIDENCE INTERVALS*

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Summary. Let $\{Z_i: -\infty < i < +\infty\}$ be a strictly stationary α-mixing sequence with unknown marginal distributions and unknown dependence structure. Suppose that, given data $\vec{Z}_m^i := (Z_{i+1}, Z_{i+2}, \ldots, Z_{i+m})$, the statistic $s_m^i := s_m(\vec{Z}_m^i)$ is a point estimator of the unknown parameter θ. If a sample series $\vec{Z}_n^0$ is available, then the subseries values $s_m^i (0 \leqslant i < i+m \leqslant n)$ may be used to construct a nonparametric confidence interval on θ *via* either Student's distribution or *via* the Typical Value principle. The asymptotic justification for both methods rests upon a more general result which provides necessary and sufficient conditions for asymptotic multivariate normality of subseries values.

1. Introduction. The jackknife (Tukey [9]) and the typical-value principle (Hartigan [6]) both employ subsample values of a general statistic as the building blocks of confidence intervals on an unknown parameter. The idea behind the jackknife is that the "pseudovalues" (which are based upon subsamples of the data) are approximately i.i.d. normal random variables; hence, they can be used to construct approximate confidence intervals based on Student's distribution. The idea behind the typical-value principle is that certain sets of subsample values of a statistic are (approximately) "typical values" for the unknown parameter; i.e. each of the intervals between the ordered subsample values will include the unknown parameter with (approximately) equal probability.

The jackknife and the typical-value principle are both examples of the same fundamental strategy for constructing confidence intervals. This strategy is to provide an *omnibus* procedure, phrased only in terms of a general statistic and the corresponding target parameter which it estimates. (This approach is also shared by Efron's [3] bootstrapped confidence intervals based on the percentile method). Omnibus procedures have strong practical and intuitive appeal,

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because they are nonparametric, they are simple to use, and they can be applied to a variety of situations – without requiring that new theory be developed from first principles in each new scenario. (Hartigan [6, 7] and Efron [2] argue convincingly along these lines.) These omnibus procedures were never intended to supersede the other confidence procedures which already existed for certain specific situations. On the contrary, simple situations (where other confidence procedures already existed) were frequently used as the motivating cases in the developing the omnibus procedures. (For example, Hartigan [6] dealt extensively with the sample mean from a continuous symmetric distribution in his development of the typical-value principle; Tukey's [9] jackknife was motivated by the sample mean from a normal distribution.) The extension of the omnibus procedures to more complicated situations was usually justified by asymptotic arguments. In particular, Hartigan's [7] theoretical justifications for the jackknife and typical-value methods both rested on a more basic result, which itself was phrased in terms of a general statistic. That result (his Theorem 2) gave conditions under which (possibly overlapping) subsample values of a general statistic are asymptotically multivariate normal (with possibly nonzero covariances).

All of the work discussed above deals with subsample values of a general statistic computed from i.i.d. data. If there is dependence in the data, then there is an even greater need for omnibus confidence interval techniques that free the user from the parametric modeling and theoretical analysis. Indeed, the dependence structure would provide one more unknown in the model and one more complication in the theory. The present paper provides omnibus confidence interval procedures for the dependent case, in the same spirit as the jackknife and typical-value procedures (as discussed above). Our procedures are analogous to the jackknife and typical-value methods in physical form: they are phrased in terms of general statistics, but they employ "subseries" of the data rather than "subsamples" (the former referring only to subsamples composed of *successive* observations). As in the i.i.d. case, the theoretical justification for our procedures is asymptotic. On the one hand, equi-lengthed non-overlapping subseries values of a general statistic are shown to be asymptotically distributed as i.i.d. normal random variables. This justifies the use of confidence intervals based on Student's distribution, analogously to the jackknife procedure. On the other hand, certain sets of linear combinations of subseries values behave asymptotically like a set of typical values. This justifies the use of these entities in constructing confidence intervals *via* the typical-value principle. As in the i.i.d. case, the fundamental property underlying our theoretical justifications is the property of asymptotic multivariate normality. Our basic result (Theorem 1) establishes necessary and sufficient conditions under which (possibly overlapping) subseries values of a general statistic are asymptotically multivariate normal (with possibly nonzero covariances).

Quite surprisingly, the conditions in Theorem 1 are virtually identical to those in Hartigan's [7] Theorem 2, even though the former permits dependence in the data while the latter forbids it.

The results presented below assume no knowledge of the marginal distributions of the data beyond stationarity. The only assumption about the dependence is that it be α-mixing (a relatively weak assumption in the mixing hierarchy [8]). The literature contains no other analogies to the jackknife or the typical-value methods for dependent data. Freedman [4] has considered applying the bootstrap to a linear model with autoregressive component; but, as he emphasizes, the bootstrap calculations assume that the user has correctly specified the form of the underlying autoregressive model.

The next section presents basic notation and definitions. Section 3 contains the fundamental asymptotic multivariate normality result for subseries values of a general statistic from an α-mixing sequence (Theorem 1). Corollaries 1 and 2 (in Section 4) provide the justifications for confidence intervals based on Student's distribution and the typical-value principle. Section 4 also provides several examples. The proof of Theorem 1 is deferred to Section 5.

2. Definitions and notation. Let $\{Z_i(\omega): -\infty < i < +\infty\}$ be a strictly stationary sequence of real-valued random variables (r.v.) defined on probability space (Ω, F, P). Let F_p^+ (F_q^- respectively) be the σ-field generated by $\{Z_p(\omega), Z_{p+1}(\omega), \dots\}$ ($\{\dots, Z_{q-1}(\omega), Z_q(\omega)\}$ respectively).

For $N \geqslant 1$ write $\alpha(N) := \sup\{|P\{A \cap B\} - P\{A\}P\{B\}| : A \in F_N^+, B \in F_0^-\}$, and define *$\alpha$-mixing* to mean $\lim_{N\to\infty} \alpha(N) = 0$.

Let $t_m(z_1, \dots, z_m)$ be a function from $R^m \to R^1$, defined for each $m \geqslant 1$ so that $t_m(Z_1(\omega), \dots, Z_m(\omega))$ is F-measurable. Suppressing the argument ω of $Z_i(\cdot)$ from here on, we denote $\vec{Z}_m^i := (Z_{i+1}, Z_{i+2}, \dots, Z_{i+m})$ and $t_m^i := t_m(\vec{Z}_m^i)$.

E, V, and C denote expectation, variance, and covariance, respectively. Indicator functions are denoted by $I\{\cdot\}$. For $B \geqslant 0$ write ${}_BX := X \cdot I\{|X| < B\}$ and ${}^BX := X - {}_BX$.

Let $\{a_n\}$ be a sequence of real vectors, and let A be a set of conditions to be satisfied by the a_n's as $n \to \infty$ (e.g. $|a_n| \to \infty$). Then the notation

$$\lim_A x_{a_n} = x$$

means that, for a single finite constant x, $\lim_{n\to\infty} x_{a_n} = x$ for *all* sequences $\{a_n\}$ satisfying A.

3. Main result. For each $n \geqslant 1$ the data $\vec{Z}_n^0$ from $\{Z_i\}$ is available. Consider a k-vector of subseries values $\vec{T}_n := (T_{1n}, T_{2n}, \dots, T_{kn})$, where $T_{in} := t_{r_{in}}^{m_{in}}$. In studying the asymptotic multivariate distribution of $\vec{T}_n$, it is natural to make certain restrictions on the indices k, m_{in}, r_{in}.

(i) $k \geqslant 1$ is fixed.

(ii) $0 \leqslant m_{in} < m_{in} + r_{in} \leqslant n$ for each $i \in \{1, \ldots, k\}$ and each $n \geqslant 1$.

Condition (ii) requires that each subseries is in fact contained in $\vec{Z}_n^0$.

(iii) $r_{in} \to \infty$ as $n \to \infty$ for each $i \in \{1, \ldots, k\}$.

Condition (iii) prohibits subseries which are asymptotically negligible.

It turns out that the asymptotic covariances between the T_{in}'s depend precisely upon the limiting proportions of overlap between subseries. Therefore it is convenient to pool the $2k$ integers $\{m_{in}, m_{in} + r_{in}: 1 \leqslant i \leqslant k\}$ for fixed n, and to order them and relabel them as $C_n := \{c_{in}: 1 \leqslant i \leqslant 2k\}$, where $0 \leqslant c_{1n} \leqslant c_{2n} \leqslant \ldots \leqslant c_{2k,n} \leqslant n$. Thus

$$c_{1n} = \min_{1 \leqslant i \leqslant k} \{m_{in}\} \quad \text{and} \quad c_{2k,n} = \max_{1 \leqslant i \leqslant k} \{m_{in} + r_{in}\}.$$

To avoid notational complications, we assume that the ranks of $\{m_{in}, m_{in} + r_{in}: 1 \leqslant i \leqslant k\}$ remain the same for all n. That is, if we define $I_n(i)$ to be the rank of m_{in} in C_n (i.e. $c_{I_n(i),n} = m_{in}$), and if we similarly define $J_n(i)$ to be the rank of $m_{in} + r_{in}$ in C_n, then

(iv) $I_n(i) \equiv I(i)$ and $J_n(i) \equiv J(i)$ for all n, for each $i \in \{1, \ldots, k\}$.

The condition guaranteeing asymptotically constant overlap between subseries is

(v) $(c_{j+1,n} - c_{jn})/r_{in} \to \gamma_{ji}^2$ as $n \to \infty$ for each $i \in \{1, \ldots, k\}$ and each $j \in \{I(i), \ldots, J(i) - 1\}$.

Any collection of indices $\{k; (m_{in}, r_{in}): 1 \leqslant i \leqslant k, n \geqslant 1\}$ satisfying conditions (i) through (v) will be called *regular*. Given a regular collection of indices, the matrix Σ is defined to have entries

$$\Sigma_{ij} := \sum_{l = \max\{I(i), I(j)\}}^{\min\{J(i), J(j)\} - 1} \gamma_{li} \gamma_{lj}, \quad i, j \in \{1, \ldots, k\},$$

where empty sums are zero. In particular, $\Sigma_{ii} = 1 \ \forall i$.

The statistics defined by $\{t_m(\cdot): m \geqslant 1\}$ will be called *central with parameter* σ^2 if:

(I) $$\lim_{A \to \infty} \overline{\lim_{n \to \infty}} A^2 P\{|t_n^0| \geqslant A\} = 0$$

and

(II) $$\lim_{A \to \infty} \overline{\lim_{n \to \infty}} A |E\{{}_A t_n^0\}| = 0$$

and

(III) $$\lim_{A \to \infty} \overline{\lim_{u_n/w_n \to \varrho^2,\, w_n \geqslant v_n + u_n \geqslant u_n \to \infty}} |E\{{}_A t_{w_n}^0 \cdot {}_A t_{u_n}^{v_n}\} - \varrho \sigma^2| = 0 \quad \forall \varrho^2 \in [0, 1].$$

These conditions are virtually identical to those set forth by Hartigan [7] in his definition of centrality for the independent case. Condition (I) controls the tails of t_n^0's distribution; (II) centers the statistic; (III) requires the statistic to have covariance behaviour analogous to the sample mean of i.i.d. r.v.s: the squared correlation between the statistic and its subseries value should be approximately equal to the proportion of overlap. Besides having some intuitive appeal, and besides being the obvious analogy to the i.i.d. case, our centrality conditions are also the minimal sufficient conditions for asymptotic multivariate normality of the subseries values of a general statistic from an α-mixing sequence. Therefore it is natural to formulate our theoretical results for general statistics in terms of centrality.

THEOREM 1. *Let $\{Z_i\}$ be α-mixing, and let σ^2 be a positive constant. $\{t_m(\cdot): m \geq 1\}$ are central with parameter σ^2 iff $\vec{T}_n \xrightarrow{D} N_k(\vec{0}, \sigma^2 \Sigma)$ as $n \to \infty$ whenever $\{k; (m_{in}, r_{in}): 1 \leq i \leq k,\ n \geq 1\}$ is a regular collection of indices.*

For a specific statistic, centrality may entail assumptions about the rate of mixing and the marginal distribution of Z_i. These assumptions vary greatly from one situation to the next (see, for example, Corollaries 7 through 10 of [1], which establish centrality for sample means, sample percentiles, and smooth functions of central statistics). Therefore these assumptions cannot be explicitly incorporated into a unified general-statistic theory. In fact, one reason for formulating a result in terms of general statistics is that the implicit conditions for many special cases can be deduced from that single result.

As a practical matter, the following alternative conditions (which are analogous but more restrictive) are known to imply centrality under α-mixing [1]:

(I′) $\{t_n^0\}$ are uniformly squared-integrable

and

(II′) $$\lim_{n\to\infty} E\{t_n^0\} = 0$$

and

(III′) $$\lim_{w_n \geq v_n + u_n \geq u_n \to \infty} (w_n/u_n)^{1/2}\, C\{t_{w_n}^0, t_{u_n}^{v_n}\} = \sigma^2.$$

These sufficient conditions are used in Corollaries 8 through 10 of [1] to deduce centrality of the sample mean and sample fractiles. Furthermore, these sufficient conditions are evidently not excessively restrictive: they are satisfied (for the sample mean) by the standard conditions of Ibragimov and Linnik's [8] Theorem 18.5.3.

4. Applications to confidence intervals. Throughout this section assume the following set-up: The statistic $s_m^i := s_m(\vec{Z}_m^i)$ is wholly computable from the data $\vec{Z}_m^i$, and does not depend upon any unknown parameters. Furthermore, s_m^i estimates the unknown parameter θ. Finally, put $t_m^i := (s_m^i - \theta)\, m^{1/2}$. The notation of Section 3 remains unchanged.

COROLLARY 1. *Let* $r_{in} = r_n$ *and* $m_{in} = (i-1)r_n \forall i \in \{1, \ldots, k\}$ *and* $\forall n$, *with* $r_n/n \to \varrho^2 > 0$, $1 \leqslant r_n \leqslant kr_n \leqslant n \forall n$. *If* $\{Z_i\}$ *is* α-*mixing and* $\{t_m(\cdot)\}$ *are central with parameter* $\sigma^2 > 0$, *then* $\vec{T}_n \xrightarrow{D} N_k(\vec{0}, \sigma^2 I_k)$ *as* $n \to \infty$.

Proof. Immediate from Theorem 1.

Analogously to [7] justification for the jackknife in the independent case, Corollary 1 provides the asymptotic justification in the α-mixing case for treating

$$S_n := (\bar{s}_n - \theta) k^{1/2} / \Big[\sum_{i=0}^{k-1} [s_{r_n}^{ir_n} - \bar{s}_n]^2/(k-1)\Big]^{1/2}$$

as an r.v. with Student's distribution on $k-1$ degrees of freedom. (Here $\bar{s}_n := \sum_{i=0}^{k-1} s_{r_n}^{ir_n}/k$.)

Observe that S_n is free of the nuisance parameter σ^2, and hence may be used as a pivot for constructing a confidence interval on θ.

Corollary 2 shows that, even under α-mixing, the subseries values s_m^i can be used to construct a set of statistics which are (asymptotically) typical values for θ.

COROLLARY 2. *Let* $m_{1n} \equiv 0$ *and* $m_{i+1,n} = m_{in} + r_{in}$ $\forall i \in \{1, \ldots, k-1\}$ *and* $\forall n$, *with* $r_{in}/n \to \varrho_i^2 > 0$ $\forall i \in \{1, \ldots, k\}$, $1 \leqslant r_{jn} < \sum_{i=1}^{k} r_{in} \leqslant n$ $\forall j \in \{1, \ldots, k\}$ *and* $\forall n$.

Let $l \in \{1, \ldots, k\}$ *be arbitrary but fixed, and write* $K := \{i \in \{1, \ldots, k\}$ s.t. $i \neq l\}$. *Define*:

$$V_{in} := \big[s_{r_{in}}^{m_{in}} r_{in}^{1/2} + s_{r_{ln}}^{m_{ln}} r_{ln}^{1/2}\big]/(r_{in}^{1/2} + r_{ln}^{1/2}) \quad \textit{for each } i \in K,\ \forall n.$$

If $\{Z_i\}$ *is* α-*mixing and* $\{t_m(\cdot)\}$ *are central with parameter* $\sigma^2 > 0$, *then*

$$\lim_{n\to\infty} P\{\sum_{i \in K} I\{V_{in} < \theta\} = N\} = 1/k \quad \textit{for each } N \in \{0, 1, \ldots, k-1\}.$$

In particular,

$$P\{\min_{i \in K} V_{in} < \theta \leqslant \max_{i \in K} V_{in}\} \to 1 - 2/k \quad \textit{as } n \to \infty.$$

Proof. By Theorem 1, $\vec{T}_n \xrightarrow{D} N_k(\vec{0}, \sigma^2 I_k)$ as $n \to \infty$. For each $i \in K$, write $\tilde{T}_{in} := T_{in} + T_{ln}$; also write $\vec{\tilde{T}}_n := (\tilde{T}_{in} : i \in K)$. Then $\vec{\tilde{T}}_n \xrightarrow{D} N_{k-1}(\vec{0}, \tilde{\Sigma})$, where $\tilde{\Sigma} := \sigma^2 (I_{k-1} + [1]_{(k-1)\times(k-1)})$. Using the logic in the second paragraph of Hartigan's [7] proof of his Theorem 6, we may conclude that the event $A_{Nn} := \{$exactly N of the $\tilde{T}_{in}$'s, $i \in K$, are less than 0$\}$ has asymptotic probability $1/k$ (for each $N \in \{0, 1, \ldots, k-1\}$). But since $\tilde{T}_{in} = (V_{in} - \theta)(r_{in}^{1/2} + r_{ln}^{1/2})$, the event A_{Nn} is equivalent to the event $\sum_{i \in K} I\{V_{in} < \theta\} = N$.

According to Corollaries 1 and 2, we may construct both types of confidence intervals for θ (using only the subseries statistics s_m^i) whenever $\{Z_i\}$ is α-mixing and $\{t_m(\cdot)\}$ are central with $\sigma^2 > 0$. We now describe several standard scenarios where these conditions are met. Note that the normal, double-exponential, and Cauchy AR(1) sequences are not only α-mixing but also (see [5]) satisfy

$$\sum_{k=1}^{\infty} (\alpha(k))^{\varepsilon} < \infty \quad \text{for } 0 < \varepsilon \leqslant 1.$$

Example 1. If $s_m(\cdot)$ is the sample mean and $\theta = E\{Z_0\}$, then in either of the following two scenarios we find that $\{t_m(\cdot)\}$ are central with $\sigma^2 > 0$:

1.a. For some $\delta > 0$,

$$E\{|Z_0|^{2+\delta}\} < \infty, \quad \sum_{k=1}^{\infty} (\alpha(k))^{\delta/(2+\delta)} < \infty, \quad \text{and} \quad \sum_{i=-\infty}^{\infty} C\{Z_0, Z_i\} > 0.$$

1.b. Z_0 is bounded,

$$\sum_{k=1}^{\infty} \alpha(k) < \infty \quad \text{and} \quad \sum_{i=-\infty}^{\infty} C\{Z_0, Z_i\} > 0.$$

Example 2. If $s_m(\cdot)$ is the π^{th} sample percentile and θ is the π^{th} percentile of Z_0's marginal distribution F, then the following regularity conditions guarantee that $\{t_m(\cdot)\}$ are central with $\sigma^2 > 0$: F is absolutely continuous and strictly increasing, with derivative f satisfying $f(\theta) > 0$; also $\pi \in (0, 1)$,

$$\sum_{k=1}^{\infty} \alpha(k) < \infty, \quad \text{and} \sum_{i=-\infty}^{\infty} (P\{Z_0 \leqslant \theta, Z_i \leqslant \theta\} - \pi^2) > 0.$$

Example 3. Suppose $\{Z_i\}$ is α-mixing and $\tilde{t}_m(\cdot) := (\tilde{s}_m(\cdot) - \tilde{\theta}) m^{1/2}$ is central with parameter $\tilde{\sigma}^2 > 0$. Then the transformed statistic $s_m(\cdot) := f(\tilde{s}_m(\cdot))$ estimates the transformed parameter $\theta := f(\tilde{\theta})$, and $t_m(\cdot) := (s_m(\cdot) - \theta) m^{1/2}$ is central with $\sigma^2 > 0$ provided that f is differentiable at $\tilde{\theta}$ and $f'(\tilde{\theta}) \neq 0$ (for details see [1]).

5. Proof of Theorem 1. Without loss of generality, put $\sigma^2 = 1$. First we prove that centrality is sufficient for asymptotic multivariate normality. For each $i \in \{2, 3, \ldots, 2k\}$ define $\{d_{in}: n \geqslant 1\}$ and $\{w_{in}: n \geqslant 1\}$ as follows: $d_{in} := c_{in} - c_{i-1,n}$. If $d_{in} \not\to \infty$ as $n \to \infty$ then $w_{in} \equiv 0\ \forall n$; otherwise we can choose w_{in} s.t. $0 \leqslant w_{in} \leqslant d_{in} \forall n$, $w_{in} \to \infty$, and $w_{in}/d_{in} \to 0$ as $n \to \infty$. For each $l \in \{1, 2, \ldots, 2k-1\}$ and each n, write

$$\tilde{t}_{ln} := t^{c_{ln}}_{d_{l+1,n} - w_{l+1,n}}.$$

We shall begin by showing that

$$B_{in}:=|T_{in}-\sum_{l=I(i)}^{J(i)-1}\gamma_{li}\cdot\tilde{t}_{ln}|\xrightarrow{P}0 \quad \text{as } n\to\infty \text{ for each } i\in\{1,\dots,k\}.$$

Observe that

$$(1)\qquad B_{in}\leqslant|^{A}T_{in}|+\sum_{l=I(i)}^{J(i)-1}\gamma_{li}|^{A}\tilde{t}_{ln}|+|_{A}T_{in}-\sum_{l=I(i)}^{J(i)-1}\gamma_{li}\cdot{}_{A}\tilde{t}_{ln}| \quad \text{for } A>0.$$

Let $\varepsilon>0$ be given.

By (I),

$$\lim_{A\to\infty}\overline{\lim_{n\to\infty}}P\{|^{A}T_{in}|>\varepsilon\}=0 \quad\text{and}\quad \lim_{A\to\infty}\overline{\lim_{n\to\infty}}P\{\gamma_{li}|^{A}\tilde{t}_{ln}|>\varepsilon\}=0$$

for each l. Let $B_{in}(A)$ denote the term within the last modulus of equation (1). We have

$$(2)\qquad P\{|B_{in}(A)|>\varepsilon\}\leqslant\varepsilon^{-2}[|E\{({}_{A}T_{in})^{2}\}-1|+$$
$$+\sum_{l=I(i)}^{J(i)-1}(2\gamma_{li}|E\{{}_{A}T_{in}\cdot{}_{A}\tilde{t}_{ln}\}-\gamma_{li}|+\gamma_{li}^{2}|E\{({}_{A}\tilde{t}_{ln})^{2}\}-1|)+$$
$$+2\sum_{I(i)\leqslant l<l'\leqslant J(i)-1}\gamma_{li}\cdot\gamma_{l'i}|E\{{}_{A}\tilde{t}_{ln}\cdot{}_{A}\tilde{t}_{l'n}\}|].$$

Now take $\lim_{A\to\infty}\overline{\lim_{n\to\infty}}$ of both sides in equation (2). Except for the terms within the last summation, each term on the r.h.s. of (2) vanishes by (III). Those remaining summands with $\gamma_{li}\cdot\gamma_{l'i}\neq 0$ are each dominated by $|C\{{}_{A}\tilde{t}_{ln},{}_{A}\tilde{t}_{l'n}\}|+|E\{{}_{A}\tilde{t}_{ln}\}E\{{}_{A}\tilde{t}_{l'n}\}|$. Now taking $\lim_{A\to\infty}\overline{\lim_{n\to\infty}}$, we obtain zero, since the covariance term is bounded by $4A^{2}\alpha(c_{l'n}-c_{l+1,n}+w_{l+1,n})$ (see [8], p. 306), and since (II) applies to the expectation terms. Combining these results establishes $B_{in}\xrightarrow{P}0$ for each i.

Let $\lambda:=(\lambda_1,\lambda_2,\dots,\lambda_k)\in R^{k}$ be fixed. It will suffice to consider the asymptotic distribution of

$$G_n:=\sum_{i=1}^{k}\sum_{l=I(i)}^{J(i)-1}\lambda_i\gamma_{li}\tilde{t}_{ln}$$

in place of that of $\lambda\cdot\vec{T}_n$, because

$$|\lambda\cdot\vec{T}_n-G_n|\leqslant\sum_{i=1}^{k}|\lambda_i|B_{in}\xrightarrow{P}0.$$

Write

$$G_n=\sum_{l=1}^{2k-1}\tilde{t}_{ln}\cdot g_l,\quad\text{where } g_l:=\sum_{i=1}^{k}\lambda_i\gamma_{li}I\{I(i)\leqslant l\leqslant J(i)-1\}.$$

Define r.v.'s. $\{\tilde{t}'_{ln}: 1 \leqslant l \leqslant 2k-1, n \geqslant 1\}$ to have the same marginal distributions as $\{\tilde{t}_{ln}: 1 \leqslant l \leqslant 2k-1, n \geqslant 1\}$, but with $\{\tilde{t}'_{ln}: 1 \leqslant l \leqslant 2k-1\}$ independent for fixed $n \geqslant 1$. Let

$$G'_n := \sum_{l=1}^{2k-1} \tilde{t}'_{ln} \cdot g_l.$$

Then, for every $u \in R$,

$$|E\{\exp\{iuG_n\}\} - E\{\exp\{iuG'_n\}\}| \leqslant 16 \sum_{l=2}^{2k-1} \alpha(w_{ln}) I\{\lim_{N\to\infty} d_{lN} = \infty\} \to 0 \text{ as } n \to \infty,$$

by the argument of Ibragimov and Linnik ([8], p. 338). Hence we may simply consider the asymptotic distribution of G'_n in place of that of $\lambda \cdot \vec{T}_n$.

By Theorem 6 of [1], each $\tilde{t}'_{ln}$ is marginally asymptotically $N(0, 1)$ (provided $\lim_{n\to\infty} d_{l+1,n} = \infty$). And since $\{\tilde{t}'_{ln}: 1 \leqslant l \leqslant 2k-1\}$ are independent, we may conclude that

$$G'_n \xrightarrow{D} N(0, \sum_{l=1}^{2k-1} g_l^2).$$

Observe that

$$\sum_{l=1}^{2k-1} g_l^2 = \sum_{i=1}^{k} \sum_{j=1}^{k} \lambda_i \lambda_j \Sigma_{ij}$$

for Σ_{ij} as defined in Section 3. Since this argument holds for each $\lambda \in R^k$, the sufficiency proof is completed.

To prove that centrality is necessary, consider any collection of indices with $k = 2$, $m_{1n} = 0 \leqslant m_{2n} < m_{2n} + r_{2n} \leqslant r_{1n} \leqslant n \,\forall n$,

$$\lim_{n\to\infty} r_{2n}/r_{1n} = \varrho^2, \ \lim_{n\to\infty} m_{2n}/r_{1n} = \mu^2, \text{ and } \lim_{n\to\infty} r_{in} = \infty \text{ for } i = 1, 2.$$

Any such collection is regular, with $I(1) = 1$, $J(1) = 4$, $I(2) = 2$, $J(2) = 3$, $\gamma_{11}^2 = \mu^2$, $\gamma_{21}^2 = \varrho^2$, $\gamma_{31}^2 = 1 - \mu^2 - \varrho^2$, and $\gamma_{22}^2 = 1$. Therefore, by hypothesis,

$$(3) \qquad (t_{r_{1n}}^{0}, t_{r_{2n}}^{m_{2n}}) \xrightarrow{D} N_2(0, 0; 1, 1; \varrho) \quad \text{as } n \to \infty.$$

The argument in the final paragraph of [1] can now be used to show that (3) holds even if m_{2n}/r_{1n} is not convergent. Finally, centrality is implied by the fact that (3) holds whenever $r_{2n}/r_{1n} \to \varrho^2$ and $r_{1n} \geqslant m_{2n} + r_{2n} \geqslant r_{2n} \to \infty$ (see [1], Theorem 6).

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