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ON A LIMIT THEOREM AND INVARIANCE PRINCIPLE FOR SYMMETRIC STATISTICS*

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Abstract. The paper contains direct proofs of two important theorems. First of them (Theorem 0.6) was formulated and proved by Dynkin and Mandelbaum [2], the second one (Theorem 1.2) — by Mandelbaum and Taqqu [3].

0. Introduction. The purpose of this note is to give a direct proof of some recent important results of Dynkin and Mandelbaum [2]. This also provides immediately the results in [3] with a very simple proof. This is achieved by avoiding the use of Poisson process.

Let us set up some notation. Let (X, Σ, μ) be a probability space and (X^k, Σ^k, μ^k) be the k-fold produce probability space. Let $h_k(x_1, \ldots, x_k)$ be a symmetric function of k variables. We call it *canonical* if

$$\int h_k(x_1, \ldots, x_{k-1}, y)\, d\mu = 0 \text{ for all } x_1, \ldots, x_{k-1} \in X^{k-1}.$$

Let $X_1, \ldots, X_n$ be an i.i.d. X-valued random variable on a probability space with distribution μ. As in [2], define

$$\sigma_k^n(h_k) = \begin{cases} \sum_{1 \leqslant s_1 < \ldots < s_k \leqslant n} h_k(X_{s_1}, \ldots, X_{s_k}) & \text{for } k \leqslant n, \\ 0 & \text{for } k > n. \end{cases}$$

Let

$$H = \{(h_0, h_1, \ldots): h_k \text{ canonical and } \sum_{k=1}^{\infty} \frac{1}{k!} \|h_k\|_2^2 < \infty\},$$

where h_0 is a constant and $\| \|_2$ is the norm in $L^2(X^k, \Sigma^k, \mu^k)$. On H define

$$\|h\|^2 = \sum_{k=0}^{\infty} \|h_k\|_2^2/k!.$$

H is the so-called *exponential* (Foch) space of $L_0^2(X, \Sigma, \mu)$ ($\varphi \in L^2(X, \Sigma, \mu)$ with $E\varphi(X) = 0$). It is a Hilbert space under coordinate addition, scalar

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multiplication and $\|\,\|$. For each $\varphi \in L_0^2(X, F, \mu)$, $h^\varphi \in H$ with $h_k^\varphi = \varphi(x_1), \ldots, \varphi(x_k)$. It can be easily seen that $\mathrm{sp}\{h^\varphi : \varphi \in L_0^2(X, F, \mu)\}$ is dense in H. Define, for each $h \in H$,

$$Y_n(h) = \sum_{k=0}^{\infty} n^{-k/2} \sigma_k^n(h_k). \tag{0.1}$$

Since $\sigma_k^n(h_k) = 0$ for $k > n$, this is a finite sum. Also, let

$$Y_n^t(h) = \sum_{k=0}^{\infty} n^{-k/2} \sigma_k^{[nt]}(h_k). \tag{0.2}$$

The main purpose is to show directly that

$$Y_n(h) \xrightarrow{D} \sum_{k=0}^{\infty} \frac{I_k(h_k)}{k!},$$

where $\xrightarrow{D}$ denotes convergence in distribution and $I_k(h_k)$ denotes Ito-Wiener multiple integral of h_k with respect to Gaussian random measure W with $EW(A)W(A') = \mu(A \cap A')$.

In the next section we discuss the convergence of $Y_n^t(h)$. We observe that for $\varphi \in L_0^2(X, \Sigma, \mu)$

$$Y_n(h^\varphi) = \sum_{k=0}^{n} n^{-k/2} \sum_{1 \leqslant s_1 < \ldots < s_k \leqslant n} \varphi(X_{s_1}) \ldots \varphi(X_{s_k})$$

$$= \sum_{k=0}^{n} \sum_{1 \leqslant s_1 < \ldots < s_k \leqslant n} \frac{\varphi(X_{s_1})}{\sqrt{n}} \ldots \frac{\varphi(X_{s_k})}{\sqrt{n}} = \prod_{1}^{n} \left(1 + \frac{\varphi(X_i)}{\sqrt{n}}\right).$$

Let us observe that, for any $\varepsilon > 0$,

$$\sum_j P(|\varphi(X_j)| > \sqrt{\varepsilon j}) = \sum_j P(|\varphi(X_1)|^2 > \varepsilon j) \leqslant C\|\varphi\|_2^2 < \infty.$$

Hence by Borel-Cantelli lemma, a.s. (for $j \leqslant n$) $|\varphi(X_j)| \leqslant \sqrt{\varepsilon j} \leqslant \sqrt{\varepsilon}\sqrt{n}$ for $j \geqslant$ some $N(\omega)$ $(N(\omega) < \infty)$. But

$$\prod_{1}^{n} \left(1 + \frac{\varphi(X_j)}{\sqrt{n}}\right) = \prod_{1}^{N(\omega)} \left(1 + \frac{\varphi(X_j)}{\sqrt{n}}\right) \prod_{N(\omega)}^{n} \left(1 + \frac{\varphi(X_j)}{\sqrt{n}}\right)$$

giving for a.s. w, so

$$\lim_n Y_n(h^\varphi) = \lim_n \prod_{N(\omega)}^{n} \left(1 + \frac{\varphi(X_j)}{\sqrt{n}}\right).$$

Thus WLOG, we can assume for n large $|\varphi(X_j)/\sqrt{n}| < 1$ a.s. for all $j \leqslant n$ and

$$Y_n(h^\varphi) = \prod_{1}^{n} \left(1 + \frac{\varphi(X_j)}{\sqrt{n}}\right).$$

Taking log on both sides and expanding $\log(1+x)$ we have

$$\log\prod_1^n\left(1+\frac{\varphi(X_j)}{\sqrt{n}}\right)=\sum_1^n\frac{\varphi(X_j)}{\sqrt{n}}-\frac{1}{2}\sum_1^n\frac{\varphi(X_j)^2}{n}+\varepsilon_n(\varphi),$$

where $\varepsilon_n(\varphi)\xrightarrow{P}0$ by the WLLN and since $\max|\varphi(X_j)/\sqrt{n}|\xrightarrow{P}0$ by Chebychev's Inequality, i.e. the $(Y_n(h^\varphi))\xrightarrow{D}\exp[I_1(\varphi)-\frac{1}{2}\|\varphi\|_2^2]$. Using Cramér-Wold device and the above argument we get

(0.3) LEMMA. *For any finite subset* $\{\varphi_1\ldots\varphi_k\}\subseteq L^2(X,\Sigma,\mu)$

$$\left(Y_n(h^{\varphi_1}),\ldots,Y_n(h^{\varphi_k})\right)\xrightarrow{D}\left(\exp(I_1(\varphi_1))-\tfrac{1}{2}\|\varphi_1\|_2^2,\ldots,\exp\left(I_1(\varphi_k)-\tfrac{1}{2}\|\varphi_k\|_2^2\right)\right).$$

As a consequence, we get for $\{\varphi_i,\ i\in I\}$ a finite subset of $L^2(X,\Sigma,\mu)$ and $\{c_i,\ i\in I\}\subseteq \boldsymbol{R}$,

$$Y_n\left(\sum_{i\in I}c_ih^{\varphi_i}\right)\xrightarrow{D}\sum_{k=0}^{\infty}\frac{I_k\left(\left[\sum_{i\in I}c_ih^{\varphi_i}\right]_k\right)}{k!}. \tag{0.3$'$}$$

We now observe that for $h,h'\in H$,

$$E\left[Y_n(h)-Y_n(h')\right]^2=\sum_k\binom{n}{k}n^{-k}\|h_k-h_k'\|^2\leqslant E\|h-h'\|^2, \tag{0.4}$$

since $E\sigma_k^n(h_k-h_k')\sigma_l^n(h_l-h_l')=\binom{n}{k}\|h_k-h_k'\|^2\delta_{kl}$ (by [2], p. 744). Also,

$$E\left(\sum_{k=0}^{\infty}I_k(h_k)/k!-\sum_{k=0}^{\infty}\frac{I_k(h_k')}{k!}\right)^2=\|h-h'\|^2. \tag{0.5}$$

Thus we get

(0.6) THEOREM. *For any* $h\in H$,

$$Y_n(h)\xrightarrow{D}W(h)=\sum_{k=0}^{\infty}\frac{I_k(h_k)}{k!}.$$

Proof. Let $h\in H$ and $\varepsilon>0$. Choose

$$h'=\sum_{i\in I}c_ih^{\varphi_i}$$

such that $\|h-h'\|^2<\varepsilon/2$. Now consider, for $t\in\boldsymbol{R}$,

$$|E(e^{itY_n(h)}-e^{itW(h)})|\leqslant E|e^{itY_n(h)}-e^{itY_n(h')}|+E|e^{itY_n(h')}-e^{itW(h')}|+E|e^{itW(h')}-e^{itW(h)}|.$$

Using Schwartz's Inequality and the fact $|e^{ix}-1|\leqslant|x|$, we get that the first and third terms of the above inequality are dominated by $t^2E\|h-h'\|^2$ using (0.4) and (0.5). Hence, by (0.3)$'$,

$$\overline{\lim}|Ee^{itY_n(h)}-Ee^{itW(h)}|\leqslant \varepsilon/2.$$

As ε is arbitrary, we get the result.

Finally, we make some observations to be used later:

$$(0.7)\quad Y_n^t(h^\varphi)=\sum_{k=0}^{[nt]} n^{-k/2}\sum_{1\leqslant s_1<\ldots<s_k\leqslant[nt]}\varphi(X_{s_1})\ldots\varphi(X_{s_k})=\prod_1^{[nt]}\left(1+\frac{\varphi(X_i)}{\sqrt{n}}\right).$$

Also, $\min(t,s)\ \mu(A\cap A')$ is a covariance on $[0,\infty)\times\Sigma$ giving that there exists a centered Gaussian process $W(t,A)$ with $EW(t,A)\ W(s,A')=\min(t,s)\ \mu(A\cap A')$. Let, for $T<\infty$,

$$H_T=\left\{(h_0,h_1,\ldots,)\in H:\ \sum_{k=0} T^k\frac{\|h_k\|^2}{k!}<\infty\right\}.$$

1. Invariance Principle. Let $D[0,T]$, $T\leqslant\infty$, be the space of right continuous functions on $[0,T]$ ($[0,\infty)$) with left limits at each $t\leqslant T$. The space $D[0,T]$ is endowed with Skorohod topology [1]. The topology in $D[0,\infty)$ is the one described in Whitt [4]. We note that

$$X_{[nt]}=\sum_1^{[nt]}\left(\frac{\varphi^2(X_i)-E\varphi^2}{n}\right)$$

has stationary independent increments. So, for $\varepsilon>0$,

$$P(\sup_{0\leqslant t\leqslant T}|X_{[nt]}|>\varepsilon)\leqslant C\,P(|X_{[nT]}|\geqslant\varepsilon)\to 0$$

by the weak law of large numbers. Using this, the arguments preceding Lemma 0.3, invariance principle and Cramér-Wold device we get the following analogue of Lemma 0.3:

LEMMA 1.1. $(Y_n^t(h^{\varphi_1}),\ldots,Y_n^t(h^{\varphi_k}))\xrightarrow{D_{k,T}}(\exp(I_1^t(\varphi_j)-\frac{1}{2}t\|\varphi_j\|^2),\ j=1,\ldots,k)$, *where* $I^t(\varphi_j)=\iint 1_{(0,t]}(u)\,\varphi_j(x)\,W_k(du,dx)$.

Here $\xrightarrow{D_{k,T}}$ *denotes convergence in* $D^k[0,T]$ *with respect to product topology.*

We note that $W(t,A)$ is a Brownian motion for each $A\in\Sigma$. Thus we can choose $I^t(\varphi)$ continuous for each φ and a martingale in t as $I^t(\varphi)=\int\varphi(x)\,W(t,dx)$. We get, for $\{c_1,\ldots,c_k\}\subseteq\boldsymbol{R}$ (k finite),

$$Y^t\Big(\sum_{j=1}^k c_j h^{\varphi_j}\Big)\to\sum_{j=1}^k c_j\exp\big(I^t(\varphi_j)-\tfrac{1}{2}t\|\varphi_j\|^2\big).$$

Let $\varphi\in L_0^2(X,\Sigma,\mu)$, $\|\varphi\|=1$, and write

$$(\varphi^k)^t=\varphi(x_1)\ldots\varphi(x_k)\,1_{(0,t]}(u_1)\ldots 1_{(0,t]}(u_k).$$

Define $I_k(\varphi^k)^t=k!\,H_k(t,I(\varphi))$, where H_k is Hermite polynomial, i.e.

$$\sum_{k=0}^{\infty}\gamma^k H_k(t,x)=\exp(\gamma x-\tfrac{1}{2}\gamma^2 t).$$

For $\varphi \in L_0^2(X, \Sigma, \mu)$, $\|\varphi\| = 1$, we define for $((h^\varphi)^t = (1, \varphi^t, (\varphi^2)^t, \ldots)$,

$$W(h^\varphi)^t = \sum_{k=0}^{\infty} \frac{I_k(\varphi^k)^t}{k!},$$

and extend it linearly to $(\Sigma c_j (h^{\varphi_j})^t)$. It is a martingale.

Let $h \in H_T$, $\{h(n)\}$ a sequence in sp $\{(h^\varphi)^t$, φ in CONS in $L_0^2(X, \Sigma, \mu)\} \subseteq H_T$; then

$$P(\sup_{t \leqslant T} |W^t(h(n)-h(m))| \geqslant \varepsilon) \leqslant E|W^T(h(m)-h(n))|^2$$

$$= \sum_{k=0}^{\infty} T^k \frac{\|h_k(m)-h_k(n)\|^2}{k!},$$

using Doob's inequality and argument as in (0.5). Define, for $h \in H^t$, $W^t(h) = -\lim W^t(h_n)$, where the limit is uniform on compact for $h_n \to h$. Then $W^t(h)$ is right continuous martingale and has the same distribution as $\Sigma_k I_k^t(h_k)/k!$. Now we derive the main theorem of [3].

THEOREM 1.2. *$Y_n^t(h) \xrightarrow{D} W^t(h)$ in $D[0, T]$ for $h \in H^T$ for each $T < \infty$.*

Proof. Let $h \in H$ and $\varepsilon > 0$, choose $h_k' \in \mathrm{sp}\{h^\varphi\colon \varphi \in L_0^2(X, \Sigma, \mu)\} \ni h_k \to h$.

Now define $X_{nk}^\cdot = Y_n^\cdot(h_k')$, $Z_n^\cdot = Y_n^\cdot(h)$, $X_k^\cdot = W^\cdot(h_k')$ and $X = W^\cdot(h)$. Then $X_{n,k}^\cdot \xrightarrow{D} X_k^\cdot$ as $n \to \infty$ in $D[0, T]$ for each $T < \infty$ by Lemma 1.1. Also $X_k^\cdot \xrightarrow{D} X$ as $n \to \infty$ in $D[0, T]$ for each $T < \infty$. In addition,

$$P(\sup_{0 \leqslant t \leqslant T} |X_{nk}^\cdot - Z_n^\cdot| \geqslant \varepsilon) \leqslant E|Y_n^T(h-h_k')|^2 \leqslant T\|h-h_k'\|$$

giving

$$\lim_{k \to \infty} \overline{\lim_n} P(\varrho(X_{nk}^\cdot, Z_n^\cdot) \geqslant \varepsilon) \to 0$$

with ϱ being the Skorohod metric on $D[0, T]$. This implies (by [1], Thm 4.2, p. 25) that $Z_n^\cdot \xrightarrow{D} W^\cdot(h)$ in $D[0, T]$ $(T < \infty)$ giving the result.

Remark. In the above arguments we may use an interpolated version of $Y_n^t(h)$ from the beginning and use appropriate version of Donsker's Invariance Principle to conclude above convergence occurs in $D[0, T]$ in sup norm giving $W^t(h)$ continuous.

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