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COMPLEMENTS ON DECOUPLING INEQUALITIES FOR MULTILINEAR FUNCTIONS IN STABLE RANDOM VECTORS*

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Abstract. Let B and V be real separable Banach spaces and d be a positive integer. Let M: $B^d \mapsto V$ be a measurable symmetric multilinear function, and let X be a B-valued symmetric p-stable random vector. It is shown that if $0 < q < p/2$, then the finiteness of $E\|M(X, \ldots, X)\|_V^q$ is not sufficient for the validity of the important part of the decoupling inequalities. A natural condition, in terms of the spectral measure of X and an algebraic equation involving M, is proposed and it is proved that this condition ensures decoupling inequalities for all $q \in (0, p)$. This result complements de Acosta's decoupling inequalities for multilinear functions in B-valued symmetric p-stable random vectors.

1. Introduction. Recently, several authors [3, 4, 6, 7] have studied decoupling inequalities for multilinear functions in random variables both for their intrinsic value as well as for their applications (e.g., multiple stochastic integrals). A. de Acosta [1] has proved decoupling inequalities for multilinear functions in Banach space valued p-stable random vectors, $0 < p < 2$. There are two main features which distinguish [1] from other papers where various decoupling inequalities are proved: (i) The random vectors considered for the decoupling inequalities in [1] are genuinely infinite-dimensional; and (ii) the decoupling inequalities for q^{th} order moments of multilinear random function are provided where q not only takes values in the interval $[1, \min\{1, p\})$ but also it may take certain values less than 1 (the concave case).

To discuss the result of [1] more precisely, let M be a measurable *multilinear function* (i.e., linear in each variable when the other variables are held fixed) from d-product of a separable Banach space B into another separable Banach space V, and let X be a B-valued symmetric p-stable random vector. A. de Acosta introduced the condition

$$(\mathrm{A}_q) \qquad E\,\|\tilde{M}(X)\|_V^q < \infty,$$

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where $q \in (0, p)$ is fixed and $\tilde{M}(x) = M(x, \ldots, x)$. Under the additional condition that M is symmetric (i.e., $M(x_1, \ldots, x_d) = M(x_{\pi(1)}, \ldots, x_{\pi(d)})$ for every permutation π of $(1, 2, \ldots, d)$), he showed that if $q \in (p/2, p)$, then (A_q) ensures the decoupling inequalities

$$(\mathrm{DI}_q) \qquad cE\,\|M(X_1, \ldots, X_d)\|_V^q \leqslant E\,\|\tilde{M}(X)\|_V^q \leqslant CE\,\|M(X_1, \ldots, X_d)\|_V^q,$$

where $X_1, \ldots, X_d$ are independent copies of X and the constants c and C depend only on p, q and d (and not on B, V, M or X). However, it turns out that if $q \in (0, p/2)$, then (A_q) is not sufficient (contrary to the assertion in [1]) for ensuring the important and difficult part (i.e., the right-hand side) of (DI_q). We substantiate this by presenting a counterexample; the proof in [1] for this case has an oversight, which we also point out. This material constitutes the contents of Section 2 of this note.

We must point out here that de Acosta's proof of the validity of the right side of (DI_q), for the case $q \in (0, p/2)$, is indeed correct if one makes the stronger assumption that (A_r) is satisfied for some $r \in (p/2, p)$. In view of this remark and the discussion above, a version of de Acosta's theorem, for which the proof given in [1] is valid, can be stated as follows:

Let $q \in (0, p)$; assume $E\,\|\tilde{M}(X)\|_V^q < \infty$ if $q \in (p/2, p)$, and $E\,\|\tilde{M}(X)\|_V^r < \infty$, for some $r = r(q) \in (p/2, p)$ if $q \in (0, p/2]$; then the decoupling inequalities (DI_q) hold. Thus, in particular, if (A_r) is satisfied for all $r \in (0, p)$, then the inequalities (DI_q) are valid for all $q \in (0, p)$.

The central part of this note is contained in Section 3 where we propose a *natural condition* (in terms of the spectral measure σ_X of X and an algebraic equation involving M) and prove that this condition ensures (DI_q) for *all* values of $q \in (0, p)$ (Theorem 3.1); this complements the above noted de Acosta's result. A motivation for introducing our condition and some reflections on its essence are discussed in the beginning of Section 3. Here we would like to mention two of its desirable features. First, it is easily accessible, through algebraic and analytical methods, and can be verified much more readily in certain specific cases comparing to the verification of (A_q) (see Proposition 3.7 and Example 3.8). Second, this condition is not only sufficient for (DI_q) to hold, for all $q \in (0, p)$, but is also necessary provided M is bilinear and σ_X is supported by finitely many points (Remark 3.9). We suspect that this later fact is true in general. In addition to the contents of Section 3 noted above, Section 3 also contains a corollary and another remark and statements of three propositions. These propositions are needed for the proof of Theorem 3.1; their proofs are relegated to an appendix which constitutes the contents of Section 4.

The main part of the proof (i.e., the proof of the right-hand side of (DI_q)) given in [1] is very simple indeed both for the ideas used and the details involved. The main ideas used in our proof of Theorem 3.1 are also simple; however, in order to carry out the details of the proof, we have to rely upon several more advanced results and the concepts from probability theory and functional analysis. This makes our proof non-elementary and, in fact, quite lengthy. Our proof, on the other hand, has the advantage that the methods used in it have the potential of being adopted for studying decoupling inequalities for certain other infinite-dimensional random vectors (e.g., non-symmetric stable, semistable). Furthermore, our methods of proof (and our condition) shed some light in clarifying the relation between de Acosta's somewhat "abstract" formulation of decoupling inequalities and the works [3, 4, 7] which discuss decoupling inequalities for more "concrete" multilinear functions in random variables.

2. Two Examples. Before discussing the examples, we introduce some notations and conventions which will remain fixed throughout this note. All Banach spaces considered here are assumed to be defined over the field R of real numbers; all measures on a metric space are assumed to be defined on its Borel σ-algebra; for a measure μ on a separable metric space, $\operatorname{supp}(\mu)$ will denote the support of μ; the measurability of a function from one metric space to another metric space would always mean relative to their Borel σ-algebras. Throughout, N and $\theta(p) \equiv \theta = (\theta_1, \theta_2, \ldots)$ denote, respectively, the set of positive integers and a sequence of independent symmetric standard p-stable random variables for a fixed $p \in (0, 2)$.

Now we are ready to present our first example which, as we noted in the Introduction, asserts that (A_q) is not sufficient to ensure the right-hand side inequality in (DI_q) for any $q \in (0, p/2)$.

Example 2.1. Fix any $p \in (0, 2)$; and choose real numbers $a_j \in (0, 1)$ satisfying

$$\sum_{j=1}^{\infty} a_j^p (1 + \log 1/a_j) < \infty \quad \text{and} \quad \sum_{j=1}^{\infty} a_j^{p/2} = \infty. \tag{2.1}$$

Let $B = l_2$, $V = R$, $d = 2$ and $X = \sum_{j=1}^{\infty} j^{-2/p} \theta_j e_j$, where $\{e_j\}$ is the standard basis of l_2; finally, for every $n \in N$, let M_n: $l_2 \times l_2 \mapsto R$ be the bilinear form defined by

$$M_n(x, y) = \sum_{j=1}^{n} a_j j^{4/p} \langle x, e_j \rangle \langle y, e_j \rangle.$$

Now, if $q \in (0, p/2)$, then

$$E\,|\tilde{M}_n(X)|^q = E\,\Big|\sum_{j=1}^{n} a_j \theta_j^2\Big|^q \leqslant \sum_{j=1}^{n} a_j^q E\,|\theta_j|^{2q} < \infty;$$

so (A_q) is satisfied for every $q \in (0, p/2)$. Let now $q \in (0, p/2)$ be fixed; then if the right-hand side of (DI_q) were valid, we would have

$$E\,|\tilde{M}_n(X)|^q = E\Big(\sum_{j=1}^{n} a_j \theta_j^2\Big)^q \leqslant CE\,|M_n(X, X')| = CE\,\Big|\sum_{j=1}^{n} a_j \theta_j \theta_j'\Big|^q, \tag{2.2}$$

where $C = C(p, q, 2)$ is independent of n, $\theta' = (\theta_j')$ is an independent copy of θ and

$$X' = \sum_{j=1}^{\infty} j^{-2/p} \theta'_j e_j.$$

Since, for every n,

$$E|\tilde{M}_n(X)|^q = E\Big(\sum_{j=1}^{n} (\sqrt{a_j}\theta_j)^2\Big)^{2q/2} = E\,\|(\sqrt{a_1}\,\theta_1, \ldots, \sqrt{a_n}\,\theta_n, 0, 0, \ldots)\|_{l_2}^{2q},$$

using the monotone convergence theorem, (2.2) and a property of stable random variables, we have

$$\begin{aligned}
(2.3)\qquad E\,\|(\sqrt{a_1}\,\theta_1, \ldots, \sqrt{a_n}\,\theta_n, \ldots)\|_{l_2}^{2q} & \\
&= \sup_n E\,\|(\sqrt{a_1}\,\theta_1, \ldots, \sqrt{a_n}\,\theta_n, 0, 0, \ldots)\|_{l_2}^{2q} \\
&\leqslant C \sup_n E\Big|\sum_{j=1}^{n} a_j \theta_j \theta'_j\Big|^q \\
&= C \sup_n E_\theta E_{\theta'} \Big|\sum_{j=1}^{n} a_j \theta_j \theta'_j\Big|^q \\
&= CE|\theta_1|^q \sup_n E\Big(\sum_{j=1}^{n} |a_j \theta_j|^p\Big)^{q/p} \\
&= CE|\theta_1|^q E\,\|(a_1\theta_1, \ldots, a_n\theta_n, \ldots)\|_{l_p}^q.
\end{aligned}$$

Now, since, by the first condition in (2.1) and Proposition 26.3.3 of [11], $\sum_{j=1}^{\infty} |a_j\theta_j|^p < \infty$ a.s., the right-hand side and hence the left-hand side of (2.3) is finite. Thus $\sum_{j=1}^{\infty} |\sqrt{a_j}\theta_j|^2 < \infty$ a.s., therefore, by the quoted propositions of [11], $\sum_{j=1}^{\infty} a_j^{p/2} < \infty$; but this contradicts (2.1).

As pointed out in the Introduction and implied by the above example, there is an oversight in the proof given in [1] for ensuring the right-hand side of (DI_q) under (A_q), for every fixed $q \in (0, p/2)$. This oversight is pointed out via the following simple example.

Example 2.2. Towards the end of the proof of the main theorem given in [1], it is incorrectly assumed that if, for a fixed $q \in (0, p/2)$, (A_q) is satisfied, then one can find an $r \in (p/2, p)$ such that (A_r) is satisfied (i.e., $E\,\|\tilde{M}(X)\|_V^r < \infty$). But this is false: Let $B = R^2$, $V = R$, $d = 2$, $M(x, y) = \langle Ax, y\rangle$, where

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad X = \begin{pmatrix} \theta_1 \\ \theta_1 \end{pmatrix}$$

(recall that θ_1 is a standard symmetric p-stable random variable); then $E|\tilde{M}(X)|^r = E|\theta_1|^{2r} = \infty$ for every $r \in (p/2, p)$ and yet $E|\tilde{M}(X)|^q < \infty$ for all $q \in (0, p/2)$.

3. The Decoupling Inequalities. Example 2.1 raises the obvious question: Under what natural condition(s) on the p-stable random vector X and the multilinear function M, can one ensure decoupling inequalities?

A careful look at de Acosta's paper [1] and the works [3, 4, 7] (which deal with decoupling inequalities for more concrete multilinear functions in independent random variables) reveals: (i) All the papers [3, 4, 7] require that the function, defining the concrete multilinear function, assume value zero on the "diagonals"; and, (ii) this condition is essential for ensuring decoupling inequalities, for all $q \in (0, p)$; and yet an analog of which is lacking in de Acosta's formulation. Note that it is precisely the violation of this crucial condition in Example 2.1 which, in most part, forces the right side of (DI_q) to fail. These observations and Example 2.2 suggest that in order for ensuring (DI_q), for all $q \in (0, p)$, one must find a condition (in terms of M and distributional properties of X) which is a suitable substitute of the above noted condition for the abstract multilinear functions and which, at the same time, does not allow a phenomenon like in Example 2.2 to occur! A condition which meets these two requirements is the following:

Let X and M be as in the second paragraph of Introduction; the condition we require of M (relative to X) is that, *for every* x_j, $j = 1, \ldots, d$, *belonging to* $\operatorname{supp}(\sigma_X)$,

$$(\mathrm{DC}) \qquad M(x_1, \ldots, x_d) = 0, \quad \textit{whenever } x_i = x_j \textit{ for some } i \neq j.$$

In order to gain some insight about (DC), let $B = R^m$, $d \in N$, $m \geqslant d$, $V = R$, and $M(x_1, \ldots, x_d) = \sum f(i_1, \ldots, i_d) x_{1i_1} \ldots x_{di_d}$, where $x_j = (x_{j1}, \ldots, x_{jm})$, $j = 1, \ldots, d$, and $f(i_1, \ldots, i_d) \in R$; finally let $X = (\xi_1, \ldots, \xi_m)$, where ξ_j's are symmetric independent p-stable random variables. Then, recalling that σ_X is concentrated on $\{\pm e_j\colon j = 1, \ldots, m\}$, where e_j has 1 at the j^{th} entry and 0 elsewhere, one verifies easily that M satisfies (DC) if and only if the function f assumes value zero at the diagonals; i.e., $f(i_1, \ldots, i_d) = 0$, whenever two or more i_r's are identical. Thus, in this case (DC) coincides with the condition mentioned above and used in the papers [3, 4, 7].

Now we record a few more notations and definitions which we will need in the sequel. Let $V, B_1, \ldots, B_d$ be separable Banach spaces and f be a map from $B_1 \times \ldots \times B_d$ into V; then we say that f is *separately measurable* (resp. *separately continuous*) if f is measurable (resp. continuous) in each variable when the other variables are held fixed. Let $d, k_n \in N$; then we shall use the notation $J_{k_n,d}$ for the set $\{1, \ldots, k_n\}^d$; and the notations $D_{k_n,d}$ and $L_{k_n,d}$ for the d-tuples in $J_{k_n,d}$ with distinct coordinates and for the set $J_{k_n,d} \setminus D_{k_n,d}$, respectively. Whenever d is known from the context, we shall delete the letter d from these notations. Now we are ready to state the main result of this note, complementing the earlier quoted result of de Acosta.

THEOREM 3.1. *Let $p \in (0, 2)$, $d \in N$, and let B and V be any separable Banach spaces; let X be a B-valued symmetric p-stable random vector. Then, for every symmetric separately measurable multilinear function $M: B^d \mapsto V$ satisfying* (DC), *decoupling inequalities* (DI_q) *hold for all* $q \in (0, p)$.

(*As a consequence of this and Corollary* 3.5 *that follows*, $E \|\tilde{M}(X)\|_V^q < \infty$ *for all* $q \in (0, p)$).

For the proof of this theorem, we shall need the following three results. The first of these (Proposition 3.2) is a restatement of Theorem 6.6 of [8] and is included here for ready reference and because we make extensive use of the notation introduced in its statement for our proof of Theorem 3.1. The second result (Proposition 3.3) seems well known, but since we are unable to find a reference for it, we have included it here along with a short proof. The third result (Proposition 3.4) seems new and interesting. In the following, we first state these propositions, and also a corollary; then, using these, we present the proof of Theorem 3.1. The proofs of these propositions will be given, as noted earlier, in the appendix.

PROPOSITION 3.2. *Let μ be a symmetric p-stable prob. measure on a separable Banach space B with spectral measure σ on ∂S, the boundary of the unit sphere of B. Then, for every $n \in N$, one can construct a partition* $\{A_1^{(n)}, \ldots, A_{k_n}^{(n)}, A_{k_n+1}^{(n)}\}$ *of* $\mathrm{supp}(\sigma)$, *such that*

$$(3.1)\qquad \sigma(A_j^{(n)}) > 0,\ \text{diameter}(A_j^{(n)}) < n^{-1}, j = 1, \ldots, k_n;\ \sigma(A_{k_n+1}^{(n)}) < n^{-1},$$

and the B-valued random vectors

$$Y_n = \sum_{j=1}^{k_n} [\sigma(A_j^{(n)})]^{-1} \Big(\int_{A_j^{(n)}} x\sigma(dx) \Big) \Lambda(A_j^{(n)})$$

converge in prob. and in $L_B^q \equiv L^q(\Omega, F, P; B)$, *for every* $q \in (0, p)$, *to a random vector Y with* $\mathscr{L}(Y) = \mu$, *where Λ is any (real) symmetric p-stable random measure on ∂S with control measure σ.*

PROPOSITION 3.3. *Let $q \in (0, \infty)$ and $d \in N$; and let $r^{(1)}, \ldots, r^{(d)}$ be independent copies of a sequence of Rademacher functions. Then there exist positive constants $c_0 \equiv c_0(q)$ and $C_0 \equiv C_0(q)$ (depending only on q) such that, for every $n \in N$ and for every separable Hilbert space H, the following inequalities hold:*

$$(3.2)\qquad c_0^d \Big(\sum_{i \in J_{n,d}} \|a_{i_1,\ldots,i_d}\|_H^2 \Big)^{q/2} \leqslant E \Big\| \sum_{i \in J_{n,d}} a_{i_1,\ldots,i_d} r_{i_1}^{(1)} \ldots r_{i_d}^{(d)} \Big\|_H^q \leqslant C_0^d \Big(\sum_{i \in J_{n,d}} \|a_{i_1,\ldots,i_d}\|_H^2 \Big)^{q/2},$$

and

$$(3.3)\qquad \|f\|_{\min}^q \left(\frac{c_0}{C_0}\right)^d E \Big\| \sum_{i \in J_{n,d}} a_{i_1,\ldots,i_d} r_{i_1}^{(1)} \ldots r_{i_d}^{(d)} \Big\|_H^q \leqslant E \Big\| \sum_{i \in J_{n,d}} a_{i_1,\ldots,i_d} f_{i_1,\ldots,i_d} r_{i_1}^{(1)} \ldots r_{i_d}^{(d)} \Big\|_H^q \leqslant \|f\|_{\max}^q \left(\frac{C_0}{c_0}\right)^d E \Big\| \sum_{i \in J_{n,d}} a_{i_1,\ldots,i_d} r_{i_1}^{(1)} \ldots r_{i_d}^{(d)} \Big\|_H^q,$$

where $a_{i_1,\ldots,i_d}$ and $f_{i_1,\ldots,i_d}$ are arbitrary elements of H and R, respectively, and

$$\|f\|_{\max} = \max_{i \in J_{n,d}} |f_{i_1,\ldots,i_d}| \quad \text{and} \quad \|f\|_{\min} = \min_{i \in J_{n,d}} |f_{i_1,\ldots,i_d}|.$$

(Inequalities (3.3) *can be referred to as a contraction principle.)*

PROPOSITION 3.4. *Let $B_1, \ldots, B_d$ and V be separable Banach spaces and let $M: B_1 \times \ldots \times B_d \mapsto V$ be a separately measurable multilinear function; then M is (jointly) continuous and hence*

$$\|M(x_1, \ldots, x_d)\|_V \leqslant \text{Const.}(M) \|x_1\|_{B_1} \ldots \|x_d\|_{B_d}, \tag{3.4}$$

for every $(x_1, \ldots, x_d) \in B_1 \times \ldots \times B_d$.

This proposition immediately yields the following result:

COROLLARY 3.5. *Let B and V be separable Banach spaces, and let $X_1, \ldots, X_d$ be B-valued independent random vectors satisfying $E\|X_j\|^q < \infty$, $j = 1, \ldots, d$, for some $q > 0$. Then, for every separately measurable multilinear function M: $B^d \mapsto V$, one has $E\|M(X_1, \ldots, X_d)\|^q < \infty$.* (Compare this with Lemma (a) of [1].)

Now we are ready to present our proof of Theorem 3.1.

Proof of Theorem 3.1. As we noted earlier in this note, the difficult part of the proof is to establish the right-hand side of (DI_q) for all $q \in (0, p)$. Once this is established, the left-hand side inequality follows from de Acosta's proof, because the right-hand side of (DI_q) and Corollary 3.5 imply (A_q) for all $q \in (0, p)$.

Now we shall establish the validity of the right-hand side of (DI_q) for all $q \in (0, p)$. This proof is divided into two parts; in the first part we prove the result under the additional assumption that V is finite-dimensional, and, in the second, we prove it for the general V. The proof of the first part itself is divided into four steps, for simplicity.

Part I. We assume, in addition, that V is finite-dimensional; and let $\mu = \mathscr{L}(X)$, and $\sigma = \sigma_X$. Then, using the notations introduced in the statement of Proposition 3.2, we have $\mu = \mathscr{L}(Y) (= \mathscr{L}(X))$, and

$$Y = P\text{-}\lim_n Y_n = P\text{-}\lim_n \sum_{j=1}^{k_n} [\sigma(A_j^{(n)})]^{-1} \Big(\int_{A_j^{(n)}} x\sigma(dx)\Big) \Lambda(A_j^{(n)}).$$

Hence, since M: $B^d \mapsto V$ is jointly continuous, by Proposition 3.4 we have

$$\tilde{M}(Y) = P\text{-}\lim_n \tilde{M}(Y_n) = P\text{-}\lim_n \sum_{i \in J_{k_n}} a^{(n)}_{i_1,\ldots,i_d} \Lambda(A_{i_1}^{(n)}) \ldots \Lambda(A_{i_d}^{(n)}), \tag{3.5}$$

where

$$a^{(n)}_{i_1,\ldots,i_d} = [\sigma(A_{i_1}^{(n)}) \ldots \sigma(A_{i_d}^{(n)})]^{-1} M\Big(\int_{A_{i_1}^{(n)}} x\sigma(dx), \ldots, \int_{A_{i_d}^{(n)}} x\sigma(dx)\Big).$$

The *first* step in this proof is to show that

$$\tilde{M}(Y) = P\text{-}\lim_n \sum_{i \in D_{k_n}} a^{(n)}_{i_1,\ldots,i_d} \Lambda(A_{i_1}^{(n)}) \ldots \Lambda(A_{i_d}^{(n)}). \tag{3.6}$$

Recalling that $L_{k_n} = J_{k_n} \setminus D_{k_n}$, (3.6) is equivalent to showing that

$$P\text{-}\lim_n \sum_{\boldsymbol{i} \in L_{k_n}} a^{(n)}_{i_1, \ldots, i_d} \Lambda(A^{(n)}_{i_1}) \ldots \Lambda(A^{(n)}_{i_d}) = 0;$$

which, since V is finite-dimensional, is in turn equivalent to showing that

$$P\text{-}\lim_n \sum_{\boldsymbol{i} \in L_{k_n}} y(a^{(n)}_{i_1, \ldots, i_d}) \Lambda(A^{(n)}_{i_1}) \ldots \Lambda(A^{(n)}_{i_d}) = 0 \tag{3.7}$$

for every fixed $y \in V^*$, the topological dual of V. We now fix a $y \in V^*$; and prove (3.7). We first note the identity

$$M\Big(\int_{A^{(n)}_{i_1}} x\sigma(dx), \ldots, \int_{A^{(n)}_{i_d}} x\sigma(dx)\Big) = \int_{A^{(n)}_{i_1}} \Big(\ldots\Big(\int_{A^{(n)}_{i_d}} M(x_{i_1}, \ldots, x_{i_d})\sigma(dx_{i_d})\Big)\ldots\Big)\sigma(dx_{i_1});$$

this can be proved using standard properties of Bochner's integrals. This identity along with (DC) yield

$$\begin{aligned} a^{(n)}_{i_1, \ldots, i_d} = {} & [\sigma(A^{(n)}_{i_1}) \ldots \sigma(A^{(n)}_{i_d})]^{-1} \times \\ & \times \int_{A^{(n)}_{i_1}} \Big(\ldots\Big(\int_{A^{(n)}_{i_d}} (M(x_{i_1}, \ldots, x_{i_d}) \\ & - M(x_{i_1}, \ldots, x_{i_r}, \ldots, x_{i_r}, \ldots, x_{i_d}))\sigma(dx_{i_d})\Big)\ldots\Big)\sigma(dx_{i_1}) \end{aligned}$$

for every $\boldsymbol{i} \in L_{k_n}$ with $i_r = i_s$, $r < s$. The last equation, along with the facts that diameter$(A^{(n)}_{i_j}) \leqslant n^{-1}$, $A^{(n)}_{i_j} \subseteq \operatorname{supp}(\sigma) \subseteq \partial S$ (see (3.1)) and (3.4), immediately yield $\|a^{(n)}_{i_1, \ldots, i_d}\|_V \leqslant \text{Const.}(M) n^{-1}$ for every $\boldsymbol{i} \in L_{k_n}$ and n; hence, for every n and $\boldsymbol{i} \in L_{k_n}$,

$$|y(a^{(n)}_{i_1, \ldots, i_d})| \leqslant \text{Const.}(M) n^{-1} \|y\|_{V^*}. \tag{3.8}$$

Now let g be a Borel measurable map from $[0, 1] \to \partial S$ such that $\text{Leb}(g^{-1}(A)) = \sigma(A)/\sigma(\partial S)$ for every Borel set A of ∂S, and let $\lambda'(\cdot) = \sigma(\partial S)\,\text{Leb}(\cdot)$, and Λ' be a symmetric p-stable random measure on $[0, 1]$ with control measure λ'. Then, since $\sigma(A) = \lambda'(g^{-1}(A))$, it follows that

$$\begin{aligned} \mathscr{L}\Big(\sum_{\boldsymbol{i} \in L_{k_n}} y(a^{(n)}_{i_1, \ldots, i_d}) \Lambda(A^{(n)}_{i_1}) \ldots \Lambda(A^{(n)}_{i_d})\Big) \\ = \mathscr{L}\Big(\sum_{\boldsymbol{i} \in L_{k_n}} y(a^{(n)}_{i_1, \ldots, i_d}) \Lambda'(C^{(n)}_{i_1}) \ldots \Lambda'(C^{(n)}_{i_d})\Big) \quad \text{for every } n, \end{aligned} \tag{3.9}$$

where $C^{(n)}_{i_j} = g^{-1}(A^{(n)}_{i_j})$, $j = 1, \ldots, d$. Let $\Lambda'^{(d)}$ denote the product random measure of d identical random measures Λ' (see Theorem 4.1 of [10]). Then, by defining

$$f_n(t_1, \ldots, t_d) = \begin{cases} y(a^{(n)}_{i_1, \ldots, i_d}) & \text{if } (t_1, \ldots, t_d) \in C^{(n)}_{i_1} \times \ldots \times C^{(n)}_{i_d},\ \boldsymbol{i} \in L_{k_n}, \\ 0 & \text{otherwise,} \end{cases}$$

we can write

$$\begin{aligned} \sum_{\boldsymbol{i} \in L_{k_n}} y(a^{(n)}_{i_1, \ldots, i_d}) \Lambda'(C^{(n)}_{i_1}) \ldots \Lambda'(C^{(n)}_{i_d}) \\ = \int \ldots \int_{[0,1]^d} f_n(t_1, \ldots, t_n) \Lambda'^{(d)}(dt_1, \ldots, dt_d) \quad \text{for every } n \in N. \end{aligned} \tag{3.10}$$

Now, since by (3.8), $f_n(t_1, \ldots, t_d) \to 0$ uniformly on $[0, 1]^d$ as $n \to \infty$, it follows, by the argument similar to that for showing the countable additivity of the product random measure in Theorem 4.1 of [10], that the right-hand side of (3.10) converges to zero in prob. Consequently, by (3.9), the validity of (3.7) (and hence of (3.6)) is established.

The *second* step is to show that

$$(3.11)\qquad E\| \sum_{i \in L_{k_n}} a^{(n)}_{i_1,\ldots,i_d} \Lambda_1(A^{(n)}_{i_1}) \ldots \Lambda_d(A^{(n)}_{i_d})\|^q_V \to 0 \quad \text{as } n \to \infty,$$

for every $q \in (0, p)$, where $\Lambda_1, \ldots, \Lambda_d$ are independent copies of Λ. Let r be a sequence of independent Rademacher random variables, and let $r^{(1)}, \ldots, r^{(d)}$, and $\theta^{(1)}, \ldots, \theta^{(d)}$ be independent random vectors with $\mathscr{L}(r^{(i)}) = \mathscr{L}(r)$ and $\mathscr{L}(\theta^{(i)}) = \mathscr{L}(\theta)$, $i = 1, \ldots, d$. Then, for every fixed $y \in V^*$ and $q \in (0, p)$,

$$\begin{aligned} E| \sum_{i \in L_{k_n}} & y(a^{(n)}_{i_1,\ldots,i_d}) \Lambda_1(A^{(n)}_{i_1}) \ldots \Lambda_d(A^{(n)}_{i_d})|^q \\ &= E| \sum_{i \in L_{k_n}} y(a^{(n)}_{i_1,\ldots,i_d}) \sigma(A^{(n)}_{i_1})^{1/p} \theta^{(1)}_{i_1} \ldots \sigma(A^{(n)}_{i_d})^{1/p} \theta^{(d)}_{i_d}|^q \\ &= EE_0| \sum_{i \in J_{k_n}} b^{(n)}_{i_1,\ldots,i_d} \sigma(A^{(n)}_{i_1})^{1/p} \theta^{(1)}_{i_1} r^{(1)}_{i_1} \ldots \sigma(A^{(n)}_{i_d})^{1/p} \theta^{(d)}_{i_d} r^{(d)}_{i_d}|^q, \end{aligned}$$

where E_0 denotes the expectation relative to Rademacher sequences, and $b^{(n)}_{i_1,\ldots,i_d} = y(a^{(n)}_{i_1,\ldots,i_d})$ if $i \in L_{k_n}$, and $b^{(n)}_{i_1,\ldots,i_d} = 0$ otherwise. Now, by (3.3) and (3.8), we see that the last expression

$$\begin{aligned} &\leqslant \left(\frac{C_0}{c_0}\right)^d \left(\frac{\text{Const.}(M)\|y\|_{V^*}}{n}\right)^q E| \sum_{i \in J_{k_n}} \Lambda_1(A^{(n)}_{i_1}) \ldots \Lambda_d(A^{(n)}_{i_d})|^q \\ &= \left(\frac{C_0}{c_0}\right)^d \left(\frac{\text{Const.}(M)\|y\|_{V^*}}{n}\right)^q E|\Lambda_1(A^{(n)}) \ldots \Lambda_d(A^{(n)})|^q \\ &\leqslant \left(\frac{C_0}{c_0}\right)^d \left(\frac{\text{Const.}(M)\|y\|_{V^*}}{n}\right)^q 2^d (E|\Lambda(\partial S)|^q)^d, \quad \text{where } A^{(n)} = \bigcup_{j=1}^{k_n} A^{(n)}_j. \end{aligned}$$

Clearly, the expression on the right-hand side of the last inequality converges to 0 as $n \to \infty$; this shows that

$$E| \sum_{i \in L_{k_n}} y(a^{(n)}_{i_1,\ldots,i_d}) \Lambda_1(A^{(n)}_{i_1}) \ldots \Lambda_d(A^{(n)}_{i_d})|^q \to 0 \quad \text{as } n \to \infty,$$

for every fixed $y \in V^*$; thus, since V is finite-dimensional, the proof of (3.11) is immediate.

Let

$$Y^{(n)}_j = \sum_{i=1}^{k_n} [\sigma(A^{(n)}_i)]^{-1} \Big(\int_{A^{(n)}_i} x\sigma(dx)\Big) \Lambda_j(A^{(n)}_i), \quad j = 1, \ldots, d,$$

where Λ_j's are as above. Then, by Proposition 3.2, $\{Y^{(n)}_j\}$ converges in L^q_B to a random vector, say Y_j, for each $j = 1, \ldots, d$.

The *third* step in this part of the proof is to show that, for every $q \in (0, p)$,

(3.12) $$E\,\|M(Y_1^{(n)}, \ldots, Y_d^{(n)}) - M(Y_1, \ldots, Y_d)\|_V^q \to 0 \quad \text{as } n \to \infty.$$

To see this, we note that

$$E\,\|M(Y_1^{(n)}, \ldots, Y_d^{(n)}) - M(Y_1, \ldots, Y_d)\|_V^q$$

$$= E\,\Big\|\sum_{i=1}^{d} \big(M(Y_1, \ldots, Y_{i-1}, Y_i^{(n)}, \ldots, Y_d^{(n)}) - M(Y_1, \ldots, Y_i, Y_{i+1}^{(n)}, \ldots, Y_d^{(n)})\big)\Big\|_V^q$$

$$\leqslant \text{Const.}(q, d) \sum_{i=1}^{d} E\,\|M(Y_1, \ldots, Y_{i-1}, Y_i^{(n)} - Y_i, Y_{i+1}^{(n)}, \ldots, Y_d^{(n)})\|_V^q$$

$$\leqslant \text{Const.}(q, d, M) \sum_{i=1}^{d} \Big(\prod_{j=1}^{i-1} E\,\|Y_j\|_B^q\Big)\big(E\,\|Y_i^{(n)} - Y_i\|_B^q\big)\Big(\prod_{j=i+1}^{d} E\,\|Y_j^{(n)}\|_B^q\Big)$$

(by independence). But since $E\,\|Y_j^{(n)}\|_B^q$ and $E\,\|Y_j\|_B^q$, $j = 1, \ldots, d$ and $n = 1, 2, \ldots$, are all uniformly bounded and $E\,\|Y_i^{(n)} - Y_i\|^q \to 0$ for all $i = 1, \ldots, d$, as $n \to \infty$, the last expression in the above inequalities converges to zero as $n \to \infty$, which proves (3.12).

In this *fourth* and final step, we complete the proof of the right-hand side of (DI_q) for all $q \in (0, p)$, when V is finite-dimensional. From (3.6) and the fact that $\mathscr{L}(X) = \mathscr{L}(Y)$, we have, for any fixed $q \in (0, p)$,

$$E\,\|\tilde{M}(X)\|_V^q \leqslant \liminf_{n\to\infty} E\,\Big\|\sum_{\mathbf{i}\in D_{k_n}} a_{i_1,\ldots,i_d}^{(n)} \Lambda(A_{i_1}^{(n)}) \ldots \Lambda(A_{i_d}^{(n)})\Big\|_V^q.$$

The right-hand side of this inequality, by Theorem 3.1 of Krakowiak and Szulga [3] or by de Acosta's theorem quoted in Section 1 (note that $E\,\|\sum_{\mathbf{i}\in D_{k_n}} a_{i_1,\ldots,i_d}^{(n)} \Lambda(A_{i_1}^{(n)}) \ldots \Lambda(A_{i_d}^{(n)})\|_V^q < \infty$ for all $q \in (0, p)$)

$$\leqslant C(p, q, d) \liminf_{n\to\infty} E\,\Big\|\sum_{\mathbf{i}\in D_{k_n}} a_{i_1,\ldots,i_d}^{(n)} \Lambda_1(A_{i_1}^{(n)}) \ldots \Lambda_d(A_{i_d}^{(n)})\Big\|_V^q$$

$$= C(p, q, d) \liminf_{n\to\infty} E\,\Big\|M(Y_1^{(n)}, \ldots, Y_d^{(n)}) - \sum_{\mathbf{i}\in L_{k_n}} a_{i_1,\ldots,i_d}^{(n)} \Lambda_1(A_{i_1}^{(n)}) \ldots \Lambda_d(A_{i_d}^{(n)})\Big\|_V^q,$$

where $Y_j^{(n)}$'s and Λ_j's are as defined above. But since, by (3.11) and (3.12), the last expression above is equal to $C(p, q, d)\,E\,\|M(Y_1, \ldots, Y_d)\|_V^q$ and since $\mathscr{L}(X_i) = \mathscr{L}(Y_i)$, $i = 1, \ldots, d$, the proof of the right-hand inequality of (DI_q), for $q \in (0, p)$, in the case where V is finite-dimensional, is complete.

Part II. For the general V; first we note that, by the Banach–Mazur theorem, V can be assumed as a subspace of $C[0, 1]$. Let $\{b_j\}$ be the standard Schauder basis of $C[0, 1]$, and $\{\beta_j\}$ be the corresponding coordinate functionals. Then, as is well known

(3.13) $$\|P_k(x)\|_\infty \nearrow \|x\|_\infty \quad \text{for every } x \in C[0, 1],$$

where $P_k(x) = \sum_{j=1}^{k} b_j \beta_j(x)$, and $\|\cdot\|_\infty$ denotes the supremum norm. Let $P_k(V) = V_k$ and $M_k = P_k \circ M$, $k = 1, 2, \ldots$ Then V_k is finite-dimensional and

M_k satisfies (DC) for every $k \in N$. Thus, from what we have proved above,

$$E\,\|\tilde{M}_k(X)\|_\infty^q \leqslant C(p, q, d)\, E\,\|M_k(X_1, \ldots, X_d)\|_\infty^q \quad \text{for all } k.$$

The proof of the right inequality of (DI_q), for every $q \in (0, p)$, now follows by the monotone convergence theorem and (3.13). Finally, note that in view of Corollary 3.5, we also get, as a consequence of this inequality, that $E\,\|\tilde{M}(X)\|_\infty^q < \infty$ for all $q \in (0, p)$.

Remark 3.6. A careful look at de Acosta's proof in [1] reveals the following important fact: Assuming that (A_r) holds for all $r \in (0, p)$, the right-hand side of (DI_q) holds, for all $q \in (0, p)$, *without* the assumption of symmetry on M. In view of this and in view of the fact that we needed symmetry of M in our proof of Theorem 3.1 *only* at one place, namely where we used de Acosta's Theorem, it follows that under (DC) the right-hand side of (DI_q) is valid without the symmetry assumption on M! We may also point out here that the left-hand side of (DI_q) is false without the symmetry assumption on M. In fact, take $B = R^2$, $V = R$, $d = 2$, $M(x, y) = \langle Ax, y \rangle$, where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad X = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}, \quad Y = \begin{pmatrix} \theta_1' \\ \theta_2' \end{pmatrix};$$

recall that θ_1 and θ_2 are independent symmetric p-stable random variables and Y is an independent copy of X. Then $\tilde{M}(X) \equiv 0$ and thus $E\,|\tilde{M}(X)|^q = 0$ for any $q \in (0, p)$ and yet $E\,|M(x, y)|^q = E\,|\theta_1 \theta_2' - \theta_2 \theta_1'|^q > 0$.

We now show that, in certain specific cases, verification of (DC) is much easier comparing to the verification of (A_q). This is done in Example 3.8, where we use a condition equivalent to (DC). This new condition is given *not* in terms of the support of the spectral measure of the p-stable random vector X, but rather in terms of the support of any σ-finite measure ν on the Banach space appearing in the representation of the characteristic function of $\mathscr{L}(X)$. Often it is easier to have an explicit description of the support of ν (comparing to the description of the support of σ_X); this observation makes this new condition somewhat more useful. We begin with a proposition which provides the equivalence of the two conditions:

PROPOSITION 3.7. *Let X, B, V and d be as in Theorem* 3.1 *and let $M\colon B^d \to V$ be a (not necessarily symmetric) separately measurable multilinear function. Let ν be a σ-finite measure on B with $\nu(\{0\}) = 0$ and such that*

$$\hat{\mathscr{L}}(X)(y) = \exp\{-\int_B |y(x)|^p \,\nu(dx)\} \quad \textit{for all } y \in B^*.$$

Then (DC)*-condition is equivalent to the following:*

(DC′) *For every $x_1, \ldots, x_d$, belonging to* $\operatorname{supp}(\nu)$, $M(x_1, \ldots, x_d) = 0$, *whenever $x_i = x_j$ for some $i \neq j$.*

Proof. We can write

$$\mathscr{L}(X)(y) = \exp\{-\int_B |y(x/\|x\|)|^p \|x\|^p \nu(dx)\} = \exp\{-\int_{\partial S} |y(x)|^p \sigma_0(dx)\},$$

where

$$\sigma_0(A) = \int_{\{x \neq 0: x/\|x\| \in A\}} \|x\|^p \nu(dx) < \infty$$

for every Borel set $A \subset \partial S$. Thus, $\sigma_X(A) = \frac{1}{2}\sigma_0(A) + \frac{1}{2}\sigma_0(-A)$ and

$$\operatorname{supp}(\sigma_X) = \{x\colon x \text{ or } -x \in \operatorname{supp}(\sigma_0)\}. \tag{3.14}$$

Now assume that (DC) does not hold. We shall show that (DC′) does not hold either. We have, by our assumption, $M(x_1, \ldots, x_d) \neq 0$ for some $x_1, \ldots, x_d \in \operatorname{supp}(\sigma_X)$ with $x_i = x_j$ for some $i \neq j$. So, by (3.14), there exist $\varepsilon_1, \ldots, \varepsilon_d = \pm 1$, with $\varepsilon_i = \varepsilon_j$, such that $z_k = \varepsilon_k x_k \in \operatorname{supp}(\sigma_0)$, $1 \leqslant k \leqslant d$. Hence, $M(z_1, \ldots, z_d) = \varepsilon_1 \ldots \varepsilon_d M(x_1, \ldots, x_d) \neq 0$, and by the continuity of M (see Proposition 3.4) there exists an $\varepsilon > 0$ such that

$$M(u_1, \ldots, u_d) \neq 0 \quad \text{for every } (u_1, \ldots, u_d) \in U_1 \times \ldots \times U_d, \tag{3.15}$$

where $U_k = \{u \in S\colon \|u - z_k\| < \varepsilon\}$. Set $G_k = \{x \neq 0\colon x/\|x\| \in U_k\}$, $1 \leqslant k \leqslant d$. Since $z_k \in \operatorname{supp}(\sigma_0)$,

$$0 < \sigma_0(U_k) = \int_{G_k} \|x\|^p \nu(dx), \quad 1 \leqslant k \leqslant d,$$

and since G_k are open, $G_k \cap \operatorname{supp}(\nu) \neq \emptyset$; and hence we can choose $v_k \in G_k \cap \operatorname{supp}(\nu)$ such that $v_i = v_j$ (note that $G_i = G_j$). Then, by (3.15),

$$M(v_1, \ldots, v_d) = \|v_1\| \ldots \|v_d\| M(v_1/\|v_1\|, \ldots, v_d/\|v_d\|) \neq 0,$$

which contradicts (DC′).

Suppose now that (DC′) fails, so that $M(x_1, \ldots, x_d) \neq 0$ for some $x_1, \ldots, x_d \in \operatorname{supp}(\nu)$ with $x_i = x_j$ for some $i \neq j$. Note that $\varepsilon_0 \equiv \min\{\|x_1\|, \ldots, \|x_d\|\} > 0$. By the continuity of M, there exists a positive $\varepsilon < \varepsilon_0/2$ such that

$$M(v_1, \ldots, v_d) \neq 0 \quad \text{for every } (v_1, \ldots, v_d) \in H_1 \times \ldots \times H_d, \tag{3.16}$$

where $H_k = \{v\colon \|v - x_k\| < \varepsilon\}$. Put

$$W_k = \{v/\|v\|\colon v \in H_k\}, \quad 1 \leqslant k \leqslant d \tag{3.17}$$

(note $0 \notin H_k$). Since $x_k \in \operatorname{supp}(\nu)$, $x_k \neq 0$, we have

$$\sigma_0(W_k) = \int_{\{x \neq 0: x/\|x\| \in W_k\}} \|x\|^p \nu(dx) \geqslant \int_{H_k} \|x\|^p \nu(dx) > 0.$$

Hence $W_k \cap \operatorname{supp}(\sigma_0) \neq \emptyset$, and so we can choose $u_k \in W_k \cap \operatorname{supp}(\sigma_0)$ with $u_i = u_j$ (note $W_i = W_j$). In view of (3.17), $u_k = v_k/\|v_k\|$ for some $v_k \in H_k$ and, by (3.16),

$$M(u_1, \ldots, u_d) = (\|v_1\| \ldots \|v_d\|)^{-1} M(v_1, \ldots, v_d) \neq 0.$$

Since $\operatorname{supp}(\sigma_0) \subseteq \operatorname{supp}(\sigma_X)$, this contradicts (DC), and the proof is completed.

Example 3.8. Let $M: (L^2[0, T])^d \to R$ be given by

$$M(f_1, \ldots, f_d) = \int_0^T \ldots \int_0^T K(t_1, \ldots, t_d) f(t_1) \ldots f(t_d)\, dt_1 \ldots dt_d,$$

$f_1, \ldots, f_d \in L^2[0, T]$, where $K \in L^2([0, T]^d)$. Let $X(t)$, $t \geqslant 0$, be a symmetric p-stable Lévy process such that $E\exp\{iuX(t)\} = \exp\{-t|u|^p\}$, $u \in R$, $t > 0$. Let X be the random vector induced by $\{X(t)\colon 0 \leqslant t \leqslant T\}$ in $L^2[0, T]$. Since

$$X(t) = \int_0^T I_{[0,t]}(s)\, dX(s),$$

by Lemma 7.1 in [9] we get, for every $f \in L^2[0, T]$,

$$E\exp\{i\langle f, x\rangle\} = E\exp\{i\int_0^T f(t)X(t)\,dt\} = E\exp\{i\int_0^T [\int_0^T f(t) I_{[s,T]}(t)\,dt]\,dX(s)\}$$

$$= \exp\{-\int_0^T |\langle f, I_{[s,T]}\rangle|^p\, ds\} = \exp\{-\int_{L^2[0,T]} |\langle f, g\rangle|^p\, \nu(dg)\},$$

where $\nu(A) = \operatorname{Leb}(\{s \in [0, T]\colon I_{[s,T]} \in A\})$ for every Borel set A of $L^2[0, T]$. Since it is easy to see that $\operatorname{supp}(\nu) = \{I_{[s,T]}\}_{s \in [0,T]}$, it follows, by the above proposition, that (DC) is equivalent to the following: for every $s_1, \ldots, s_d \in [0, T]$

$$M(I_{[s_1,T]}, \ldots, I_{[s_d,T]}) = \int_{s_1}^T \ldots \int_{s_d}^T K(t_1, \ldots, t_d)\, dt_1 \ldots dt_d = 0, \tag{3.18}$$

whenever $s_i = s_j$ for some $i \neq j$.

Clearly, every antisymmetric function K satisfies (3.18), but one can also find many "non-antisymmetric" functions which satisfy (3.18) as well. Thus the verification of (DC′) and hence also of (DC) is easy; on the other hand, it is not clear directly as to why (3.18) implies $E|\tilde{M}(X)|^q < \infty$. Of course, we know, via Theorem 3.1, that all the moments $E|\tilde{M}(X)|^q$ are in fact finite (Remark 3.6 is pertinent here).

Remark 3.9. (a) We noted in the Introduction that our condition (DC) is *not only* sufficient but also necessary for (DI$_q$) to hold, for all $q \in (0, p)$, provided M and σ_X satisfy additional hypotheses. In fact, one can say more: Let B and V be separable Banach spaces and M be a separately measurable symmetric bilinear function: $B^2 \mapsto V$. Let X be a symmetric B-valued p-stable random vector with σ_X supported on a finite set. Then the following statements are equivalent:

(I) (DI$_q$) holds for all $q \in (0, p)$;

(II) (A$_q$) is satisfied for all $q \in (0, p)$;

(III) (A_{q_0}) is satisfied for some $q_0 \in (p/2, p)$;
(IV) (DC) is satisfied.

The proofs of these equivalences follow from Theorem 3.1 and the following observation which shows (III)$\Rightarrow$(IV). Namely, let $\mathrm{supp}(\sigma_X) = \{x_1, \ldots, x_n\}$. Then $\mathscr{L}(X) = \mathscr{L}(\sum_{j=1}^{n} m_j^{1/p} \theta_j x_j)$, where $0 < m_j = \sigma_X\{x_j\}$. Thus $\tilde{M}(X) = \sum_{i,j=1}^{n} m_i^{1/p} m_j^{1/p} \theta_i \theta_j M(x_i, x_j)$. Now, since

$$E \left\| \sum_{i \neq j} m_i^{1/p} m_j^{1/p} \theta_i \theta_j M(x_i, x_j) \right\|_V^q < \infty \quad \text{for every } q \in (0, p),$$

(III) is equivalent to $E \left\| \sum_{i=1}^{n} m_i^{2/p} \theta_i^2 M(x_i, x_i) \right\|_V^{q_0} < \infty$ for some fixed $q_0 \in (p/2, p)$. But, by independence, this implies that $E |\theta_i^2|^{q_0} \|M(x_i, x_i)\|_V^{q_0} < \infty$ for all $1 \leqslant i \leqslant n$. Thus, since $q_0 > p/2$, $M(x_i, x_i) = 0$ for all $1 \leqslant i \leqslant n$.

The answer to the question whether (I) implies (IV), in general, is unknown to us, but, as we noted in the Introduction, we suspect it to be affirmative.

(b) The proof of Theorem 3.1 can be considerably shortened and simplified if the Banach space B is assumed to be of stable type p. Indeed, in this case, one can choose $x_1^{(n)}, \ldots, x_{k_n}^{(n)} \in \mathrm{supp}(\sigma_X)$ such that the random vectors $Y_n \equiv \sum_{j=1}^{k_n} x_j^{(n)} \Lambda(A_j^{(n)})$ converge in L_B^q and $\mathscr{L}(\lim_n Y_n) = \mathscr{L}(X)$ [5] (see the proof of Theorem 3.1 for the explanation of the notation used here). Then (DC) implies that

$$\tilde{M}(Y_n) = \sum_{i \in D_{k_n}} M(x_{i_1}^{(n)}, \ldots, x_{i_d}^{(n)}) \Lambda(A_{i_1}^{(n)}) \ldots \Lambda(A_{i_d}^{(n)}),$$

which yields (3.6) immediately; further, (DC) also implies that

$$M(Y_1^{(n)}, \ldots, Y_d^{(n)}) = \sum_{i \in D_{k_n}} M(x_{i_1}, \ldots, x_{i_d}) \Lambda_1(A_{i_1}^{(n)}) \ldots \Lambda_d(A_{i_d}^{(n)}),$$

where

$$Y_r^{(n)} = \sum_{j=1}^{k_n} x_j^{(n)} \Lambda_r(A_j^{(n)}), \qquad r = 1, \ldots, d.$$

With these observations at hand, one completes the proof of Theorem 3.1 for general V quickly by making use of steps three and four given in Part I of the proof. The above observations make it possible to bypass the lengthy steps one and two of Part I of the proof. In the case of a general Banach space B (i.e., the one which is not of stable type p), these two steps are necessary, because, in this case, the elements corresponding to $x_j^{(n)}$ are

$$y_j^{(n)} \equiv [\sigma(A_j^{(n)})]^{-1} \int_{A_j^{(n)}} x \sigma(dx), \qquad j = 1, \ldots, k_n,$$

which are known to belong to the closed convex hull of $\mathrm{supp}(\sigma_X)$ and not necessarily to $\mathrm{supp}(\sigma_X)$. Consequently, it does not follow, as in the case above, that $M(y_{i_1}^{(n)}, \ldots, y_{i_d}^{(n)}) = 0$ for $i \in L_{k_n}$, and this forces one to use steps one and two which makes the proof considerably long.

4. Appendix. As mentioned before, we present the proof of Propositions 3.2, 3.3, and 3.4 in this appendix.

Proof of Proposition 3.2. Let

$$Z_n = Y_n + [\sigma(A_{k_n+1}^{(n)})]^{-1} \Big(\int_{A_{k_n+1}^{(n)}} x\sigma(dx) \Big) \Lambda(A_{k_n+1}^{(n)}).$$

Then, from Theorem 6.6 of [8], $\{Z_n\}$ converges in prob. and in L_B^q to a random vector Y with $\mathscr{L}(Y) = \mu$ for every $q \in (0, p)$. Thus, since

$$\Big\| [\sigma(A_{k_n+1}^{(n)})]^{-1} \Big(\int_{A_{k_n+1}^{(n)}} x\sigma(dx) \Big) \Lambda(A_{k_n+1}^{(n)}) \Big\|_B \leqslant |\Lambda(A_{k_n+1}^{(n)})|,$$

and $|\Lambda(A_{k_n+1}^{(n)})| \to 0$ in prob. (because $\sigma(A_{k_n+1}^{(n)}) \leqslant n^{-1} \to 0$) as $n \to \infty$, we have

$$Y = P\text{-}\lim_{n \to \infty} Y_n.$$

Further, since

$$E\|Y_n - Y\|_B^q \leqslant 2^q \{E|\Lambda(A_{k_n+1}^{(n)})|^q + E\|Z_n - Y\|_B^q\}$$

and $E\|Z_n - Y\|_B^q \to 0$, $E|\Lambda(A_{k_n+1}^{(n)})|^q \to 0$, as $n \to \infty$, we also infer that $\{Y_n\}$ converges to Y in L_B^q for every $q \in (0, p)$.

Proof of Proposition 3.3. For simplicity of the notation, we shall abbreviate (3.2) by

$$(4.1) \qquad E\Big\| \sum_{\boldsymbol{i} \in J_{n,d}} a_{i_1,\ldots,i_d} r_{i_1}^{(1)} \ldots r_{i_d}^{(d)} \Big\|_H^q \overset{C_0^d}{\underset{c_0^d}{\simeq}} \Big(\sum_{\boldsymbol{i} \in J_{n,d}} \|a_{i_1,\ldots,i_d}\|_H^2 \Big)^{q/2}.$$

We shall prove the result by induction on d. If $d = 1$, then (4.1) reduces to

$$E\Big\| \sum_{\boldsymbol{i} \in J_{n,1}} a_i r_i^{(1)} \Big\|_H^q \overset{C_0}{\underset{c_0}{\simeq}} \Big(\sum_{\boldsymbol{i} \in J_{n,1}} \|a_i\|_H^2 \Big)^{q/2},$$

which is the well known Khintchine's inequality. Note that c_0 and C_0 depend only on q and not on n, H or a_i's. Assume that (4.1) holds for a $d \geqslant 1$. Let $a_{i_1,\ldots,i_{d+1}} \in H$, $\boldsymbol{i} \in J_{n,d+1}$. Then

$$(4.2) \quad E\Big\| \sum_{\boldsymbol{i} \in J_{n,d+1}} a_{i_1,\ldots,i_{d+1}} r_{i_1}^{(1)} \ldots r_{i_{d+1}}^{(d+1)} \Big\|_H^q$$

$$= E\Big\| \sum_{i_{d+1}=1}^{n} \Big(\sum_{\boldsymbol{i} \in J_{n,d}} a_{i_1,\ldots,i_d,i_{d+1}} r_{i_1}^{(1)} \ldots r_{i_d}^{(d)} \Big) r_{i_{d+1}}^{(d+1)} \Big\|_H^q$$

$$\overset{C_0}{\underset{c_0}{\simeq}} E\Big(\sum_{i_{d+1}=1}^{n} \Big\| \sum_{\boldsymbol{i} \in J_{n,d}} a_{i_1,\ldots,i_d,i_{d+1}} r_{i_1}^{(1)} \ldots r_{i_d}^{(d)} \Big\|_H^2 \Big)^{q/2}$$

by induction with $d = 1$. Now let $\mathscr{H}$ be the Hilbert space with elements $x = (x_1, \ldots, x_n)$, $x_j \in H$, and the inner product $\langle x, y \rangle_{\mathscr{H}} = \sum_{i=1}^n \langle x_i, y_i \rangle_H$. Then, the last expression above can be written as

$$E\Big\| \sum_{\boldsymbol{i} \in J_{n,d}} b_{i_1,\ldots,i_d} r_{i_1}^{(1)} \ldots r_{i_d}^{(d)} \Big\|_{\mathscr{H}}^q,$$

where $b_{i_1,\dots,i_d} = (a_{i_1,\dots,i_d,1}, \dots, a_{i_1,\dots,i_d,n})$; and, by the induction hypotheses, we have

$$(4.3)\qquad E\|\sum_{i\in J_{n,d}} b_{i_1,\dots,i_d} r_{i_1}^{(1)}\dots r_{i_d}^{(d)}\|_{\mathscr{H}}^q \overset{C_0^d}{\underset{c_0^d}{\simeq}} (\sum_{i\in J_{n,d}} \|b_{i_1,\dots,i_d}\|_{\mathscr{H}}^2)^{q/2}.$$

Then (4.2) and (4.3) yield

$$E\|\sum_{i\in J_{n,d+1}} a_{i_1,\dots,i_{d+1}} r_{i_1}^{(1)}\dots r_{i_{d+1}}^{(d)}\|_H^q \overset{C_0^{d+1}}{\underset{c_0^{d+1}}{\simeq}} (\sum_{i\in J_{n,d+1}} \|a_{i_1,\dots,i_{d+1}}\|_H^2)^{q/2}.$$

This proves (3.2); the proof of (3.3) now easily follows by using (3.2) and observing that

$$E\|\sum_{i\in J_{n,d}} a_{i_1,\dots,i_d} f_{i_1,\dots,i_d} r_{i_1}^{(1)}\dots r_{i_d}^{(d)}\|_H^q \overset{C_0^d}{\underset{c_0^d}{\simeq}} (\sum_{i\in J_{n,d}} \|a_{i_1,\dots,i_d} f_{i_1,\dots,i_d}\|_H^2)$$

and

$$\|f\|_{\min}^q (\sum_{i\in J_{n,d}} \|a_{i_1,\dots,i_d}\|_H^2)^{q/2} \leqslant (\sum_{i\in J_{n,d}} \|a_{i_1,\dots,i_d} f_{i_1,\dots,i_d}\|_H^2)$$
$$\leqslant \|f\|_{\max}^q (\sum_{i\in J_{n,d}} \|a_{i_1,\dots,i_d}\|_H^2)^{q/2}.$$

Proof of Proposition 3.4. By a theorem of Banach (Theorem 3, [2, p. 15]) it follows immediately that M is separately continuous. Using this and an induction argument, we will show that M is jointly continuous.

If $d = 2$, the result is well known (see, e.g., [12, p. 142]). Assume that the result is true for a $d \geqslant 2$. We will show that it is true for $d+1$. Let $B_d \otimes_\pi B_{d+1}$ denote the tensor product of B_d and B_{d+1} endowed with the π-topology [12, p. 434] and let $B_d \hat{\otimes}_\pi B_{d+1}$ be the completion of $B_d \otimes_\pi B_{d+1}$ in this topology.

For any fixed $(x_1, \dots, x_{d-1}) \in B_1 \times \dots \times B_{d-1}$, the map

$$M(x_1, \dots, x_{d-1}, \cdot, \cdot): B_d \times B_{d+1} \mapsto V$$

is a separately, and hence jointly, continuous bilinear map. Hence, by the definitions of the tensor product, the π-topology and Proposition 43.4 of [12, p. 438], there exists a unique continuous linear map

$$b_{x_1,\dots,x_{d-1}}: B_d \hat{\otimes}_\pi B_{d+1} \mapsto V$$

such that

$$(4.4)\qquad M(x_1, \dots, x_{d-1}, x_d, x_{d+1})$$
$$= b_{x_1,\dots,x_{d-1}}(x_d \otimes x_{d+1}) \quad \text{for every } (x_d, x_{d+1}) \in B_d \times B_{d+1}.$$

Now define the map $\hat{M}$: $B_1 \times \dots \times B_{d-1} \times B_d \hat{\otimes} B_{d+1} \mapsto V$ by

$$\hat{M}(x_1, \dots, x_{d-1}, y) = b_{x_1,\dots,x_{d-1}}(y).$$

Then using (4.4), the fact that $B_d \hat{\otimes}_\pi B_{d+1}$ is the π-closure of the span of $\{x_d \otimes x_{d+1}: (x_d, x_{d+1}) \in B_d \times B_{d+1}\}$, it follows that $\hat{M}$ is multilinear; and, since

M is separately continuous, by the Banach Steinhaus theorem we have that $\hat{M}$ is a separately continuous multilinear function:

$$B_1 \times \ldots \times B_{d-1} \times B_d \hat{\otimes}_\pi B_{d+1} \mapsto V.$$

Thus, by the induction hypothesis, $\hat{M}$ is jointly continuous. But, since $(x_d, x_{d+1}) \mapsto x_d \otimes x_{d+1}$ is continuous from $B_d \times B_{d+1} \mapsto B_d \hat{\otimes}_\pi B_{d+1}$ [12, p. 434], it follows that M is jointly continuous. This completes the proof of the first part; the proof of (3.4) now follows by the joint continuity of M at the zero element of $B_1 \times \ldots \times B_d$. ∎

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