PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 12, Fasc. 2 (1991), pp. 311–317

ADMISSIBLE TRANSLATIONS OF THE BROWNIAN MOTION ON A LIE GROUP

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Abstract. The paper provides a new proof of Shigekawa's theorem characterizing admissible translations of the Wiener measure on a Lie group. We prove Shigekawa's conditions to be necessary finding the "derivative" of the translation as a linear functional on a Hilbert space, applying integrals of 1-forms along the paths of stochastic processes. We use the classical Girsanov theorem as the main tool while obtaining the sufficiency in a straightforward way. No advanced theorems concerning absolute continuity of measures induced by stochastic processes are used, as was in Shigekawa's original proof.

In a series of papers [1], [2] and [3] Cameron and Martin investigated transformations of the Wiener measure. Such a transformation is called *admissible* if it produces a measure which is equivalent to the previous one. The simplest version of the renowned Cameron–Martin theorem characterizes admissible translations of the Wiener measure.

The theorem has been generalized in many ways. In [8] Shigekawa presented an analogue of the Cameron–Martin theorem for the Brownian motion on a Lie group. He mentioned [8, Remark 2] that the case of right translations is strongly connected with transformations of the Brownian motion on a Riemannian symmetric space. In [9] he formulated sufficient conditions for a class of such transformations to be admissible. As an attempt to approach to show their necessity we give here a new proof of Shigekawa's theorem for right translations on a Lie group.

The necessity of Shigekawa's conditions is proved in a more immediate way by constructing a linear functional on a function space. The sufficiency is a simple consequence of the classical Girsanov theorem.

At first let us fix some notation and recall basic definitions and facts. Throughout the paper, G will stand for a d-dimensional Lie group. $A_1, \ldots, A_d$ denote a fixed basis of its Lie algebra $\mathfrak{g}$.

Let $T > 0$ and let $X = (X_t)$, $0 \leqslant t \leqslant T$, be a Brownian motion on G (see [5] for the definition). We assume it is continuous and starts at the unit e a.s.

Since we shall be interested in transformations of the law of X, we may assume that it is simply the coordinate process on $W(G)$, the space of continuous G-valued functions over $[0; T]$ endowed with its cylindrical σ-field $\mathscr{B}(W(G))$ with the probability measure P. Let $\Omega = W(G)$ and let $\mathscr{F}$ be the completion of $\mathscr{B}(W(G))$ with respect to P. $(\mathscr{F}_t)$ will denote the natural filtration connected with X. The probability space which will be used below is $(\Omega, \mathscr{F}, P)$.

Without loss of generality we may assume that the generator of X on $C_c^\infty(G)$ is $\frac{1}{2}\sum_{\alpha=1}^{c} A_\alpha A_\alpha$, where $1 \leqslant c \leqslant d$. Here $C_c^\infty(G)$ denotes the set of smooth real-valued functions having compact support on G. Thus, X satisfies the stochastic differential equation

$$dX_t = \sum_\alpha A_\alpha \circ dB_t^\alpha, \tag{1}$$

where B is a c-dimensional $(\mathscr{F}_t)$-Brownian motion. (The terminology concerning stochastic analysis may be found in [6]; see also [7]–[9].)

Suppose that (g_t) is a continuous G-valued function over $[0; T]$. We say that it is *absolutely continuous* (has *bounded variation*) if $(f(g_t))$ has this property for every $f \in C^\infty(G)$. A function of bounded variation is clearly a G-valued semimartingale. If (g_t) is absolutely continuous, there exist a tangent vector $\dot{g}_t \in T_{g_t}(G)$ and $\zeta_t \in \boldsymbol{g}$ such that $\zeta_t(g_t) = \dot{g}_t$ for almost every $t \in [0; T]$. Hence (g_t) satisfies the differential equation

$$dg_t = \zeta_t dt. \tag{2}$$

We define an inner product $B(\cdot, \cdot)$ on $\boldsymbol{g}$ by putting $B(A_i, A_j) = \delta_{ij}$. Let V denote the linear subspace of $\boldsymbol{g}$ spanned by $A_1, \ldots, A_c$. Let H be the set of all $b \in G$ such that $\mathrm{Ad}(g)V \subset V$ and $\mathrm{Ad}(g)|_V$ is an isometry, i.e.,

$$B(\mathrm{Ad}(g)A, \mathrm{Ad}(g)A) = B(A, A) \quad \text{ for } A \in V.$$

If $g \in H$ and $(\mathrm{Ad}(g)_j^i)_{i,j=1,\ldots,d}$ is the matrix of the operator $\mathrm{Ad}(g)$ with respect to the basis $A_1, \ldots, A_d$, then the matrix $(\mathrm{Ad}(g)_\beta^\alpha)_{\alpha,\beta=1,\ldots,c}$ is orthogonal.

LEMMA. *H is a closed subgroup of G. Let $\boldsymbol{h}$ be its Lie algebra and $A \in \boldsymbol{g}$. Then $A \in \boldsymbol{h}$ iff $(\mathrm{ad}\, A)V \subset V$ and $\mathrm{ad}\, A|_V$ is skew-symmetric, i.e.,*

$$B((\mathrm{ad}\, A)A', A'') + B(A', (\mathrm{ad}\, A)A'') = 0 \quad \textit{for every } A', A'' \in V.$$

The proof is standard and uses the exponential mapping (cf. [8, Lemma 3.2]).

Let $\boldsymbol{k} = \boldsymbol{h} \cap V$. If $A', A'' \in \boldsymbol{k}$, then $[A', A''] = (\mathrm{ad}\, A')A'' \in V$. This shows that $\boldsymbol{k}$ is a Lie subalgebra of $\boldsymbol{g}$.

Now we are able to formulate the theorem [8, Theorem 3.2]. Let (g_t) be a continuous trajectory in G, $g_0 = e$. We define a process Z, $Z_t = X_t g_t$, where X is our Brownian motion.

THEOREM. *The laws of the processes X and Z are equivalent (i.e., mutually absolutely continuous) iff the following three conditions are satisfied:*

(i) *(g_t) is absolutely continuous,*

(ii) *$\zeta_t \in \boldsymbol{k}$ a.e.,*

(iii) $$\int_0^T (\zeta_t^i)^2 dt < \infty \quad \textit{for } i = 1, \ldots, d,$$

where ζ_t^i *are components of* ζ_t *in* (2), *i.e.*, $\zeta_t = \sum_i \zeta_t^i A_i$.

Moreover, if Q *denotes the law of* Z, *then the Radon–Nikodým derivative is given by*

(iv) $$\frac{dQ}{dP} = \exp\left(\sum_{\alpha=1}^{c} \int_0^T \zeta_t^\alpha dB_t^\alpha - \frac{1}{2}\sum_{\alpha=1}^{c} \int_0^T (\zeta_t^\alpha)^2 dt\right).$$

Remark. The arguments of the functions appearing in (iv) are trajectories. On the left-hand side of the equation they are treated as points of the measurable space $(W(G), \mathscr{B}(W(G)))$. And on the right-hand side, they mean elementary events of the probability space $(\Omega, \mathscr{F}, P)$.

Proof of the necessity. Let us assume that P and Q are equivalent (on $W(G)$). Besides the space $(\Omega, \mathscr{F}, P)$ we can consider the complete probability space $(\Omega, \mathscr{F}, Q)$ with the filtration $(\mathscr{F}_t)$.

Let $\tilde{X}_t = X_t g_t^{-1}$; the law of the process $\tilde{X}$ under the measure Q is P. Hence $\tilde{X}$ is a Q-Brownian motion on G, i.e., a Brownian motion on the probability space $(\Omega, \mathscr{F}, Q)$, whose generator is given by $\frac{1}{2}\sum_\alpha A_\alpha A_\alpha$. Since the natural filtration for $\tilde{X}$ is $(\mathscr{F}_t)$, there exists a c-dimensional $(\mathscr{F}_t)$-Brownian motion on the space $(\Omega, \mathscr{F}, Q)$ (briefly, Q-Brownian motion) $\tilde{B} = (\tilde{B}^\alpha)$ such that $\tilde{X}$ satisfies

(3) $$d\tilde{X}_t = \sum_\alpha A_\alpha \circ d\tilde{B}_t^\alpha.$$

Let $\mathscr{H}$ denote the Hilbert space $(L^2[0; T])^d$. We shall identify its elements, of the form $(h^1, \ldots, h^d)$, $h^i \in L^2[0; T]$, with the functions h: $[0, T] \to g$, $h_t = \sum_i h_t^i A_i$.

$L^0(P)$ will denote the space of random variables on $(\Omega, \mathscr{F}, P)$ which are finite a.s., with convergence in probability. $L^0(Q)$ gets a similar meaning. The equivalence of P and Q yields that both $L^0(P)$ and $L^0(Q)$ have the same elements and the same topologies. We define a mapping Λ: $\mathscr{H} \to L^0(P)$ by

(4) $$\Lambda h = \sum_\alpha (P)\int_0^T h_t^\alpha dB_t^\alpha - \sum_\alpha \sum_i (Q)\int_0^T \mathrm{Ad}(g_t^{-1})_\alpha^i h_t^i d\tilde{B}_t^\alpha.$$

The first expression on the right-hand side is an element of $L^2(P)$, and the other is from $L^2(Q)$ ($(P)\int$ and $(Q)\int$ denote the stochastic integrals with respect to the appropriate measures). The $L^2(P)$-norm of the first term equals $(\sum \int_0^T (h_t^\alpha)^2 dt)^{1/2}$. The $L^2(Q)$-norm of the latter is estimated by the product of a constant and $(\sum \int_0^T (h_t^i)^2 dt)^{1/2}$. Therefore, Λ is a continuous linear operator.

Using again the equivalence of the measures P and Q, we see that both the processes $\tilde{X}$ and $(g_t) = (\tilde{X}_t^{-1} X_t)$ are P-semimartingales (see, e.g., [4, Theorem 13.12]). Thus, (g_t) is a function of bounded variation.

Let $h \in \mathscr{H}$ be continuous and have bounded variation (i.e., its components are of bounded variation). Then on the right-hand side of (4) we may omit (P) and (Q), since the integrals $(P)\int$ and $(Q)\int$ are the same [4, Theorem 13.15].

For a continuous function of bounded variation h, h^* denotes the form

$$h_t^* = \sum_i h_t^i \omega^i,$$

where $\omega^1, \ldots, \omega^d$ is the dual basis of $A_1, \ldots, A_d$. If $g \in G$, then $\mathrm{Ad}(g)'$ will denote the adjoint operator of $\mathrm{Ad}(g)$, which is an endomorphism of $\mathfrak{g}^*$. We shall use the integral of 1-forms along the paths of diffusion processes (see [6] and [7]). We want to express the integral $\int_0^T (\mathrm{Ad}(g_t^{-1})' h_t^*) \circ d\tilde{X}_t$ in terms of integrals along the processes (X_t) and (g_t). Lemma 3.4 from [7] will be essential.

If the function $\phi\colon G \times G \to G$ is given by $\phi(x, y) = xy^{-1}$, then $\tilde{X}_t = \phi(X_t, g_t)$. Let us put

$$\phi_x = \phi(x, \cdot)\colon G \to G \quad \text{and} \quad {}_y\phi = \phi(\cdot, y)\colon G \to G \quad \text{for any } x, y \in G.$$

Let $\delta\phi_x$ and $\delta_y\phi$ denote the co-tangent mappings of ϕ_x and ${}_y\phi$, respectively. Then for every left invariant 1-form ω we have

$$\delta\phi_x(\omega)(y) = -(\mathrm{Ad}(y)'\omega)(y), \qquad \delta_y\phi(\omega) = \mathrm{Ad}(y)'\omega.$$

Hence the following holds:

$$\begin{aligned}\int_0^T (\mathrm{Ad}(g_t^{-1})' h_t^*) \circ d\tilde{X}_t &= \sum_{i,j} \int_0^T \mathrm{Ad}(g_t^{-1})_j^i h_t^i \omega^j \circ d\tilde{X}_t \\ &= -\sum_{i,j} \int_0^T \mathrm{Ad}(g_t^{-1})_j^i h_t^i (\mathrm{Ad}(g_t)'\omega^j) dg_t \\ &\qquad + \sum_{i,j} \int_0^T \mathrm{Ad}(g_t^{-1})_j^i h_t^i (\mathrm{Ad}(g_t)'\omega^j) \circ dX_t \\ &= -\int_0^T h_t^* dg_t + \int_0^T h_t^* \circ dX_t,\end{aligned}$$

i.e.,

$$\int_0^T h_t^* \circ dX_t - \int_0^T (\mathrm{Ad}(g_t^{-1})' h_t^*) \circ d\tilde{X}_t = \int_0^T h_t^* dg_t. \tag{5}$$

X and $\tilde{X}$ satisfy the equations (1) and (3), respectively, which gives

$$\int_0^T h_t^* \circ dX_t = \sum_\alpha \int_0^T h_t^\alpha dB_t^\alpha,$$

$$\int_0^T (\mathrm{Ad}(g_t^{-1})' h_t^*) \circ d\tilde{X}_t = \sum_\alpha \sum_i \int_0^T \mathrm{Ad}(g_t^{-1})_\alpha^i h_t^i d\tilde{B}_t^\alpha.$$

Combining the last formula with (5) yields, for every continuous function of bounded variation $h \in \mathscr{H}$,

$$\Lambda h = \int_0^T h_t^* \, dg_t \quad P\text{-a.s.} \tag{6}$$

Thus, for $h \in \mathscr{H}$ the function Λh is constant, in other words, Λ is a continuous linear functional on $\mathscr{H}$. Consequently, there exists $\zeta \in \mathscr{H}$ such that

$$\Lambda h = \langle h, \zeta \rangle, \qquad h \in \mathscr{H}. \tag{7}$$

$\langle \cdot, \cdot \rangle$ denotes here the inner product in $\mathscr{H}$.

Let $f \in C^\infty(G)$ and let ω^f be the 1-form defined by the formula $\omega^f(v) = v(f)$, $v \in T(G)$. Then for every t, ω_t will denote the left invariant 1-form for which $\omega^f(g_t) = \omega_t(g_t)$. $t \mapsto \omega_t \in g^*$ is a continuous mapping of bounded variation, as its components are $A_i f(g_t)$. Hence, using (6) and (7), we obtain

$$f(g_t) - f(g_0) = \int_0^t \omega^f \, dg_s = \int_0^t \omega_s \, dg_s = \sum_i \int_0^t A_i f(g_s) \zeta_s^i \, ds = \int_0^t \zeta_s f(g_s) \, ds.$$

Therefore, the function (g_t) is absolutely continuous and satisfies (2).

Summing up the formulae (4) and (7) we get

$$\sum_\alpha (P) \int_0^T h_t^\alpha \, dB_t^\alpha - \sum_\alpha \sum_i (Q) \int_0^T \mathrm{Ad}(g_t^{-1})_\alpha^i h_t^i \, d\tilde{B}_t^\alpha = \sum_i \int_0^T h_t^i \zeta_t^i \, dt, \qquad h \in \mathscr{H}. \tag{8}$$

Let us fix I, $c+1 \leqslant I \leqslant d$, and put $h^i = \delta_I^i I_{[0;s]}$. The equality (8) yields

$$- \sum_\alpha (Q) \int_0^s \mathrm{Ad}(g_t^{-1})_\alpha^I \, d\tilde{B}_t^\alpha = \int_0^s \zeta_t^I \, dt, \qquad s \in [0; T]. \tag{9}$$

Since on the left-hand side we have a martingale, and a function of bounded variation stands on the right, the both sides equal zero. In particular,

$$\zeta_t^I = 0 \quad \text{for a.e. } t, \ I = c+1, \ldots, d. \tag{10}$$

Computing the Q-quadratic variation of the left-hand side in the formula (9) we get

$$\sum_\alpha \int_0^s (\mathrm{Ad}(g_t^{-1})_\alpha^I)^2 \, dt = 0,$$

so $\mathrm{Ad}(g_t^{-1}) V \subset V$ for every t.

The formula (8) may be now rewritten in the form

$$\sum_\alpha (P) \int_0^T h_t^\alpha \, dB_t^\alpha - \sum_{\alpha,\beta} (Q) \int_0^T \mathrm{Ad}(g_t^{-1})_\alpha^\beta h_t^\beta \, d\tilde{B}_t^\alpha = \sum_\alpha \int_0^T h_t^\alpha \zeta_t^\alpha \, dt. \tag{11}$$

For every $\gamma = 1, \ldots, c$, putting $h^\alpha = \delta^\alpha_\gamma I_{[0;s]}$ in (11) we easily get

(12) $$dB^\gamma_t - \sum_\alpha \mathrm{Ad}(g_t^{-1})^\gamma_\alpha d\tilde{B}^\alpha_t = \zeta^\gamma_t dt.$$

Hence $dt = \sum_\alpha (\mathrm{Ad}(g_t^{-1})^\gamma_\alpha)^2 dt$, i.e., $\sum_\alpha (\mathrm{Ad}(g_t^{-1})^\gamma_\alpha)^2 = 1$ for every t.

Now, if $\gamma \neq \delta$, then (12) yields

$$dB^\gamma_t - dB^\delta_t - \sum_\alpha (\mathrm{Ad}(g_t^{-1})^\gamma_\alpha - \mathrm{Ad}(g_t^{-1})^\delta_\alpha) d\tilde{B}^\alpha_t = (\zeta^\gamma_t - \zeta^\delta_t) dt.$$

Then we add the last term of the left-hand side to both sides and compute the quadratic variation. Applying the previous result we get the equations

$$2dt = \sum_\alpha (\mathrm{Ad}(g_t^{-1})^\gamma_\alpha - \mathrm{Ad}(g_t^{-1})^\delta_\alpha)^2 dt = 2dt - 2\sum_\alpha \mathrm{Ad}(g_t^{-1})^\gamma_\alpha \mathrm{Ad}(g_t^{-1})^\delta_\alpha dt,$$

$$\sum_\alpha \mathrm{Ad}(g_t^{-1})^\gamma_\alpha \mathrm{Ad}(g_t^{-1})^\delta_\alpha = 0.$$

Thus, $g_t \in H$ for every t or, equivalently, $\zeta_t \in \boldsymbol{h}$ for a.e. t. Because of (10), we see that $\zeta_t \in \boldsymbol{k}$ for a.e. t, which completes the proof.

Proof of the sufficiency. Suppose that (g_t) is a function that satisfies the conditions (i)–(iii) of the theorem.

Let $\tilde{X}$ be defined as before. We shall use formulae for products and inverses of semimartingales (see, e.g., [8] or [9]). By (1) and (2) we observe that $\tilde{X}$ satisfies the stochastic differential equation

(13) $$d\tilde{X}_t = \sum_\alpha \mathrm{Ad}(g_t) A_\alpha \circ dB^\alpha_t - \mathrm{Ad}(g_t)\zeta_t dt$$

(with respect to the measure P).

Since $\zeta_t \in \boldsymbol{k}$ for a.e. t, $g_t \in H$ for every t. Therefore, we can rewrite (13) as

$$d\tilde{X}_t = \sum_{\alpha,\beta} \mathrm{Ad}(g_t)^\beta_\alpha A_\beta \circ dB^\alpha_t - \sum_{\alpha,\beta} \mathrm{Ad}(g_t)^\alpha_\beta \zeta^\beta_t A_\alpha dt$$

or

(14) $$d\tilde{X}_t = \sum_\alpha A_\alpha \circ d\tilde{B}^\alpha_t,$$

where

(15) $$\tilde{B}^\alpha_t = \sum_\beta \int_0^t \mathrm{Ad}(g_s)^\alpha_\beta dB^\beta_s - \sum_\beta \int_0^t \mathrm{Ad}(g_s)^\alpha_\beta \zeta^\beta_s ds.$$

Let

(16) $$\hat{B}^\alpha_t = \sum_\beta \int_0^t \mathrm{Ad}(g_s)^\alpha_\beta dB^\beta_s, \quad \alpha = 1, \ldots, c.$$

Since $(\mathrm{Ad}(g_t)^\alpha_\beta)$ is an orthogonal matrix a.e., $\hat{B} = (\hat{B}^\alpha)$ is a c-dimensional P-Brownian motion. Applying the formula (16) to (15), we may write

$$\tilde{B}^\alpha_t = \hat{B}^\alpha_t - \sum_\beta \int_0^t \mathrm{Ad}(g_s)^\alpha_\beta \zeta^\beta_s ds.$$

We define a measure Q' on $(\Omega, \mathscr{F})$ by the density

$$\frac{dQ'}{dP} = \exp\Big(\sum_{\alpha,\beta} \int_0^T \mathrm{Ad}(g_t)^{\alpha}_{\beta} \zeta^{\beta}_t \, d\hat{B}^{\alpha}_t - \frac{1}{2} \sum_{\alpha} \int_0^T \big(\sum_{\beta} \mathrm{Ad}(g_t)^{\alpha}_{\beta} \zeta^{\beta}_t\big)^2 dt\Big) \tag{17}$$

$$= \exp\Big(\sum_{\alpha} \int_0^T \zeta^{\alpha}_t \, dB^{\alpha}_t - \frac{1}{2} \sum_{\alpha} \int_0^T (\zeta^{\alpha}_t)^2 dt\Big).$$

(We used here (16) and the orthogonality of $(\mathrm{Ad}(g_t)^{\alpha}_{\beta})$.)

Now the Girsanov theorem (see, e.g., [4, Theorem 13.25]) yields that $\tilde{B} = (\tilde{B}^{\alpha})$ is a Q'-Brownian motion. Since $\tilde{X}$ satisfies (14) and the measures P and Q' are equivalent, $\tilde{X}$ is a Q'-Brownian motion on G, whose generator is $\frac{1}{2}\sum_{\alpha} A_{\alpha} A_{\alpha}$. Hence, its law under Q' is P and, consequently, $Q' = Q$. The measures P and Q are thus equivalent, and (17) gives the formula (iv). The proof of the theorem is complete.

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Received on 6.3.1990;
revised version on 20.2.1991