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A STOCHASTIC TAYLOR FORMULA FOR TWO-PARAMETER STOCHASTIC PROCESSES

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Abstract. The purpose of the present paper is to prove a stochastic Taylor formula for two-parameter processes which extends the results of W. Wagner and E. Platen in the one-parameter case (cf. [5]–[7]).

1. Introduction and notation. Let $(\Omega, \mathscr{F}, P)$ be a complete probability system and let $\leqslant$ be the natural ordering in R_+^2, i.e., for $(s_i, t_i) \in R_+^2$, $i = 1, 2$, $(s_1, t_1) \leqslant (s_2, t_2)$ if $s_1 \leqslant s_2$ and $t_1 \leqslant t_2$. An integrable process $M = \{M_z, z \in R_+^2\}$ is said to be a *martingale* (w.r.t. an increasing family $\{\mathscr{F}_z, z \in R_+^2\}$ of sub-σ-fields of $\mathscr{F}$) if it is $\{\mathscr{F}_z\}$-adapted and $\mathrm{E}(M_{z'} \mid \mathscr{F}_z) = M_z$ for any $z \leqslant z'$. In this paper we assume that the family $\{\mathscr{F}_z\}$ satisfies the usual conditions in [3] or [4]. Given $p \geqslant 1$ let m_c^p be the class of all continuous martingales M such that $M_z = 0$ on the axes and $\sup_z \mathrm{E}|M_z|^p < +\infty$. If $p \geqslant 2$ and $M \in m_c^p$, we denote by $\tilde{M}$ and $\langle M \rangle_z$, $\langle \tilde{M} \rangle_z$, $\langle M_{s.} \rangle_t$, $\langle M_{.t} \rangle_s$ the martingale and the continuous versions of quadratic variations of the martingales (cf. [3]).

Let $B = \{1, 2, \ldots, 7\}$ and $A = \{\emptyset\} \cup \left(\bigcup_{l=1}^{\infty} B^l\right)$, where B^l denotes the l-fold Cartesian product of the set B. Further, let φ and ψ be functions from the set B into $\{1, 2, 3, 4\}$ such that

$$\varphi(1) = 1, \quad \varphi(2) = \ldots = \varphi(5) = 2, \quad \varphi(6) = 3, \quad \varphi(7) = 4,$$

$$\psi(1) = \psi(2) = 1, \quad \psi(3) = \ldots = \psi(7) = 2.$$

Given $\alpha \in A$, set

$$|\alpha| = \begin{cases} 0 & \text{if } \alpha = \emptyset, \\ l & \text{if } \alpha \in B^l,\ l \geqslant 1, \end{cases}$$

and if $\alpha = (\alpha_1, \ldots, \alpha_l) \in A \setminus \{\emptyset\}$, we put

$$\|\alpha\| = \sum_{i=1}^{l} \varphi(\alpha_i), \quad \psi(\alpha) = \sum_{i=1}^{l} \psi(\alpha_i),$$

$$\alpha- = \begin{cases} \emptyset & \text{if } l = 1, \\ (\alpha_1, \ldots, \alpha_{l-1}) & \text{if } l \geqslant 2. \end{cases}$$

The composition of the vectors $\alpha = (\alpha_1, \ldots, \alpha_l)$ and $\beta = (\beta_1, \ldots, \beta_k)$ is defined by $\alpha * \beta = (\alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_k)$.

For a continuous stochastic function h defined on $\boldsymbol{R}^2_+$ and adapted to $\mathscr{F}_z$ we put

$$I_1(h, s, t) = \int_{R_{st}} h(z) dM_z, \qquad I_2(h, s, t) = \int_{R_{st}} h(z) d\tilde{M}_z,$$

$$I_3(h, s, t) = \tfrac{1}{2} \int_0^s h(x, t) d\langle M_{.t}\rangle_x, \qquad I_4(h, s, t) = \tfrac{1}{2} \int_0^t h(s, y) d\langle M_{s.}\rangle_y,$$

(1.1)

$$I_5(h, s, t) = -\tfrac{1}{2} \int_{R_{st}} h(z) d\langle M\rangle_z, \qquad I_6(h, s, t) = -\int_{R_{st}} h(z) d\langle M, \tilde{M}\rangle_z,$$

$$I_7(h, s, t) = -\tfrac{1}{4} \int_{R_{st}} h(z) d\langle \tilde{M}\rangle_z,$$

where $R_{st} = \{z \in \boldsymbol{R}^2_+ : z \leqslant (s, t)\}$.

If $\alpha \in A$ is a multi-index, we define inductively a multiple stochastic integral I_α by

$$I_\alpha(h, z) = \begin{cases} h(z) & \text{if } \alpha = \emptyset, \\ I_{\alpha-}(I_{\alpha_l}(h, \cdot), z) & \text{if } \alpha = (\alpha_1, \ldots, \alpha_l),\ l \geqslant 1. \end{cases} \tag{1.2}$$

In what follows we shall use the following Itô formula for functions of two-parameter martingales due to Nualart [4].

Suppose that $f: \boldsymbol{R} \to \boldsymbol{R}$ is a function belonging to C^4 and $f(0) = 0$ and suppose that $\tilde{M}$ is a martingale in m_c^4. Then for any $(s, t) \in \boldsymbol{R}^2_+$ we have

$$\begin{aligned} f(M_{st}) = {} & \int_{R_{st}} f'(M_z) dM_z + \int_{R_{st}} f''(M_z) d\tilde{M}_z + \tfrac{1}{2} \int_0^t f''(M_{x,t}) d\langle M_{.t}\rangle_x \\ & + \tfrac{1}{2} \int_0^t f''(M_{sy}) d\langle M_{s.}\rangle_y - \tfrac{1}{2} \int_{R_{st}} f''(M_z) d\langle M\rangle_z \\ & - \int_{R_{st}} f'''(M_z) d\langle M, \tilde{M}\rangle_z - \tfrac{1}{4} \int_{R_{st}} f^{\mathrm{IV}}(M_z) d\langle \tilde{M}\rangle_z. \end{aligned} \tag{1.3}$$

Using the notation (1.1) we can rewrite (1.3) as follows:

$$f(M_z) = \sum_{|\alpha|=1} I_\alpha(D^{\|\alpha\|} f \circ M, z). \tag{1.3'}$$

From the above formula we shall introduce a wide class of M-differentiable processes and prove a Taylor formula for this class. We also obtain estimations of errors for such an expansion and apply these results to the problem of the approximation of stochastic processes by stochastic polynomials.

DEFINITION 1.1. Let $M \in m_c^4$. A stochastic process $f = \{f(z), z \in \boldsymbol{R}^2_+\}$ is said to be *n-times M-differentiable* (or *belonging to the class* C^n_M) if there exist continuous stochastic processes $\{f_\alpha(z), 1 \leqslant |\alpha| \leqslant n\}$ such that for any $z \in \boldsymbol{R}^2_+$:

$$f(z) = f(0) + \sum_{\alpha \in B} I_\alpha(f_\alpha, z), \tag{1.4}$$

$$f_\alpha(z) = f_\alpha(0) + \sum_{\beta \in B} I_\beta(f_{\alpha * \beta}, z) \quad (1 \leqslant |\alpha| < n). \tag{1.5}$$

2. Stochastic Taylor formula.

THEOREM 2.1. *Suppose that $f = \{f(z), z \in R_+^2\}$ is an n-times M-differentiable stochastic process where $M \in m_c^4$, $n \geqslant 1$. Then for each $z \in R_+^2$:*

$$f(z) = f(0) + \sum_{0<|\alpha|<n} I_\alpha(f_\alpha(0), z) + \sum_{|\alpha|=n} I_\alpha(f_\alpha, z). \tag{2.1}$$

Proof (by induction). (i) For $n = 1$, (2.1) follows from (1.4).

(ii) Suppose that (2.1) is true for any $n = m$. Then

$$f(z) = f(0) + \sum_{0<|\alpha|<m} I_\alpha(f_\alpha(0), z) + \sum_{|\alpha|=m} I_\alpha(f_\alpha, z). \tag{2.2}$$

According to Definition 1.1 and induction assumption, for each $\alpha \in B^m$ f_α is 1-time M-differentiable. Hence

$$f_\alpha(z') = f_\alpha(0) + \sum_{\beta \in B} I_\beta(f_{\alpha*\beta}, z') \quad (\alpha \in B^m). \tag{2.3}$$

From (2.2) and (2.3) we have

$$\begin{aligned} f(z) &= f(0) + \sum_{0<|\alpha|<m} I_\alpha(f_\alpha(0), z) + \sum_{|\alpha|=m} I_\alpha(f_\alpha(0), z) + \sum_{|\alpha|=m} \sum_{\beta \in B} I_\alpha(I_\beta(f_{\alpha*\beta}, z)) \\ &= f(0) + \sum_{0<|\alpha|<m+1} I_\alpha(f_\alpha(0), z) + \sum_{\alpha*\beta} I_{\alpha*\beta}(f_{\alpha*\beta}, z) \\ &= f(0) + \sum_{0<|\alpha|<m+1} I_\alpha(f_\alpha(0), z) + \sum_{|\alpha|=m+1} I_\alpha(f_\alpha, z), \end{aligned}$$

which shows that (2.1) also holds for $n = m+1$. The proof is complete.

THEOREM 2.2. *Suppose that $M \in m_c^4$ and let C^m be the class of all real functions defined on R_+^2, m-times continuously differentiable. Then*

$$\{f \circ M, f \in C^{4n}\} \subset C_M^n \quad (n \geqslant 1) \tag{2.4}$$

and

$$f(M_z) = f(0) + \sum_{0<|\alpha|<n} I_\alpha(D^{\|\alpha\|} f(0), z) + \sum_{|\alpha|=n} I_\alpha(D^{\|\alpha\|} f \circ M, z) \tag{2.5}$$

for any $f \in C^{4n}$ and any $z \in R_+^2$.

Proof (by induction). (a) Suppose that $f \in C^4$; then $g = f - f(0) \in C^4$ and $g(0) = 0$.

Applying the two-parameter Itô formula (1.3') for $g \circ M$, we get

$$g(M_z) = \sum_{|\alpha|=1} I_\alpha(D^{\|\alpha\|} g \circ M, z).$$

Hence

$$f(M_z) = f(0) + \sum_{|\alpha|=1} I_\alpha(D^{\|\alpha\|} f \circ M, z), \tag{2.6}$$

which shows that $F=f\circ M\in C^1_M$, i.e., (2.4) holds for $n=1$. Moreover, the functions F_α in Definition 1.1 are of the form

$$F_\alpha = D^{\|\alpha\|} f\circ M. \tag{2.7}$$

Now we suppose that (2.4) holds for $n=m\geqslant 1$ and (2.7) is true for every α with $|\alpha|\leqslant m$.

Let f be an arbitrary function in $C^{4(m+1)}$. It is clear that $D^{\|\alpha\|}f\in C^4$ for any $\alpha\in B^m$ and $F=f\circ M\in C^m_M$, and (2.7) holds for $\alpha\in\sum_{l=1}^m B^l$ (by induction assumption). Applying the Itô formula (1.3′) for each $F_\alpha = D^{\|\alpha\|}f\circ M$ with $|\alpha|=m$ we get

$$F_\alpha(z) = D^{\|\alpha\|}f(M_z) = D^{\|\alpha\|}f(0)+\sum_{|\beta|=1} I_\beta(D^{\|\beta\|}D^{\|\alpha\|}f\circ M, z)$$
$$= D^{\|\alpha\|}f(0)+\sum_{|\beta|=1} I_\beta(D^{\|\alpha*\beta\|}f\circ M, z),\qquad z\in R^2_+,$$

where $D^{\|\alpha*\beta\|}f$ is continuous.

Thus, by Definition 1.1, $F_\alpha\in C^1_M$ $(\alpha\in B^m)$, and $F=f\circ M\in C^{m+1}_M$. Moreover, for $\alpha\in B^{m+1}$, $F_\alpha = D^{\|\alpha\|}f\circ M$.

Hence (2.4) and (2.7) hold for $n=m+1$.

(b) Now (2.5) follows from (2.4), (2.7) and Theorem 2.1. The proof is complete.

Remark 2.1. Putting $I_\alpha(z):=I_\alpha(1,z)$ we get $I_\alpha(h,z)=h\cdot I_\alpha(z)$ for any $h=\text{const}$. Therefore, (2.5) can be written as follows:

$$f(M_z)=f(0)+\sum_{0<|\alpha|<n} D^{\|\alpha\|}f(0)I_\alpha(z)+\sum_{|\alpha|=n} I_\alpha(D^{\|\alpha\|}f\circ M, z).$$

EXAMPLE. (a) Let $f(x)=\sum_{k=0}^n a_k x^k$ $(a_n\neq 0)$. From (2.5) we have

$$f(M_z)=a_0+\sum_{0<\|\alpha\|\leqslant n} (\|\alpha\|!)a_{\|\alpha\|}I_\alpha(z).$$

(b) $e^{M_z}=1+\sum_{0<|\alpha|<n}I_\alpha(z)+\sum_{|\alpha|=n}I_\alpha(e^M, z)$.

3. Estimation of errors. We denote by m^4_{cd} a subclass of m^4_c such that $M\in m^4_{cd}$ iff the following domination condition is satisfied:

There exist continuous increasing non-stochastic functions $\lambda_1, \lambda_2\geqslant 0$ on R^1_+ such that

$$\langle M\rangle(B)\leqslant(\lambda_1\otimes\lambda_2)(B),\qquad \langle\tilde M\rangle(B)\leqslant(\lambda_1\otimes\lambda_2)(B)$$

for all Borel subsets B of R^2_+,

$$\langle M_{s.}\rangle_{t_2}-\langle M_{s.}\rangle_{t_1}\leqslant\lambda_1(s)(\lambda_2(t_2)-\lambda_2(t_1)),\qquad t_1\leqslant t_2,\ s\geqslant 0,$$
$$\langle M_{.t}\rangle_{s_2}-\langle M_{.t}\rangle_{s_1}\leqslant\lambda_2(t)(\lambda_1(s_2)-\lambda_1(s_1)),\qquad s_1\leqslant s_2,\ t\geqslant 0.$$

It is obvious that the class m^4_{cd} contains all two-parameter Wiener processes.

THEOREM 3.1. *Suppose that $M \in m_{cd}^4$ and $f \in C_M^m$. Furthermore, suppose that there exist numbers A_1 and B_1 such that*

$$K_\alpha := \sup_{z \leqslant z_0} \mathrm{E}\,|f_\alpha(z)|^2 \leqslant A_1 \cdot B_1^{\psi(\alpha)}, \qquad \alpha \in B^n. \tag{3.1}$$

Then

$$\sup_{z \leqslant z_0} \mathrm{E}\,\Big|f(z)-f(0)-\sum_{0<|\alpha|<n} I_\alpha(f_\alpha(0), z)\Big|^2 \leqslant A_1 \cdot B_2^n/(n!), \tag{3.2}$$

where B_2 is a positive constant depending only on $\lambda_1 \otimes \lambda_2(z_0)$ and B_1.

LEMMA 3.1. *Let $h: R_{z_0} \to R^1$ be a continuous adapted process such that $\sup_{z \leqslant z_0} \mathrm{E}\,|h(z)|^2 \leqslant K$. Then for any $\alpha \in A \setminus \{\emptyset\}$*

$$\mathrm{E}\,|I_\alpha(h, z)|^2 \leqslant K[\lambda_1 \otimes \lambda_2(z)]^{\psi(\alpha)}/(|\alpha|!) \tag{3.3}$$

for any $z \in R_{z_0}$.

Proof of Lemma 3.1. By the isometry property of the stochastic integral (cf. [1]) and the Schwarz inequality we infer that (3.3) holds for any $\alpha \in B^1$. Suppose that (3.3) is true for every $\alpha \in B^n$. Let $\bar{\alpha} = \alpha_0 * \alpha \in B^{n+1}$ be any but fixed and $\alpha_0 \in B$. Applying the same inequalities, for $M \in m_{cd}^4$ we get

$$\mathrm{E}\,|I_{\bar{\alpha}}(h, s, t)|^2 \tag{3.4}$$

$$\leqslant \begin{cases} \int\limits_{R_{st}} \mathrm{E}\,|I_\alpha(h, x, y)|^2 d\lambda_1(x) d\lambda_2(y) & \text{for } \alpha_0 = 1, 2, \\ \lambda_1(s)\lambda_2^2(t) \int\limits_0^s \mathrm{E}\,|I_\alpha(h, x, t)|^2 d\lambda_1(x) & \text{for } \alpha_0 = 3, \\ \lambda_1^2(s)\lambda_2(t) \int\limits_0^t \mathrm{E}\,|I_\alpha(h, s, y)|^2 d\lambda_2(y) & \text{for } \alpha_0 = 4, \\ \lambda_1(s)\lambda_2(t) \int\limits_{R_{st}} \mathrm{E}\,|I_\alpha(h, x, y)|^2 d\lambda_1(x) d\lambda_2(y) & \text{for } \alpha_0 = 5, 6, 7, \end{cases}$$

where λ_1, λ_2 are the same as at the beginning of Section 3.

By (3.4) and induction assumption, we get

$$\mathrm{E}\,|I_{\bar{\alpha}}(h, z)|^2 \leqslant \begin{cases} \dfrac{K[\lambda_1 \otimes \lambda_2(z)]^{\psi(\alpha)+1}}{(|\alpha|!)[\psi(\alpha)+1]^2} & \text{for } \alpha_0 = 1, 2, \\ \dfrac{K[\lambda_1 \otimes \lambda_2(z)]^{\psi(\alpha)+2}}{(|\alpha|!)[\psi(\alpha)+1]^2} & \text{for } \alpha_0 = 3, 4, \\ \dfrac{K[\lambda_1 \otimes \lambda_2(z)]^{\psi(\alpha)+2}}{(|\alpha|!)[\psi(\alpha)+1]^2} & \text{for } \alpha_0 = 5, 6, 7. \end{cases}$$

Moreover, since $\psi(\alpha) \geqslant |\alpha|$ and $\psi(\bar{\alpha}) = \psi(\alpha) + \psi(\alpha_0)$, we have

$$\mathrm{E}\,|I_{\bar{\alpha}}(h, z)|^2 \leqslant \frac{K[\lambda_1 \otimes \lambda_2(z)]^{\psi(\bar{\alpha})}}{(|\bar{\alpha}|!)} \quad \text{for any } z \leqslant z_0.$$

Hence (3.3) is true for any $\alpha \in B^{n+1}$, which completes the proof of Lemma 3.1.

Proof of Theorem 3.1. Let

$$R_n(z) := \mathrm{E}\left|f(z)-f(0)-\sum_{0<|\alpha|<n} I_\alpha(f_\alpha(0), z)\right|^2 = \mathrm{E}\left|\sum_{|\alpha|=n} I_\alpha(f_\alpha, z)\right|^2.$$

By Buniakovsky's inequality and from (3.1) and (3.3) we have

$$R_n(z) \leqslant 7^n \sum_{|\alpha|=n} \mathrm{E}|I_\alpha(f_\alpha, z)|^2$$

$$\leqslant A_1 \cdot 7^n \sum_{|\alpha|=n} [B_1 \lambda_1 \otimes \lambda_2(z)]^{\psi(\alpha)}/(n!) \quad \text{for any } z \leqslant z_0.$$

Hence, by the inequality $|\alpha| \leqslant \psi(\alpha) \leqslant 2|\alpha|$ for any $\alpha \in A$ and the equality $\mathrm{card}\{\alpha \in A\colon |\alpha| = n\} = 7^n$ for $n \geqslant 1$, we obtain

$$\sup_{z \leqslant z_0} R_n(z) \leqslant A_1 \cdot 7^{2n}[B_1 \lambda_1 \otimes \lambda_2(z_0) \vee 1]^{2n}/(n!)$$

$$= A_1 \cdot B_2^n/(n!),$$

where $B_2 = \{7[B_1 \lambda_1 \otimes \lambda_2(z_0) \vee 1]\}^2$ and $a \vee b := \max\{a, b\}$, which completes the proof of Theorem 3.1.

Remark 3.1. The expression

$$a_0 + \sum_{0<|\alpha|\leqslant n} a_\alpha I_\alpha(z), \quad \text{where } a_0, a_\alpha \in \boldsymbol{R},$$

can be considered as a *stochastic polynomial of degree n*. Thus (3.1) is a sufficient condition for the approximation of a process by stochastic polynomials.

COROLLARY 3.1. *Let $f \in C^\infty$ and suppose that there exists a constant $B_3 \geqslant 1$ such that*

$$\sup_{x \in \boldsymbol{R}} |D^k f(x)| \leqslant B_3^k \quad \text{for } k = 1, 2, \ldots \tag{3.5}$$

Then for any $M \in m_{cd}^4$ and any $z_0 \in \boldsymbol{R}_+^2$ we get

$$\lim_{n\to\infty} \sup_{z \in R_{z_0}} \mathrm{E}\left|f(M_z) - \sum_{0<|\alpha|\leqslant n} D^{\|\alpha\|} f(0) I_\alpha(z)\right|^2 = 0. \tag{3.6}$$

Proof of Corollary 3.1. By (3.5) we have

$$\sup_{z \leqslant z_0} |D^k f(M_z)| \leqslant B_3^k \quad \text{for } k \geqslant 1.$$

Hence

$$\sup_{z \leqslant z_0} \mathrm{E}|D^{\|\alpha\|} f(M_z)|^2 \leqslant B_3^{2\|\alpha\|} \leqslant B_3^{4\psi(\alpha)} \quad \text{for any } \alpha \in A \backslash \{\emptyset\}. \tag{3.7}$$

It follows from (3.7) and Theorem 3.1 that for any $z_0 \in \boldsymbol{R}_+^2$ and every $n \geqslant 1$

$$R_n := \sup_{z \leqslant z_0} \mathrm{E}\left|f(M_z) - \sum_{0<|\alpha|\leqslant n} D^{\|\alpha\|} f(0) I_\alpha(z)\right|^2 \leqslant B_4^{n+1}/((n+1)!), \tag{3.8}$$

where $B_4 = [7(B_3^4 \lambda_1 \otimes \lambda_2(z_0) \vee 1)]^2$.

By (3.8) we get $\lim_{n\to\infty} R_n = 0$, which completes the proof of Corollary 3.1.

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