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THE CLUSTER SET OF $\{S_n(2nLLn)^{-1/2};\ n\in\mathcal{N}\}$ IN BANACH SPACES

BY

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Abstract. Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed random vectors with values in a Banach space E, weak mean zero and weak second moment. Let $S_n = X_1 + \ldots + X_n$ and let K_μ be the unit ball of the reproducing kernel Hilbert space associated with $\mu = \mathscr{L}(X_1)$. We show that for any infinie set $\mathcal{N}$ of positive integers the cluster set of $\{S_n(2n\log\log n)^{-1/2};\ n\in\mathcal{N}\}$ equals almost surely αK_μ, where α satisfies $0 \leqslant \alpha \leqslant 1$ and can be determined in terms of $\mathcal{N}$ and μ by the convergence of certain series.

1. Introduction. Let E be a separable Banach space and let $X, X_1, X_2, \ldots$ be a sequence of independent identically distributed (i.i.d.) E-valued random vectors. S_n will denote $X_1+X_2+\ldots+X_n$, and a_n will denote the LIL normalizing sequence, i.e.

$$a_n = (2nLLn)^{1/2}, \quad \text{where } Ln = \begin{cases} \log n & \text{if } n \geqslant 3, \\ 1 & \text{if } n = 1, 2. \end{cases}$$

We will write $X \in WM_0^2$ if $Ef(X) = 0$ and $Ef^2(X) < \infty$ for every $f \in E^*$.

Let $X \in WM_0^2$, $\mu = \mathscr{L}(X)$. For every $f \in E^*$ define

$$Sf = \int x f(x)\, d\mu(x).$$

Then S is an operator from E^* into E. The completion of the range of S with respect to the norm $\|Sf\|_\mu = \int |f(x)|^2 d\mu(x)$ is called the *reproducing kernel Hilbert space* of μ and is denoted by H_μ. K_μ will denote the unit ball of H_μ. For details on H_μ and K_μ we refer the reader to [2].

For any sequence $(x_n)_{n=1}^\infty$, $C\big((x_n)_{n=1}^\infty\big)$ will denote its cluster set. For any $\mathcal{N} \subset N$ we define

$$B(\mathcal{N}) = \Big\{\beta \geqslant 0;\ \sum_{k=1}^{\infty} k^{-\beta^2} P\{\|S_n/a_n\| < \delta \text{ for some } n \in \mathcal{N} \cap I_k\} = \infty \quad \text{for all } \delta > 0\Big\}.$$

Alexander obtained in [1] the following cluster set result:

PROPOSITION 1.1. *Let $X \in WM_0^2$, $\gamma > 1$, $I_k = [[\gamma^k], [\gamma^{k+1}])$ for $k = 1, 2, \ldots$ and let $\alpha = \sup B(N)$, whenever $B(N) \neq \emptyset$. Then α does not depend on the* *choice of* $\gamma > 1$ *and*

$$C\left(\left(\frac{S_n}{a_n}\right)_{n=1}^{\infty}\right) = \begin{cases} \emptyset & \text{if } B(N) = \emptyset, \\ \alpha K_\mu & \text{if } B(N) \neq \emptyset. \end{cases}$$

It follows from the one-dimensional law of the iterated logarithm (LIL) that the assumption $X \in WM_0^2$ is an obvious necessary condition. The purpose of this paper is to characterize the cluster set of $(S_n/a_n)_{n\in\mathcal{N}}$ for an arbitrary infinite set $\mathcal{N}$ of positive integers. This question has been studied in the real-valued case or finite-dimensional situations by Torrång [6], Weber [7] and the author [5]. In [5] we have also obtained the following infinite-dimensional result. (See Theorem 2.4 in [5].)

PROPOSITION 1.2. *Let* $X \in WM_0^2$ *and let* $\mathcal{N}$ *be any infinite set of positive integers. If*

$$\mathrm{E}\,\|S_n/a_n\| \xrightarrow[n\to\infty]{n\in\mathcal{N}} 0, \tag{1.1}$$

then

$$C\big((S_n/a_n)_{n\in\mathcal{N}}\big) = \varepsilon_1^*(\mathcal{N})K_\mu \ \text{a.s.},$$

where

$$\varepsilon_1^*(\mathcal{N}) = \sup\left\{\varepsilon^*(\mathcal{M});\ \mathcal{M} \subset \mathcal{N},\ \mathcal{M} = \{m_k;\ k = 1, 2, \ldots\},\ \limsup_{k\to\infty} \frac{m_k}{m_{k+1}} < 1\right\}$$

and

$$\varepsilon^*(\mathcal{M}) = \inf\left\{\varepsilon > 0;\ \sum_{k=3}^{\infty} (\log m_k)^{-\varepsilon^2} < \infty\right\}. \ \blacksquare$$

Remark 1.1. (a) Since in finite dimensions the finite second moment implies (1.1), Proposition 1.2 includes the general cluster set statement for subsequences in finite-dimensional spaces.

(b) It can be shown that ε_1^* can be expressed in an alternative form

$$\varepsilon_1^*(\mathcal{N}) = \sup\left\{\beta > 0;\ \sum_{k=1}^{\infty} k^{-\beta^2} I_{\{k: I_k(\gamma)\cap\mathcal{N}\neq\emptyset\}} = \infty\right\}$$

which resembles Alexander's definition of α.

In the following result we characterize the cluster set of $\{S_n/a_n;\ n \in \mathcal{N}\}$ without assuming (1.1). It is therefore a generalization of Proposition 1.1 and Proposition 1.2.

THEOREM 1.1. *Let* $X \in WM_0^2$, $\mathcal{N}$ *be an arbitrary infinite set of positive integers,* $I_n = I_n(\gamma)$ *be defined as in Proposition* 1.1, *and let* $\alpha(\mathcal{N}) = \sup B(\mathcal{N})$ *whenever* $B(\mathcal{N}) \neq \emptyset$. *Then* $\alpha(\mathcal{N})$ *does not depend on the choice of* $\gamma > 1$ *and*

$$C\left(\left(\frac{S_n}{a_n}\right)_{n\in\mathcal{N}}\right) = \begin{cases} \emptyset & \text{if } B(\mathcal{N}) = \emptyset, \\ \alpha(\mathcal{N})K_\mu & \text{if } B(\mathcal{N}) \neq \emptyset. \end{cases}$$

Remark 1.2. If $S_n/a_n \xrightarrow[n\to\infty]{n\in\mathcal{N}} 0$ in probability, then $\alpha(\mathcal{N}) = \varepsilon_1^*(\mathcal{N})$. Indeed, for large k

$$\tfrac{1}{2} I_{\{k: I_k \cap \mathcal{N} \neq \emptyset\}} \leqslant P\{\|S_n/a_n\| < \delta \text{ for some } n \in \mathcal{N} \cap I_k\} \leqslant I_{\{k: I_k \cap \mathcal{N} \neq \emptyset\}};$$

hence, by Remark 1.1 (b), $\alpha(\mathcal{N}) = \varepsilon_1^*(\mathcal{N})$. Thus Propositions 1.1 and 1.2 as well as finite-dimensional results of [5] are all special cases of Theorem 1.1.

2. Characterization of the cluster set. The idea of the proof of Theorem 1.1 is that of Alexander's proof of Proposition 1.1. In fact, some of his lemmas can be applied here directly. The main ingredient of his proof however, i.e. Lemmas 2.15 and 2.16 in [1], can be neither applied nor their proof can be immediately altered for the case of a general set of positive integers $\mathcal{N}$. In our Lemmas 2.5 and 2.6, working with the more general index set, we were able to obtain not only more general but also simplified statements.

It is well known (see [2]) that $X \in WM_0^2$ implies $C((S_n/a_n)_{n=1}^{\infty}) \subset K_\mu$ a.s. Therefore to establish Theorem 1.1 it is enough to prove the following result (cf. Theorem 2.3 in [1]):

THEOREM 2.1. *Let $X \in WM_0^2$ and let $h \in H_\mu$. The following conditions are equivalent:*

(i) $h \in C((S_n/a_n)_{n\in\mathcal{N}})$ *a.s.*

(ii) *For every $\beta < \|h\|_\mu$ [or $\beta = 0$ if $h = 0$] and for every $\delta > 0$ there is $\gamma > 1$ such that*

$$\sum_{k=1}^{\infty} k^{-\beta^2} P\{\|S_n/a_n\| < \delta \text{ for some } n \in I_k(\gamma) \cap \mathcal{N}\} = \infty.$$

(iii) *For every $\beta < \|h\|_\mu$ [or $\beta = 0$ if $h = 0$], for every $\delta > 0$ and for every $\gamma > 1$*

$$\sum_{k=1}^{\infty} k^{-\beta^2} P\{\|S_n/a_n\| < \delta \text{ for some } n \in I_k(\gamma) \cap \mathcal{N}\} = \infty.$$

We introduce first some notation. We try to be consistent with the notation used by Alexander in [1].

We will consider bounded partitions $\Pi = (E_0, E_1, \ldots, E_J)$ of E, i.e. such that E_0 is the only unbounded set in the partition. Let $\xi_0, \xi_1, \ldots, \xi_J$ be independent and such that for every $A \in \mathcal{B}(E)$

$$P(\xi_j \in A) = P(X \in A \mid X \in E_j),$$

where $\mathcal{B}(E)$ denotes the Borel σ-algebra of subsets of E. Let $\eta_0, \eta_1, \ldots, \eta_J$ be independent of $(\xi_j)_{j=0}^J$ and such that

$$(\eta_0, \eta_1, \ldots, \eta_J) \stackrel{\mathcal{D}}{=} (I_{\{X\in E_0\}}, I_{\{X\in E_1\}}, \ldots, I_{\{X\in E_J\}}).$$

Then it is easy to verify that

$$X \stackrel{\mathcal{D}}{=} \sum_{j=0}^{J} \eta_j \xi_j.$$

Let now $\{(\xi_{jk})_{j=0}^{J};\ k = 1, 2, \ldots\}$ be a sequence of independent copies of $(\xi_j)_{j=0}^{J}$ and let $\{(\eta_{jk})_{j=0}^{J};\ k = 1, 2, \ldots\}$ be a sequence of independent copies of $(\eta_j)_{j=0}^{J}$. Assume also that $\{(\xi_{jk})_{j=0}^{J};\ k = 1, 2, \ldots\}$ and $\{(\eta_{jk})_{j=0}^{J};\ k = 1, 2, \ldots\}$ are independent. Then

$$S_n \overset{\mathscr{D}}{=} \sum_{k=1}^{n} \sum_{j=0}^{J} \eta_{jk} \xi_{jk}.$$

Let $\mathscr{I}$ be the σ-algebra generated by $\{X^{-1}(E_j)\}_{j=0}^{J}$, and assume that $P(X \in E_0) > 0$. Then we can define $\mathrm{E}(X \mid \mathscr{I})$ in Pettis' sense. For every $k = 1, 2, \ldots$ let now $\mathscr{I}_k$ denote the σ-algebra generated by $\{X_k^{-1}(E_j)\}_{j=0}^{J}$ and let $X'_k = \mathrm{E}(X \mid \mathscr{I}_k)$ and $X''_k = X_k - X'_k$. Let

$$S'_n = \sum_{i=1}^{n} X'_i \quad \text{and} \quad S''_n = \sum_{i=1}^{n} X''_i.$$

If we assume that

$$X_k = \sum_{j=0}^{J} \eta_{jk} \xi_{jk}, \ k = 1, 2, \ldots, \quad \text{and} \quad S_n = \sum_{k=1}^{n} \sum_{j=0}^{J} \eta_{jk} \xi_{jk}, \ n = 1, 2, \ldots,$$

then

$$X'_k = \sum_{j=0}^{J} \eta_{jk} \mathrm{E}\xi_{jk} \quad \text{and} \quad S'_n = \sum_{k=1}^{n} \sum_{j=0}^{J} \eta_{jk} \xi_{jk}.$$

Let for every n and $j = 1, 2, \ldots, J$

$$T_{jn} = \sum_{k=1}^{n} \eta_{jk}.$$

By comparing the Fourier transforms one can show that for every n

$$\Big(\sum_{k=1}^{n} \eta_{jk} \xi_{jk},\ \sum_{k=1}^{n} \eta_{jk} \mathrm{E}\xi_{jk}\Big)_{j=0}^{J}$$

has the same joint distribution as

$$\Big(\sum_{k=1}^{T_{jn}} \xi_{jk},\ \sum_{k=1}^{T_{jn}} \mathrm{E}\xi_{jk}\Big)_{j=0}^{J}.$$

Hence for every n

$$(S_n, S'_n) = \Big(\sum_{k=1}^{n} \sum_{j=0}^{J} \eta_{jk} \xi_{jk},\ \sum_{k=1}^{n} \sum_{j=0}^{J} \eta_{jk} \mathrm{E}\xi_{jk}\Big) \overset{\mathscr{D}}{=} \Big(\sum_{j=0}^{J} \sum_{k=1}^{T_{jn}} \xi_{jk},\ \sum_{j=0}^{J} \sum_{k=1}^{T_{jn}} \mathrm{E}\xi_{jk}\Big).$$

Therefore we can actually assume that

$$S_n = \sum_{j=0}^{J} \sum_{k=1}^{T_{jn}} \xi_{jk}, \quad S'_n = \sum_{j=0}^{J} \sum_{k=1}^{T_{jn}} \mathrm{E}\xi_{jk},$$

(2.1)

$$S''_n = \sum_{j=0}^{J} \sum_{k=1}^{T_{jn}} (\xi_{jk} - \mathrm{E}\xi_{jk}).$$

Our first lemma is a particular case of Lemma 2.4 in [1].

LEMMA 2.1. *Let $\{F_n;\ n\in\mathcal{N}\}$ be any sequence of events and let $\beta \geqslant 0$. Then the convergence or divergence of $\sum_{k=1}^{\infty} k^{-\beta} P\{F_n$ occurs for some $n\in\mathcal{N}\cap I_k(\gamma)\}$ does not depend on $\gamma > 1$.*

Proof. By an application of Lemma 2.4 in [1] with $F_n = \emptyset$ for $n \notin \mathcal{N}$. ∎

The next lemma is a generalization of Kuelbs' result and it can be proved by a slight modification of the proof of Lemma 4 in [3].

LEMMA 2.2. *Let $\gamma > 1$ and $y\in E$. Then*

$$y\in C(\{S_n/a_n;\ n\in\mathcal{N}\}) \text{ a.s.}$$

if and only if

$$\sum_{k=1}^{\infty} P\{\|S_n/a_n - y\| < \varepsilon \text{ for some } n\in\mathcal{N}\cap I_k(\gamma)\} = \infty \quad \text{for every } \varepsilon > 0. \ ∎$$

LEMMA 2.3 (cf. Proposition 2.6 in [1]). *Let Γ be a bounded partition of E, let $X\in WM_0^2$ and let $\theta > 0$. Suppose that $y\in C(\{S_n/a_n;\ n\in\mathcal{N}\})$ a.s. Then a partition Λ can be chosen so that Λ refines Γ and*

$$\sum_{k=1}^{\infty} P\{\|S_n'/a_n - y\| < \theta \text{ and } \|S_n''/a_n\| < \theta \text{ for some } n\in\mathcal{N}\cap I_k\} = \infty,$$

where S_n' and S_n'' are defined in terms of Λ. ∎

The proof is again a slight modification of the proof of Proposition 2.6 in [1] with Lemma 2.2 used instead of Lemma 4 from [3].

The next lemma follows immediately from Lemma 2.8 in [1].

LEMMA 2.4. *Let $\delta > 0$. If $\gamma - 1$ is small enough, then*

$$\sum_{k=1}^{\infty} P\left\{\left\|\frac{S_n' - S_{n_k}'}{a_{n_k}}\right\| > \delta \text{ for some } n\in\mathcal{N}\cap I_k(\gamma)\right\} < \infty,$$

where $n_k = [\gamma^k]$. ∎

Next we modify Lemma 2.12 of [1].

LEMMA 2.5. *Assume that $X\in WM_0^2$, $\theta > 0$, $0 \leqslant \mu < \beta \leqslant 1$ (or $\mu = \beta = 0$), Λ is a bounded partition of E and for every $\delta > 0$*

$$\sum_{k=1}^{\infty} k^{-\beta} P\{\|S_n/a_n\| < \delta \text{ for some } n\in\mathcal{N}\cap I_k\} = \infty.$$

Then a bounded partition Π can be chosen so that Π refines Λ and

$$\sum_{k=1}^{\infty} k^{-\mu} P\{\|S_n''/a_n\| < \theta \text{ for some } n\in\mathcal{N}\cap I_k\} = \infty. \ ∎$$

We omit the proof which is a repetition of the proof of Lemma 2.12 in [1] with Lemmas 2.1, 2.3 and 2.4 used instead of Lemma 2.4, Proposition 2.6 and Lemma 2.8 of [1], respectively.

Before we will present the last two lemmas we introduce some additional notation that are taken mostly unchanged from Alexander's paper:

$$t_{jn} = [np_j], \text{ where } p_j = P\{X \in E_j\}; \qquad m_{jn} = [8p_j^{1/2} a_n];$$

$$W_n = \{k \geq 0;\ |k - t_{0n}| \leq m_{0n}\};$$

$$R_n = \{(k_1, k_2, \ldots, k_J) \in Z_+^J;\ |k_j - t_{jn}| \leq m_{jn},\ 1 \leq j \leq J\};$$

$$Q_n = W_n \times R_n; \quad U_n = (T_{1n}, \ldots, T_{Jn});$$

$$V_n = (T_{0n}, \ldots, T_{Jn}); \qquad C_k = \bigcap_{n \in I_k} \{V_n \in Q_n\}.$$

Let $Y, Y_1, Y_2, \ldots$ be an independent copy of the sequence $X, X_1, X_2, \ldots$ and let $\tilde{X}, \tilde{X}_1, \tilde{X}_2, \ldots$ be a sequence of i.i.d. random vectors such that

$$P\{\tilde{X} \in A\} = P\{X \in A \mid X \in E_0^c\}.$$

Given a bounded partition Π of E let $Y = Y' + Y''$ and $\tilde{X} = \tilde{X}' + \tilde{X}''$ be the decompositions induced by Π. It is easy to verify that for every k

$$\tilde{X}_k \overset{\mathcal{D}}{=} \sum_{j=1}^{J} \eta_{jk} \xi_{jk}.$$

Thus for every n

$$\sum_{k=1}^{n} \tilde{X}_k \overset{\mathcal{D}}{=} \sum_{k=1}^{n} \sum_{j=1}^{J} \eta_{jk} \xi_{jk}. \tag{2.2}$$

Finally, let us write

$$A_k(\delta) = \{\|S_n''/a_n\| < \delta \text{ for some } n \in \mathcal{N} \cap I_k\},$$

$$B_k(\delta, h) = \{\|S_{n_k}'/a_{n_k} - h\| < \delta\}, \quad \text{where } n_k = [\gamma^k].$$

LEMMA 2.6. *Let* $h \in K_\mu$ *and let*

$$\sigma(\mathrm{E}h^2(X''))^{1/2} < \delta, \quad \textit{where } \sigma^2 = \sup_{\|f\| \leq 1} \mathrm{E}f^2(X),\ \delta > 0.$$

If $p_0 = P(X \in E_0) < 1/100$, *then*

$$\sum_{k=1}^{\infty} \left[P(A_k(\delta)) - P(A_k(5\delta) \mid B_k(\delta, h))\right] < \infty$$

for $\gamma - 1$ *small enough.*

Before we start the proof we make some observations. We shall assume that $p_0 = P(X \in E_0) > 0$. If $p_0 = 0$, some steps of our proof become obviously superfluous. Let $q_0 \in W_{n_k}$, i.e. $|q_0 - n_k p_0| \leq 8p_0^{1/2} a_{n_k}$. Define a stopping time

$N = \inf\{n;\ T_{0n} = q_0\}$. Note that $N \geqslant q_0$ and $T_{0n} = q_0$. It follows from (2.2) that

$$S_N \overset{\mathcal{D}}{=} \sum_{k=1}^{q_0} \xi_{0k} + \sum_{k=1}^{N-q_0} \tilde{X}_k, \quad S''_N \overset{\mathcal{D}}{=} \sum_{k=1}^{q_0} (\xi_{0k} - \mathrm{E}\xi_{0k}) + \sum_{k=1}^{N-q_0} \tilde{X}''_i,$$

$$S''_{N+n-n_k} \overset{\mathcal{D}}{=} S''_N + \sum_{i=1}^{n-n_k} Y''_i \quad \text{for } n > n_k.$$

Let

$$H_n = \sum_{i=1}^{q_0} (\xi_{0i} - \mathrm{E}\xi_{0i}) + \sum_{i=1}^{n_k - q_0} \tilde{X}''_i + \sum_{i=1}^{n-n_k} Y''_i \quad \text{for } n > n_k.$$

Observe that for k large enough we have

$$C_k \overset{\mathrm{df}}{=} \bigcap_{n \in I_k} \{V_n \in Q_n\} \subset \{|N - n_k| < n_k\},$$

and so, by Lemma 2.14 in [1],

$$\text{(2.3)} \qquad \sum_{k=1}^{\infty} P\{|N - n_k| \geqslant n_k\} \leqslant \sum_{k=1}^{\infty} P(C_k^c) < \infty.$$

Proof of Lemma 2.6. The following inequalities hold for sufficiently large k, sufficiently small $\gamma - 1$ and $1 < c < \gamma$:

$$\begin{aligned} P(A_k(\delta)) &\leqslant P(A_k(\delta) \cap \{|N - n_k| < cn_k\}) + P\{|N - n_k| \geqslant cn_k\} \\ &\leqslant P\{S''_n < \delta\gamma a_{n_k} \text{ for some } n \in \mathcal{N} \cap I_k,\ |N - n_k| < cn_k\} + P(C_k^c) \\ &\leqslant P\{\|S''_{N+n-n_k}\| < 2\delta\gamma a_{n_k} \text{ for some } n \in \mathcal{N} \cap I_k,\ |N - cn_k| < cn_k\} \\ &\quad + P\{\|\sum_{i=1}^{|N-n_k|} Y''_i\| > \delta\gamma a_{n_k},\ |N - n_k| < cn_k\} + P(C_k^c) \\ &\leqslant P\{\|H_n\| < 3\delta\gamma a_{n_k} \text{ for some } n \in \mathcal{N} \cap I_k,\ |N - n_k| < cn_k\} \\ &\quad + P\{\max_{n \in I_k} \|H_n - S_{N+n-n_k}\| > \delta\gamma a_{n_k},\ |N - n_k| < cn_k\} \\ &\quad + 2P\{\|\sum_{i=n_k}^{[(1+c)n_k]} Y''_i\| > \tfrac{1}{2}\delta\gamma a_{n_k}\} + P(C_{n_k}^c), \end{aligned}$$

where the last inequality follows from Lemma 2.7 in [1]. Thus

$$\begin{aligned} P(A_k(\delta)) &\leqslant P\{\|H_n\| < 3\delta\gamma a_{n_k} \text{ for some } n \in \mathcal{N} \cap I_k\} \\ &\quad + P\{\|\sum_{i=1}^{|N-n_k|} \tilde{X}''_i\| > \delta\gamma a_{n_k},\ |N - n_k| < cn_k\} \\ &\quad + 2P\{\|\sum_{i=n_k}^{[(1+c)n_k]} Y''_i\| > \tfrac{1}{2}\delta\gamma a_{n_k}\} + P(C_k^c) \\ &\leqslant P\{\|H_n\| < 3\delta\gamma a_{n_k} \text{ for some } n \in \mathcal{N} \cap I_k\} \\ &\quad + 2P\{\|\sum_{i=1}^{[cn_k]} \tilde{X}''_i\| > \tfrac{1}{2}\delta\gamma a_{n_k}\} + 2P\{\|\sum_{i=1}^{[cn_k]} Y''_i\| > \tfrac{1}{2}\delta\gamma a_{n_k}\} + P(C_{n_k}^c). \end{aligned}$$

Putting

$$\lambda_k = 2P\{\|\sum_{i=1}^{[cn_k]} \tilde{X}_i''\| > \tfrac{1}{2}\delta\gamma a_{n_k}\} + 2P\{\|\sum_{i=1}^{[cn_k]} Y_i''\| > \tfrac{1}{2}\delta\gamma a_{n_k}\} + P(C_{n_k}^c)$$

we see that for every $q_0 \in W_{n_k}$

$$P(A_k(\delta)) \leqslant P(A_k(4\delta) \mid T_{0n_k} = q_0) + \lambda_k, \tag{2.4}$$

where $\sum_{k=1}^{\infty} \lambda_k < \infty$ by Lemma 2.13 in [1] and (2.3). Now

$$P(A_k(4\delta) \mid T_{0n_k} = q_0) \leqslant \sum_{r\in R_{n_k}} P(A_k(4\delta) \mid V_{n_k} = (q_0, r)) \times P(U_{n_k} = r \mid T_{0n_k} = q_0) + P(U_{n_k} \notin R_{n_k} \mid T_{0n_k} = q_0), \tag{2.5}$$

where, by (2.21) in [1], $P(U_{n_k} \notin R_{n_k} \mid T_{0n_k} = q_0) \leqslant k^{-2}$ for large k.

Let $q = (q_0, r) \in Q_n$ and let $q' \in Q_{n_k}$ be such that $q_0' = q_0$. We will show that

$$P(A_k(4\delta) \mid V_{n_k} = q') \leqslant P(A_k(5\delta) \mid V_{n_k} = q) + \lambda_k', \quad \text{where } \sum_{k=1}^{\infty} \lambda_k' < \infty. \tag{2.6}$$

We have

$$\begin{aligned} P(A_k(4\delta) \mid V_{n_k} = q') &= P\{\|\sum_{j=0}^{J} \sum_{i=1}^{q_j'} (\xi_{ji} - \mathrm{E}\xi_{ji})\| < 4\delta a_{n_k} \\ &\qquad\qquad \text{for some } n \in \mathcal{N} \cap I_k\} \\ &\leqslant P\{\|\sum_{j=0}^{J} \sum_{i=1}^{q_j} (\xi_{ji} - \mathrm{E}\xi_{ji})\| < 5\delta a_{n_k} \\ &\qquad\qquad \text{for some } n \in \mathcal{N} \cap I_k\} \\ &\quad + P\{\sum_{j=1}^{J} \|\sum_{i=1}^{|q_j - q_j'|} (\xi_{ji} - \mathrm{E}\xi_{ji})\| > \delta a_{n_k}\} \\ &\leqslant P(A_k(5\delta) \mid V_{n_k} = q) \\ &\quad + \sum_{j=1}^{J} P\{\|\sum_{i=1}^{|q_j - q_j'|} (\xi_{ji} - \mathrm{E}\xi_{ji})\| > (\delta/J) a_{n_k}\} \\ &\leqslant P(A_k(5\delta) \mid V_{n_k} = q) \\ &\quad + 2\sum_{j=1}^{J} P\{\|\sum_{i=1}^{2m_{jk}} (\xi_{ji} - \mathrm{E}\xi_{ji})\| > (\delta/2J) a_{n_k}\}, \end{aligned}$$

where the last inequality is a consequence of a version of Lemma 2.7 in [1]. Putting

$$\lambda_k' = 2\sum_{j=1}^{J} P\{\|\sum_{i=1}^{2m_{jk}} (\xi_{ji} - \mathrm{E}\xi_{ji})\| > (\delta/2J) a_{n_k}\},$$

we infer from Lemma 2.13 in [1] that $\sum_{k=1}^{\infty} \lambda_k' < \infty$, and so (2.6) is proved.

Combining (2.5) and (2.6) we get for every $q = (q_0, \ldots, q_J)$

$$P(A_k(4\delta) \mid T_{0n_k} = q_0) \leqslant P(A_k(5\delta) \mid V_{n_k} = q) + \lambda_k' + k^{-2}$$

for sufficiently large k. Therefore from (2.4) we obtain

$$P(A_k(\delta)) \leqslant P(A_k(5\delta) \mid V_{n_k} = q) + \lambda_k + \lambda_k' + k^{-2}.$$

Now

$$P(A_k(\delta)) \leqslant \sum_{q \in Q_{n_k}} P(A_k(\delta)) P(V_{n_k} = q \mid B_k(\delta, h)) + P(V_{n_k} \notin Q_{n_k} \mid B_k(\delta, h)).$$

By (2.23) in [1] we have $P(V_{n_k} \notin Q_{n_k} \mid B_k(\delta, h)) \leqslant k^{-2}$ for k large enough (here the assumption $\sigma(\mathrm{E}h^2(X''))^{1/2} < \delta$ is used). Note also that for every q such that $P(V_{n_k} = q,\ B_k(\delta, h)) > 0$ we have

$$P(A_k(5\delta) \mid V_{n_k} = q) = P(A_k(5\delta) \mid V_{n_k} = q,\ B_k(\delta, h)). \tag{2.7}$$

Thus

$$\begin{aligned} P(A_k(\delta)) &\leqslant \sum_{q \in Q_{n_k}} P(A_k(5\delta) \mid V_{n_k} = q) P(V_{n_k} = q \mid B_k(\delta, h)) + \lambda_k + \lambda_k' + 2k^{-2} \\ &= \sum_{q \in Q_{n_k}} P(A_k(5\delta) \mid V_{n_k} = q,\ B_k(\delta, h)) P(V_{n_k} = q \mid B_k(\delta, h)) \\ &\qquad\qquad + \lambda_k + \lambda_k' + 2k^{-2} \\ &\leqslant P(A_k(5\delta) \mid B_k(\delta, h)) + \lambda_k + \lambda_k' + 2k^{-2}, \end{aligned}$$

where

$$\sum_{k=1}^{\infty} (\lambda_k + \lambda_k' + 2k^{-2}) < \infty. \ \blacksquare$$

LEMMA 2.7 (cf. Lemma 2.16 in [1]). *Let* $h \in K_\mu$ *and let* $\sigma(\mathrm{E}h^2(X''))^{1/2} < \delta$, *where* $\sigma^2 = \sup_{\|f\| \leqslant 1} \mathrm{E}f^2(X)$. *If* $p_0 = P(X \in E_0) < 1/100$, *then*

$$\sum_{k=1}^{\infty} [P(A_k(\delta) \mid B_k(\delta, h)) - P(A_k(5\delta))] < \infty,$$

provided $\gamma - 1$ *is sufficiently small.*

Proof. As in Lemma 2.6 we assume that $p_0 > 0$. Let $q = (q_0, q_1, \ldots, q_J) \in Q_{n_k}$; then by (2.6) for every $r \in R_{n_k}$ we have

$$P(A_k(\delta) \mid V_{n_k} = q) \leqslant P(A_k(2\delta) \mid V_{n_k} = (q_0, r)) + \lambda_k', \quad \text{where } \sum_{k=1}^{\infty} \lambda_k' < \infty.$$

Thus

$$\begin{aligned} &P(A_k(\delta) \mid V_{n_k} = q) \\ &\leqslant \sum_{r \in R_{n_k}} P(A_k(2\delta) \mid V_{n_k} = (q_0, r)) P(V_{n_k} = (q_0, r) \mid T_{0n_k} = q_0,\ U_{n_k} \in R_{n_k}) + \lambda_k' \\ &= P(A_k(2\delta) \mid T_{0n_k} = q_0,\ U_{n_k} \in R_{n_k}) + \lambda_k'. \end{aligned} \tag{2.8}$$

Now

$$P(A_k(\delta) \mid B_k(\delta, h)) \leqslant P(A_k(\delta), V_{n_k} \in Q_{n_k} \mid B_k(\delta, h)) + P(V_{n_k} \notin Q_{n_k} \mid B_k(\delta, h))$$
$$\leqslant \sum_{q \in Q_{n_k}} P(A_k(\delta), V_{n_k} = q \mid B_k(\delta, h)) + P(V_{n_k} \notin Q_{n_k} \mid B_k(\delta, h))$$
$$\leqslant \sum P(A_k(\delta) \mid V_{n_k} = q, B_k(\delta, h)) P(V_{n_k} = q \mid B_k(\delta, h)) + P(V_{n_k} \notin Q_{n_k} \mid B_k(\delta, h)),$$

where the last sum is over all $q \in Q_{n_k}$ for which $P(V_{n_k} = q, B_k(\delta, h)) > 0$. From (2.7) and (2.23) of [1] we obtain

$$P(A_k(\delta) \mid B_k(\delta, h)) \leqslant \sum_{q \in Q_{n_k}} P(A_k(\delta) \mid V_{n_k} = q) P(V_{n_k} = q \mid B_k(\delta, h)) + k^{-2},$$

and, by (2.8),

$$P(A_k(\delta) \mid B_k(\delta, h))$$
$$\leqslant \sum_{q_0 \in W_{n_k}} \sum_{r \in R_{n_k}} P(A_k(2\delta) \mid T_{0n_k} = q_0, U_{n_k} \in R_{n_k}) \times P(V_{n_k} = (q_0, r) \mid B_k(\delta, h)) + k^{-2} + \lambda'_k$$
$$\leqslant \sum_{q_0 \in W_{n_k}} P(A_k(2\delta) \mid T_{0n_k} = q_0, U_{n_k} \in R_{n_k}) \times P(T_{0n_k} = q_0, U_{n_k} \in R_{n_k} \mid B_k(\delta, h)) + k^{-2} + \lambda'_k.$$

Now by (2.22) in [1] we have

$$P(A_k(2\delta) \mid T_{0n_k} = q_0, U_{n_k} \in R_{n_k}) \leqslant \frac{P(A_k(2\delta), T_{0n_k} = q_0)}{P(T_{0n_k} = q_0) P(U_{n_k} \in R_{n_k} \mid T_{0n_k} = q_0)}$$
$$\leqslant P(A_k(2\delta) \mid T_{0n_k} = q_0) \frac{1}{1 - k^{-2}} \leqslant P(A_k(2\delta) \mid T_{0n_k} = q_0) + 2k^{-2}.$$

Therefore

$$P(A_k(\delta) \mid B_k(\delta, h))$$
$$\leqslant \sum_{q_0 \in W_{n_k}} P(A_k(2\delta) \mid T_{0n_k} = q_0) P(T_{0n_k} = q_0 \mid B_k(\delta, h)) + 3k^{-2} + \lambda'_k.$$

In a similar way to that we have proved (2.4) we can show that

$$P(A_k(2\delta) \mid T_{0n_k} = q_0) \leqslant P(A_k(5\delta)) + \lambda''_k, \quad \text{where } \sum_{k=1}^{\infty} \lambda''_k < \infty.$$

Thus

$$P(A_k(\delta) \mid B_k(\delta, h)) \leqslant P(A_k(5\delta)) + \lambda'_k + \lambda''_k + 3k^{-2},$$

where $\sum_{k=1}^{\infty} (\lambda'_k + \lambda''_k + 3k^{-2}) < \infty$. ∎

Proof of Theorem 2.1. (i) $\Rightarrow$ (ii). Let $h \in C((S_n/a_n)_{n\in\mathcal{N}})$ a.s. Since $C((S_n/a_n)_{n\in\mathcal{N}}) \subset K_\mu$ a.s., we can assume that $h \in K_\mu$. Let $\gamma - 1$ be so small that Lemmas 2.4 and 2.7 are satisfied and, for every $y \in E$, $\|y/a_n - h\| < \theta$ for some $n \in I_k$ implies

$$\|y/a_{n_k} - h\| < 2\theta.$$

We choose a partition Γ so that the assumptions of Lemma 2.7 are satisfied. By Lemma 2.3 there is a partition Λ that refines Γ and such that

$$\infty = \sum_{k=1}^{\infty} P(\|S_n'/a_n - h\| < \theta,\ \|S_n''/a_n\| < \theta \text{ for some } n \in \mathcal{N} \cap I_k)$$

$$\leqslant \sum_{k=1}^{\infty} P(\|S_n'/a_{n_k} - h\| < 2\theta,\ \|S_n''/a_n\| < \theta \text{ for some } n \in \mathcal{N} \cap I_k)$$

$$\leqslant \sum_{k=1}^{\infty} P(\|S_{n_k}'/a_{n_k} - h\| < 3\theta,\ \|S_n''/a_n\| < \theta \text{ for some } n \in \mathcal{N} \cap I_k)$$

$$+ \sum_{k=1}^{\infty} P(\|(S_n' - S_{n_k}')/a_{n_k}\| > \theta \text{ for some } n \in \mathcal{N} \cap I_k).$$

It follows then from Lemma 2.4 that

$$\sum_{k=1}^{\infty} P\big(A_k(3\theta),\ B_k(3\theta, h)\big) = \infty.$$

From Lemma 2.10 in [1] we infer that for every $\beta < \|h\|_\mu$ (or $\beta = 0$ if $h = 0$)

$$\sum_{k=1}^{\infty} k^{-\beta^2} P\big(A_k(3\theta) \mid B_k(3\theta, h)\big) = \infty.$$

Since the partition Λ also satisfies the assumption of Lemma 2.7 we obtain

$$\sum_{k=1}^{\infty} k^{-\beta^2} P\big(A_k(15\theta)\big) = \infty.$$

Since X' is finite dimensional with finite second moment, $S_n'/a_n \underset{n\to\infty}{\longrightarrow} 0$ in probability, i.e. $P(B_k(15\theta, 0)) \underset{k\to\infty}{\longrightarrow} 1$. Thus

$$\sum_{k=1}^{\infty} k^{-\beta^2} P\big(A_k(15\theta)\big) P\big(B_k(15\theta, 0)\big) = \infty.$$

By Lemma 2.6 we have

$$\infty = \sum_{k=1}^{\infty} k^{-\beta^2} P\big(A_k(75\theta) \mid B_k(15\theta, 0)\big) P\big(B_k(15\theta, 0)\big)$$

$$\leqslant \sum_{k=1}^{\infty} k^{-\beta^2} P\big(A_k(75\theta),\ B_k(15\theta, 0)\big).$$

Finally from Lemma 2.4 we conclude that

$$\infty = \sum_{k=1}^{\infty} k^{-\beta^2} P(A_k(75\theta),\ \|S_n/a_n\| < 15\theta \text{ for all } n \in \mathcal{N} \cap I_k)$$

$$\leqslant \sum_{k=1}^{\infty} k^{-\beta^2} P(\|S_n/a_n\| < 90\theta \text{ for some } n \in \mathcal{N} \cap I_k).$$

(ii) $\Rightarrow$ (iii). This is an immediate consequence of Lemma 2.1.

(iii) $\Rightarrow$ (i). If $h = 0$, this is a consequence of the Borel–Cantelli Lemma. We shall assume therefore that $h \neq 0$. Suppose that

$$\sum_{k=1}^{\infty} k^{-\beta^2} P(\|S_n/a_n\| < \delta \text{ for some } n \in \mathcal{N} \cap I_k(\gamma)) = \infty$$

for every $\beta < \|h\|_\mu$, $\delta > 0$, $\gamma > 1$. Let $\varepsilon > 0$, $\varphi = (1-\varepsilon)h$ and let $\alpha, \beta, \eta > 0$ satisfy

$$\|h\|_\mu > \beta > \alpha > (1+\eta)\|\varphi\|_\mu.$$

Let $\theta > 0$ and let Λ be a bounded partition of E such that

$$\sigma\big(\mathrm{E}\varphi^2(X'')\big)^{1/2} < \theta. \tag{2.9}$$

We choose $\gamma - 1$ so small that the following three conditions are satisfied:

(1) Lemma 2.4 applies with $\delta = \theta$.

(2) Lemma 2.6 applies with $\delta = \theta$ and $h = \varphi$.

(3) For k large enough

$$\|y/a_{n_k} - \varphi\| < 2\theta \text{ implies } \|y/a_n - \varphi\| < 3\theta \quad \text{for all } n \in I_k \text{ and every } y \in E. \tag{2.10}$$

By Lemma 2.5 there is a bounded partition Π which refines Λ and satisfies

$$\sum_{k=1}^{\infty} k^{-\alpha^2} P(A_k(\theta)) = \infty.$$

Since Π also satisfies (2.9), by Lemma 2.11 in [1] for sufficiently large k we have

$$k^{-\alpha^2} \leqslant 2P(B_k(\theta, \varphi)).$$

Thus by Lemma 2.6 we obtain

$$\sum_{k=1}^{\infty} P(A_k(5\theta), B_k(\theta, \varphi)) = \infty.$$

It follows from (2.10) and Lemma 2.4 that

$$\sum_{k=1}^{\infty} P(\|S_n''/a_n\| < 5\theta,\ \|S_n'/a_n - \varphi\| < 3\theta \text{ for some } n \in \mathcal{N} \cap I_k) = \infty.$$

Therefore

$$\sum_{k=1}^{\infty} P(\|S_n/a_n - \varphi\| < 8\theta \text{ for some } n \in \mathcal{N} \cap I_k) = \infty,$$

and so by Lemma 2.2 we have $\varphi \in C((S_n/a_n)_{n\in\mathcal{N}})$ a.s. Since ε is arbitrary and $C((S_n/a_n)_{n\in\mathcal{N}})$ is a closed set, we conclude that

$$h \in C((S_n/a_n)_{n\in\mathcal{N}}) \text{ a.s. } \blacksquare$$

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