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STOCHASTIC PROCESSES WITH LINEAR CONDITIONAL EXPECTATION AND QUADRATIC CONDITIONAL VARIANCE

BY

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Abstract. Linear conditional expectations and quadratic conditional variances determine a class of stochastic processes with independent increments. Characterizations of the Wiener, Poisson, gamma, negative binomial and hyperbolic secant processes are obtained. Also a result on existence and determination of moments by the first two conditional moments for a sequence of random variables is proved.

1. Introduction. Linear conditional expectations and non-random conditional variances with some additional assumptions on the covariance function are a characteristic property of the Gaussian process. It is the main result of a series of papers: Plucińska [11], Wesołowski [15], Bryc [1]. A similar theorem for infinite Gaussian sequences was proved by Bryc and Plucińska [3].

The conditioning applied in these papers was not only given "the past" – as in the well-known martingale characterizations – but also given the "future" states of the process. This type of conditioning leads to theorems on finite-dimensional distributions of the process with uniform assumptions – concerning moments and conditional moments of the process only. Conditions imposed on trajectories, occurring in each martingale characterization, are omitted.

Since the first results were limited to the Gaussian processes only, a natural question has arised of extending this type of characterization to other processes. This was done for the Poisson process by Bryc [1], then slightly generalized by Wesołowski [16] (see also [20]). A similar characterization for the gamma process is due to Wesołowski [18]. The case of mean-square differentiable processes was considered by Szabłowski [12].

This paper is devoted to a solution of the problem in the case where the conditional variance is a quadratic function of the increments and the conditional expectation is linear. Such a result for infinitely integrable processes was formulated in [18].

It should be stressed that we identify processes only with respect to finite-dimensional distributions. Throughout the whole paper we assume that the stochastic processes and random variables are real valued and are defined on a probability space $(\Omega, \mathcal{F}, P)$. The equations between random variables are assumed to hold with P-probability 1.

2. Characterization by linear conditional expectations and quadratic conditional variances. Let $X = (X_t)_{t\geqslant 0}$ be a square integrable process with independent increments and a strictly increasing variance function. Suppose that for any $0 \leqslant r_1 \leqslant \ldots \leqslant r_n \leqslant r < s < t$, $n = 1, 2, \ldots$, the conditional expectation $\mathrm{E}(X_s \mid X_{r_1}, \ldots, X_{r_n}, X_r, X_t)$ is a linear function and the conditional variance $\mathrm{Var}(X_s \mid X_{r_1}, \ldots, X_{r_n}, X_r, X_t)$ is a polynomial of the second order in $X_{r_1}, \ldots, X_{r_n}, X_r, X_t$. Observe that then the only possible forms of conditional expectations and variances are the following:

$$\mathrm{E}(X_s \mid Y, X_r) = A_1 X_r + A_0, \tag{1}$$

$$\mathrm{E}(X_s \mid Y, X_r, X_t) = B_2 X_t + B_1 X_r + B_0, \tag{2}$$

$$\mathrm{Var}(X_s \mid Y, X_r, X_t) = C_2 (X_t - X_r)^2 + C_1 (X_t - X_r) + C_0, \tag{3}$$

where $Y = (X_{r_1}, \ldots, X_{r_n})$ and $A_0, A_1, B_0, B_1, B_2, C_0, C_1, C_2$ are non-random and depend on $r_1, \ldots, r_n, r, s, t$.

Our aim is to obtain a kind of the converse result. We will prove that the conditions (1)–(3) imply the independence of increments of the process and the relations between C_0, C_1, C_2 determine finite-dimensional distributions of the process. In this way we get a characterization of the processes with independent increments having (these increments) distributions from the natural exponential family with quadratic variance function — see [10]. More precisely, they exhaust the infinitely divisible laws of this family. The special cases of (1)–(3) were considered earlier in [2] and [16], [18], [19].

Let μ be an infinitely divisible distribution. We say that a stochastic process X is a μ (*Wiener*) *type process* if it has independent increments $X_t - X_s$ such that the distribution of some affine function of $X_t - X_s$ is $\mu(N(0, \sqrt{t-s}))$. Now we can formulate our main result.

THEOREM 1. *Let* $X = (X_t)_{t\geqslant 0}$ $(X_0 = 0)$ *be a square integrable, stochastic continuous process such that for any* $0 \leqslant r_1 \leqslant \ldots \leqslant r_n \leqslant r < s < t$, $n = 1, 2, \ldots$, $\mathrm{E}(X_r) = m_r$, $\mathrm{cov}(X_r, X_s) = \mathrm{Var}(X_r) = \sigma_r^2 < \sigma_s^2$ *and the conditions* (1)–(3) *hold with* $B_1 B_2 \neq C_2$.

Assume that there are $0 \leqslant \underline{r}_1 \leqslant \ldots \leqslant \underline{r}_n \leqslant \underline{r} < \underline{s} < \underline{t}$ *such that for* $\underline{C}_i = C_i(\underline{r}_1, \ldots, \underline{r}_n, \underline{r}, \underline{s}, \underline{t})$, $i = 0, 1, 2$, *we have:*

(i) $\underline{C}_2 = 0 = \underline{C}_1$. *Then X is a Wiener type process and, for* $0 \leqslant s \leqslant t$, $x \in \boldsymbol{R}$,

$$\mathrm{E}\{\exp[ix(X_t - X_s)]\} = \exp[i(m_t - m_s)x - (\sigma_t^2 - \sigma_s^2)x^2/2].$$

(ii) $\underline{C}_2 = 0 \neq \underline{C}_1$. *Then X is a Poisson type process and*

$$\mathrm{E}\{\exp[ix(X_t - X_s)]\} = \exp\{ix[(m_t - m_s) + \theta(\sigma_t^2 - \sigma_s^2)] + \lambda(\sigma_t^2 - \sigma_s^2)(e^{i\varrho x} - 1)\},$$

$0 \leqslant s \leqslant t$, $x \in \boldsymbol{R}$, *where*

$$\varrho = a \neq 0, \qquad \lambda = a^{-2} > 0, \qquad \theta = -1/a.$$

(iii) $\underline{C}_2 \neq 0$ *and* $\underline{C}_1^2 > 4\underline{C}_2\underline{C}_0 > 0$. *Then X is a negative binomial type process and*

$$\mathrm{E}\{\exp[ix(X_t - X_s)]\}$$
$$= [p\exp(i\theta_1 x) + (1-p)\exp(i\theta_2 x)]^{-\varrho(\sigma_t^2 - \sigma_s^2)}\exp[i(m_t - m_s)x],$$

$0 \leqslant s \leqslant t$, $x \in \boldsymbol{R}$, *where*

$$\theta_{1,2} = [a \pm (a^2 - 2b)^{1/2}]/2, \qquad \varrho = 2/b > 0,$$
$$p = [1 - a(a^2 - 2b)^{-1/2}]/2, \qquad p(1-p) < 0.$$

(iv) $\underline{C}_2 \neq 0$ *and* $\underline{C}_1^2 = 4\underline{C}_2\underline{C}_0$. *Then X is a gamma type process and*

$$\mathrm{E}\{\exp[x(X_t - X_s)]\} = \exp\{ix[(m_t - m_s) + \theta(\sigma_t^2 - \sigma_s^2)]\}(1 - ix/\alpha)^{-\varrho(\sigma_t^2 - \sigma_s^2)},$$

$0 \leqslant s \leqslant t$, $x \in \boldsymbol{R}$, *where*

$$\theta = -a/b, \qquad \varrho = 2/b > 0, \qquad \alpha = 2/a.$$

(v) $\underline{C}_2 \neq 0$ *and* $\underline{C}_1^2 < 4\underline{C}_2\underline{C}_0$. *Then X is a hyperbolic secant type process and*

$$\mathrm{E}\{\exp[ix(X_t - X_s)]\}$$
$$= \exp\{ix[(m_t - m_s) + \theta(\sigma_t^2 - \sigma_s^2)]\}[\mathrm{ch}(\alpha x) + i\lambda\mathrm{sh}(\alpha x)]^{-\varrho(\sigma_t^2 - \sigma_s^2)},$$

$0 \leqslant s \leqslant t$, $x \in \boldsymbol{R}$, *where*

$$\theta = -a/b, \qquad \alpha = (2b - a^2)^{1/2}/2 \neq 0,$$
$$\varrho = 2/b > 0, \qquad \lambda = a(2b - a^2)^{-1/2}.$$

The quantities

$$a = \frac{2C_2(m_t - m_r) + C_1}{B_1 B_2 - C_2}, \tag{4}$$

$$b = \frac{2C_2[C_2(m_t - m_r)^2 + C_1(m_t - m_r) + C_0]}{(B_1 B_2 - C_2)^2} \tag{5}$$

do not depend on $r_1, \ldots, r_n, r, s, t$.

Remarks. 1. A characterization of the Wiener process given in [19] (Theorem 1) may be easily deduced from the above result. The assumptions imply (1)–(3) with $C_2 = C_1 = 0$, $m_t = 0$, $\sigma_t^2 = t$, $t \geqslant 0$. Consequently, the condition (i) holds.

2. As a corollary to Theorem 1 we obtain also a characterization of the Poisson process mentioned in Section 1 (see [2] and [16]). Indeed, the assumptions in this case yield (1)–(3) with $C_2 = C_0 = 0$, $m_t = t$, $\sigma_t^2 = t$, $t \geqslant 0$. Consequently, $a = C_1/B_1 B_2 = 1$ and the result follows from (ii).

3. Also a characterization of the gamma process obtained in [18] is a consequence of our result. In this case we have $C_0 = C_1 = 0$ and $C_2 = B_1 B_2/(t-r+1)$. Hence by (iv) we have the result with $a = b = 2$.

4. The negative binomial process has been also investigated in other papers as an example of a process with independent increments (see [4], [13], and [8]). The hyperbolic secant law was considered by Laha and Lukacs [7], where its infinite divisibility was proved. This law was thoroughly examined by Harkness and Harkness in [5]. In this paper it was established that it is uniquely determined by its moments. The investigations of Wang [14] lead, in a natural way, to the notion of the hyperbolic secant process.

5. The case $C_2 \neq 0$, $C_1^2 > 4C_2C_0 < 0$ is conjectured to be impossible, however its treatment in [17] does not seem to be satisfying.

The proof of Theorem 1, given in Section 4, is based on a unique determination of the moments of the process. The existence of moments of any order is investigated in Section 3.

3. Determination of higher moments by conditional moments of order up to two. In this section we give an auxiliary result which will be used in the proof of the main result. The theorem on the unique determination of the higher moments and conditional moments by conditional expectation and conditional variance seems to be also of independent interest.

We consider a more general situation. Let $(\xi_k)_{k>0}$ be a sequence of non-degenerate square integrable random variables. We introduce some denotation for $k = 1, 2, \ldots$:

$$\xi_0 = 0, \quad \mathrm{E}_{k-1} = \mathrm{E}(\cdot \mid \xi_0, \xi_1, \ldots, \xi_{k-1}),$$

$$\mathrm{E}_{k-1,k+1} = \mathrm{E}(\cdot \mid \xi_0, \xi_1, \ldots, \xi_{k-1}, \xi_{k+1}), \qquad \varrho_k = \operatorname{corr}(X_{k-1}, X_k).$$

The following theorem is a substantial extension of the result from [2], where a similar general problem was investigated.

THEOREM 2. *Assume that for the sequence* $(\xi_k)_{k>0}$ *the following conditions are fulfilled*:

(6) $$\mathrm{E}_{k-1}(\xi_k) = \alpha_k \xi_{k-1} + \beta_k,$$

(7) $$\mathrm{E}_{k-1,k+1}(\xi_k) = a_k \xi_{k+1} + b_k \xi_{k-1} + c_k,$$

(8) $$\mathrm{Var}(\xi_k \mid \xi_0, \xi_1, \ldots, \xi_{k-1}) = \gamma_k,$$

(9) $$\mathrm{E}_{k-1,k+1}(\xi_k^2) = \underline{a}_k \xi_{k+1}^2 + \underline{b}_k \xi_{k-1} \xi_{k+1} + \underline{c}_k \xi_{k-1}^2 + \underline{d}_k \xi_{k+1} + \underline{e}_k \xi_{k-1} + \underline{f}_k,$$

where the coefficients are such that for $k = 1, 2, \ldots$

(10) $$0 \neq \varrho_{k+1}^2 \neq 1,$$

(11) $$\alpha_{k+1} \underline{a}_k \neq a_k.$$

Then

1° *all the moments of the sequence* $(\xi_k)_{k>0}$ *exist;*

2° *for any* $n, k = 1, 2, \ldots,$ *the conditional moment* $\mathrm{E}_{k-1}(\xi_k^n)$ *is a polynomial of the n-th order in* ξ_{k-1} *with coefficients uniquely determined by the constants from* (6), (7) *and* (9).

Remarks. 1. The above result is a generalization of Proposition 3.1 from [2], where 1° was among the assumptions.

2. A natural conclusion of Theorem 2 is a unique determination of all the moments of the sequence $(\xi_k)_{k>0}$ by the constants from (6), (7), (9).

The proof of the above theorem is preceded by an auxiliary result on integrability.

LEMMA. *If the random variables* X, Y *and* $Z = X\mathrm{E}(Y|X)$ *are integrable and* Y *is non-negative, then also the product* XY *is integrable.*

Proof. For $R_n = |X| I(|X| < n) Y$, $n = 1, 2, \ldots,$ we have $R_n \uparrow |X| Y$ a.s. as $n \to \infty$. On the other hand,

$$\mathrm{E}(R_n) = \mathrm{E}\big(|X| I(|X| < n) \mathrm{E}(Y|X)\big) \uparrow \mathrm{E}(Z) < \infty \quad \text{as } n \to \infty.$$

Consequently, from the theorem on monotonous convergence it follows that the product XY is integrable. ∎

Proof of Theorem 2. Let us assume at first that there is $k_0 \geqslant 1$ such that ξ_{k_0} is discrete and takes less than three values. Observe that (10) implies $\alpha_{k+1} \neq 0$. Since

$$a_k = \frac{(\mathrm{Var}(\xi_k))^{1/2}}{(\mathrm{Var}(\xi_{k+1}))^{1/2}} \varrho_{k+1} \frac{1 - \varrho_k^2}{1 - \varrho_k^2 \varrho_{k+1}^2} \neq 0, \quad k \geqslant 1,$$

from (6) and (7) it follows that each ξ_k is bounded a.s., $k \geqslant 1$. Hence by (11) the result follows from Proposition 3.1 in [2].

Consequently, it suffices to consider the case where for each k the r.v. ξ_k takes at least three values.

From the identity $E_{k-1}(\xi_k) = E_{k-1}(E_{k-1,k+1}(\xi_k))$ we obtain the relation

$$\alpha_k = \alpha_{k+1}\alpha_k a_k + b_k \tag{12}$$

since ξ_k is not degenerate. Let us consider now the equation $E_{k-1}(\xi_k^2) = E_{k-1}(E_{k-1,k+1}(\xi_k^2))$. Evaluate both sides applying to the right-hand side the formula $E_{k-1}(\xi_{k+1}^2) = E_{k-1}(E_k(\xi_{k+1}^2))$. Since ξ_{k-1} takes at least three different values, the functions (of argument ω) 1, ξ_{k-1} and ξ_{k-1}^2 are linearly independent. Consequently, we obtain

$$\alpha_k^2 = \alpha_{k+1}^2\alpha_k^2 \underline{a}_k + \alpha_{k+1}\alpha_k \underline{b}_k + \underline{c}_k, \qquad k = 2, 3, \ldots \tag{13}$$

We will prove that for any $n = 1, 2, \ldots$ the random variable ξ_k is n-integrable and the following formula holds:

$$E_{k-1}(\xi_k^n) = \alpha_k^n \xi_{k-1}^n + P_{k,n-1}(\xi_{k-1}), \tag{14}$$

where $P_{k,n-1}$ is a polynomial of order $n-1$ with coefficients uniquely determined by the constants from (6), (7) and (9), $k = 1, 2, \ldots$

We apply induction with respect to n. For $n = 1, 2$ the formula (14) is implied by (6) and (8), respectively. Now let us assume that the sequence $(\xi_k)_{k>0}$ is m-integrable ($m \geqslant 2$) and the formula (14) is fulfilled for any $n = 1, \ldots, m$, $k = 1, 2, \ldots$ The proof of $(m+1)$-integrability is divided into seven steps. In the first six steps we prove that if ξ_{k-1} is $(m+1)$-integrable, then ξ_{k+1} has the same property. The seventh step proves that ξ_1 and ξ_2 are $(m+1)$-integrable.

STEP 1. *The random variables*
(a) $T_{i+2}(k) = \xi_{k-1}^{i+2}(\xi_k - \alpha_k \xi_{k-1})^{m-i-1}$,
(b) $U_{i+2}(k) = \xi_{k-1}^{i+2}(\xi_{k+1} - \alpha_{k+1}\alpha_k \xi_{k-1})^{m-i-1}$,
(c) $V_{i+1}(k) = \xi_{k-1}\xi_k^{i+1}(\xi_{k+1} - \alpha_{k+1}\xi_k)^{m-i-1}$
are integrable for $i = 0, 1, \ldots, m-2$ *and* $k = 2, 3, \ldots$

Proof. We consider only the case (a) (in (b) and (c) the argumentation is very similar).

(i) Let us take i such that $m-i-1$ is even. Then by the Lemma it suffices to show that $\xi_{k-1}^{i+2}E_{k-1}(\xi_k - \alpha_k\xi_{k-1})^{m-i-1}$ is integrable. From the induction assumption (14) we have

$$\begin{aligned} E_{k-1}(\xi_k - \alpha_k\xi_{k-1})^{m-i-1} &= \sum_{j=0}^{m-i-1} \binom{m-i-1}{j} (-\alpha_k\xi_{k-1})^{m-i-1-j} E_{k-1}(\xi_k^j) \\ &= Q_{k,m-i-2}(\xi_{k-1}), \end{aligned}$$

where Q is a polynomial of order $m-i-2$. Consequently, it follows that $\xi_{k-1}^{i+2}E_{k-1}(\xi_k - \alpha_k\xi_{k-1})^{m-i-1}$ is a polynomial of order m in ξ_{k-1} and as such is integrable.

(ii) Now we take i such that $m-i-1$ is odd. From the Schwartz inequality we have

$$\mathrm{E}\big(|\xi_{k-1}|^{i+2}|\xi_k-\alpha_k\xi_{k-1}|^{m-i-1}\big)$$
$$\leqslant \big(\mathrm{E}(|\xi_{k-1}|^{i+1}|\xi_k-\alpha_k\xi_{k-1}|^{m-i})\big)^{1/2}\big(\mathrm{E}(|\xi_{k-1}|^{i+3}|\xi_k-\alpha_k\xi_{k-1}|^{m-i-2})\big)^{1/2}.$$

It follows from (i) that the right-hand side is finite for $i=1,\ldots,m-3$. For $i=0$, m is even; then the right-hand side is also finite. For $i=m-2$ from the Jensen inequality we have

$$\mathrm{E}_{k-1}|\xi_k-\alpha_k\xi_{k-1}-\beta_k| \leqslant \big(\mathrm{Var}(\xi_k|\xi_0,\ldots,\xi_{k-1})\big)^{1/2} = \gamma_k^{1/2}.$$

Consequently, $\xi_{k-1}^m \mathrm{E}_{k-1}|\xi_k-\alpha_k\xi_{k-1}|$ is integrable and the result follows from the Lemma.

STEP 2. *The random variables*
(a) $W_{i+2}(k) = \xi_{k-1}^{i+2}\xi_{k+1}^{m-i-2}(\xi_{k+1}-\alpha_{k+1}\alpha_k\xi_{k-1})$,
(b) $Z_{i+2}(k) = \xi_{k-1}^{i+2}\xi_k^{m-i-2}(\xi_k-\alpha_k\xi_{k-1})$,
(c) $\xi_{k-1}\xi_k\xi_{k+1}^{m-2}(\xi_{k+1}-\alpha_{k+1}\xi_k)$
are integrable for $i=0,1,\ldots,m-2$ *and* $k=2,3,\ldots$

Proof. It is a consequence of Step 1 and the relations

$$W_{i+2}(k) = \sum_{j=0}^{m-i-2}\binom{m-i-2}{j}(-\alpha_{k+1}\alpha_k)^j U_{j+i+2}(k),$$

$$Z_{i+2}(k) = \sum_{j=0}^{m-i-2}\binom{m-i-2}{j}(-\alpha_k)^j T_{j+i+2}(k),$$

$$\xi_{k-1}\xi_k\xi_{k+1}^{m-2}(\xi_{k+1}-\alpha_{k+1}\xi_k) = \sum_{j=0}^{m-2}\binom{m-2}{j}(-\alpha_{k+1})^j V_{j+1}(k).$$

STEP 3. *The random variables*
(a) $\xi_{k-1}\xi_{k+1}^{m-2}(\xi_{k+1}-\alpha_{k+1}\alpha_k\xi_{k-1})^2$,
(b) $\xi_{k-1}\xi_k^{m-2}(\xi_k-\alpha_k\xi_{k-1})^2$,
are integrable for $k=2,3,\ldots$

Proof. Step 2 (c) implies the integrability of

$$\xi_{k-1}\xi_{k+1}^{m-2}[\xi_{k+1}\mathrm{E}_{k-1,k+1}(\xi_k)-\alpha_{k+1}\mathrm{E}_{k-1,k+1}(\xi_k^2)] \tag{15}$$
$$= x\xi_{k-1}\xi_{k+1}^{m-2}(\xi_{k+1}-\alpha_{k+1}\alpha_k\xi_{k-1})^2+yW_2(k)+v,$$

where v is an integrable random variable, and

$$x = a_k-\alpha_{k+1}\underline{a}_k,\qquad y = \alpha_k+\alpha_{k+1}\alpha_k a_k-\alpha_{k+1}b_k-2\alpha_{k+1}^2\alpha_k\underline{a}_k.$$

To obtain (15) we make use of the assumptions (7), (9) and the relations (12) and (13). Now from (11) we get (a).

The part (b) is a consequence of (a). It suffices to apply the induction assumption (14) to the integrable random variable $\xi_{k-1}\mathrm{E}_k[\xi_{k+1}^{m-2}(\xi_{k+1} -\alpha_{k+1}\alpha_k\xi_{k-1})^2]$.

STEP 4. *The random variables $W_1(k)$ and $Z_1(k)$ are integrable for any $k = 2, 3, \ldots$*

Proof. This in an immediate consequence of Steps 2 and 3 and the relations

$$W_1(k) = \xi_{k-1}\xi_{k+1}^{m-2}(\xi_{k+1}-\alpha_{k+1}\alpha_k\xi_{k-1})^2+\alpha_{k+1}\alpha_k W_2(k),$$
$$Z_1(k) = \xi_{k-1}\xi_k^{m-2}(\xi_k-\alpha_k\xi_{k-1})^2+\alpha_k Z_2(k).$$

STEP 5. *The random variable $W_0(k)$ is integrable for any $k = 2, 3, \ldots$*

Proof. From Step 4 we infer that the random variable $\xi_k\xi_{k+1}^{m-1}(\xi_{k+1}-\alpha_{k+1}\xi_k)$ is integrable for $k = 2, 3, \ldots$ Hence, as in Step 3 (see the equation (15)), we get the integrability of the random variable

$$\xi_{k+1}^{m-1}(\xi_{k+1}\mathrm{E}_{k-1,k+1}(\xi_k)-\alpha_{k+1}\mathrm{E}_{k-1,k+1}(\xi_k^2)) = x\xi_{k+1}^{m-1}(\xi_{k+1}-\alpha_{k+1}\alpha_k\xi_{k-1})^2+yW_1(k)+v.$$

Hence $\xi_{k+1}^{m-1}(\xi_{k+1}-\alpha_{k+1}\alpha_k\xi_{k-1})^2$ is integrable and, finally, the integrability of $W_0(k)$ follows from the relation

$$W_0(k) = \xi_{k+1}^{m-1}(\xi_{k+1}-\alpha_{k+1}\alpha_k\xi_{k-1})^2+\alpha_{k+1}\alpha_k W_1(k).$$

STEP 6. *If ξ_{k-1} is $(m+1)$-integrable, then ξ_{k+1} is also $(m+1)$-integrable, $k = 2, 3, \ldots$*

Proof. From Steps 2 (a), 4 and 5 it follows that the random variable

$$(\xi_{k+1}-\alpha_{k+1}\alpha_k\xi_{k-1})^{m+1} = \sum_{j=0}^{m}\binom{m}{j}(-\alpha_{k+1}\alpha_k)^j W_j(k)$$

is integrable. The result follows now from the identity

$$\xi_{k+1} = (\xi_{k+1}-\alpha_{k+1}\alpha_k\xi_{k-1})+\alpha_{k+1}\alpha_k\xi_{k-1}.$$

STEP 7. *The random variables ξ_1 and ξ_2 are $(m+1)$-integrable.*

Proof. The random variable $Z_1(2)$ is equal to $\xi_1\xi_2^{m-1}(\xi_2-\alpha_1\xi_1)$. From Step 4 we get the integrability of

$$\xi_2^{m-1}(\xi_2\mathrm{E}_{0,2}(\xi_1)-\alpha_2\mathrm{E}_{0,2}(\xi_1^2)) = \xi_2^{m+1}(a_1-\alpha_2\underline{a}_1)+v,$$

where v is an integrable random variable. Consequently, ξ_2 is $(m+1)$-integrable. Now we will show that $\xi_2^m(\xi_2-\alpha_2\xi_1)$ is integrable. By the Lemma, to this end it suffices to prove the integrability of $\xi_2^m\mathrm{E}_{0,2}|\xi_2-\alpha_2\xi_1|$. And this is a consequence of the Jensen inequality

$$|\xi_2|^m\mathrm{E}_{0,2}|\xi_2-\alpha_2\xi_1| \leqslant |\xi_2|^m(\mathrm{E}_{0,2}(\xi_2-\alpha_2\xi_1)^2)^{1/2} \leqslant c|\xi_2|^{m+1}+v,$$

where c is a constant and v is an integrable random variable. Hence $\xi_1 = \alpha_2^{-1}[\xi_2 - (\xi_2 - \alpha_2\xi_1)]$ is $(m+1)$-integrable.

Steps 6 and 7 imply the $(m+1)$-integrability of the sequence $(\xi_k)_{k>0}$.

In the second part of the induction argumentation we prove (14) for $n = m+1$.

Applying (7), (9) and the induction assumption (14) in the identities

$$\mathrm{E}_{k-1}[\mathrm{E}_k(\xi_{k+1}^m)\xi_k] = \mathrm{E}_{k-1}[\xi_{k+1}^m \mathrm{E}_{k-1,k+1}(\xi_k)], \tag{16}$$

$$\mathrm{E}_{k-1}[\mathrm{E}_k(\xi_{k+1}^{m-1})\xi_k^2] = \mathrm{E}_{k-1}[\xi_{k+1}^{m-1} \mathrm{E}_{k-1,k+1}(\xi_k^2)], \tag{17}$$

we obtain the following system of linear equations:

$$\alpha_{k+1}^m x - a_k y = \alpha_{k+1}^m \alpha_k^m b_k \xi_{k-1}^{m+1} + P_{k,m}(\xi_{k-1}),$$

$$\alpha_{k+1}^{m-1} x - \underline{a}_k y = \alpha_{k+1}^{m-1} \alpha_k^{m-1}(\alpha_{k+1}\alpha_k \underline{b}_k + \underline{c}_k)\xi_{k-1}^{m+1} + Q_{k,m}(\xi_{k-1})$$

with $x = \mathrm{E}_{k-1}(\xi_k^{m+1})$, $y = \mathrm{E}_{k-1}(\xi_{k+1}^{m+1})$ and the determinant $\det = \alpha_{k+1}^{m-2}(a_k - \alpha_{k+1}\underline{a}_k) \neq 0$ (from (10) and (11)), where $P_{k,m}$ and $Q_{k,m}$ are polynomials of order m and of uniquely determined coefficients, $k = 2, 3, \ldots$ We solve this system and get

$$x = \det{}^{-1}\alpha_{k+1}^{m-2}\alpha_k^{m-2}[a_k(\alpha_{k+1}\alpha_k\underline{b}_k + \underline{c}_k) - \alpha_{k+1}\alpha_k\underline{a}_k b_k]\xi_{k-1}^{m+1} + P_{k,m}(\xi_{k-1}).$$

Applying in this equation the relations (12) and (13) we get (14) for $n = m+1$. ∎

4. Proof of the main result. Without loss of generality we may assume that $m_t = 0$, $t \geqslant 0$. Let $(s_k)_{k>0}$ be a strictly increasing sequence with $s_0 = 0$. Let us put $\xi_k = X_{s_k}$, $k = 0, 1, \ldots$ The sequence $(\xi_k)_{k>0}$ is square integrable. Observe that the conditions (6), (7) and (9) hold by (1)–(3) with

$$\alpha_k = A_1, \quad \beta_k = A_0,$$

$$a_k = B_2, \quad b_k = B_1, \quad c_k = B_0,$$

$$\underline{a}_k = B_2^2 + C_2, \quad \underline{b}_k = 2(B_1B_2 - C_2), \quad \underline{c}_k = B_1^2 + C_2,$$

$$\underline{d}_k = -\underline{e}_k = C_1, \quad \underline{f}_k = c_0$$

$(r_1 = s_0, r_2 = s_1, \ldots, r_n = s_{k-2}, r = s_{k-1}, s = s_k, t = s_{k+1})$, $k = 1, 2, \ldots$ It is not difficult to evaluate the constants involved:

$$A_1 = 1, \quad A_0 = 0, \quad B_2 = \frac{\sigma_t^2 - \sigma_s^2}{\sigma_t^2 - \sigma_r^2}, \quad B_1 = 1 - B_2,$$

$$B_0 = 0, \quad \sigma_s^2 = (B_2^2 + C_2)\sigma_t^2 + (B_1^2 + 2B_1B_2 - C_2)\sigma_r^2 + C_0.$$

Hence (10) follows from the relation $\varrho_{k+1}^2 = \sigma_{s_k}^2/\sigma_{s_{k+1}}^2$, $k = 1, 2, \ldots$, and (11) is a consequence of the assumption $B_1B_2 \neq C_2$.

To prove (8) consider two equations

$$\mathrm{E}(X_s^2 \mid Y, X_r) = \mathrm{E}\big(\mathrm{E}(X_s^2 \mid Y, X_r, X_t) \mid Y, X_r\big),$$

$$\mathrm{E}\big(X_s \mathrm{E}(X_t \mid Y, X_r, X_s) \mid Y, X_r\big) = \mathrm{E}\big(\mathrm{E}(X_s \mid Y, X_r, X_t) X_t \mid Y, X_r\big).$$

The application of (1)–(3) to these formulas leads to

$$\mathrm{E}(X_t^2 \mid Y, X_r) = X_r^2 + C_0(B_1 B_2 - C_2)^{-1} \tag{18}$$

for any $0 \leqslant r_1 \leqslant \ldots \leqslant r_n < r < s < t$. Consequently, the equation (8) also follows with $\gamma_k = \sigma_s^2 - \sigma_r^2$.

Now by Theorem 2 we conclude that for any $m = 1, 2, \ldots$ the process X is m-integrable and the conditional moments $\mathrm{E}\big((X_t - X_r)^m \mid Y, X_r\big)$ are uniquely determined by $(\sigma_t^2)_{t>0}$ and the coefficients from (1)–(3).

Observe that the processes defined in the parts (i)–(v) of Theorem 1 fulfil the conditions (1)–(3) with a and b given in (4) and (5). Also each of conditional distributions of an increment for these processes is uniquely determined by the sequence of moments of the increment. Consequently, the uniqueness obtained in the first part of the proof yields that to prove our theorem it suffices to show that a and b defined by (4) and (5) do not depend on $r_1, \ldots, r_n, r, s, t, n \geqslant 1$. (Hence a and b are constant for the whole process and determine exactly one of the conditions (i)–(v) for the process not only for some $\underline{r}_1, \ldots, \underline{r}_n, \underline{r}, \underline{s}, \underline{t}$.)

To this end observe that, by (18),

$$\sigma_t^2 - \sigma_r^2 = C_0(B_1 B_2 - C_2)^{-1}.$$

Now apply (1)–(3) and (18) to

$$\mathrm{E}\big(X_s \mathrm{E}(X_t^2 \mid Y, X_r, X_s) \mid Y, X_r\big) = \mathrm{E}\big(\mathrm{E}(X_s \mid Y, X_r, X_t) X_t^2 \mid Y, X_r\big),$$

$$\mathrm{E}\big(X_s^2 \mathrm{E}(X_t \mid Y, X_r, X_s) \mid Y, X_r\big) = \mathrm{E}\big(\mathrm{E}(X_s^2 \mid Y, X_r, X_t) X_t \mid Y, X_r\big).$$

After some computations we get

$$\mathrm{E}(X_t^3 \mid Y, X_r) = X_r^3 + 3C_0(B_1 B_2 - C_2)^{-1} X_r + C_0 C_1 (B_1 B_2 - C_2)^{-2}. \tag{19}$$

Observe that, on the other hand, by

$$\mathrm{E}(X_r X_t^2) = \mathrm{E}\big(\mathrm{E}(X_r | X_t) X_t^2\big) = \mathrm{E}\big(X_r \mathrm{E}(X_t^2 | X_r)\big),$$

for any $t > 0$ we have

$$\mathrm{E}(X_t^3)\sigma_t^{-2} = \alpha = \mathrm{const}.$$

Now (19) implies $\alpha = C_1(B_1 B_2 - C_2)^{-1} = a$ by definition. Also, by (19), from the equations

$$\mathrm{E}(X_r X_t^3) = \mathrm{E}\big(\mathrm{E}(X_r | X_t) X_t^3\big) = \mathrm{E}\big(X_r \mathrm{E}(X_t^3 | X_r)\big)$$

for any $t > 0$ we have

$$\mathrm{E}(X_t^4)\sigma_t^{-2} - 3\sigma_t^2 = \beta = \mathrm{const}. \tag{20}$$

Consider once again a pair of equations

$$\mathrm{E}(X_s\mathrm{E}(X_t^3 \mid Y, X_r, X_s) \mid Y, X_r) = \mathrm{E}(\mathrm{E}(X_s \mid Y, X_r, X_t)X_t^3 \mid Y, X_r),$$

$$\mathrm{E}(X_s^2\mathrm{E}(X_t^2 \mid Y, X_r, X_s) \mid Y, X_r) = \mathrm{E}(\mathrm{E}(X_s^2 \mid Y, X_r, X_t)X_t^2 \mid Y, X_r).$$

By (1)–(3), (18) and (19) applied to the above formulas we evaluate the conditional moment of the fourth order:

$$\mathrm{E}(X_t^4 \mid Y, X_r) = X_r^4+6(\sigma_t^2-\sigma_r^2)X_r^2+4a(\sigma_t^2-\sigma_r^2)X_r$$
$$+a^2(\sigma_t^2-\sigma_r^2)+(\sigma_t^2-\sigma_r^2)+2\frac{(\sigma_t^2-\sigma_s^2)(\sigma_s^2-\sigma_r^2)}{B_1B_2-C_2}.$$

Hence (20) implies $\beta = a^2+2C_0C_2(B_1B_2-C_2)^{-2}$ and by the definition (5) we get $b = \beta-a^2 = \text{const.}$ ∎

Remark. The general recurrence formula for the conditional moments of the increments was obtained in [17].

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