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QUEUES WITH BREAKDOWNS AND CUSTOMER DISCOURAGEMENT

BY

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Abstract. This paper considers an $M/M/1$ queue with service breakdowns and customer discouragement. Each of the customers present in the system at the time of a breakdown may become discouraged and leave with a constant probability, independently of other customers. The system alternates between working and repair periods. Formulas are found for the expected queue size at the end of a working-repair cycle. The system is shown to have a stationary distribution if the probability of discouragement is positive.

This paper will consider an $M/M/1$ queue with server breakdowns and customer discouragement. Thus, the system alternates between working and repair periods. We assume that the durations, say $S_1, R_1, S_2, R_2, \ldots$, of service and repair periods are random variables independent of queue size and of each other. Moreover, we assume that $S_1, S_2, \ldots$ are exponential with $\mathrm{E}[S] = 1/\alpha$, while $R_1, R_2, \ldots$ are also exponential with $\mathrm{E}[R] = 1/\beta$.

During the working/service periods, the system operates as an $M/M/1$ queue with rates λ and μ. During the breakdown/repair periods, no customers are served.

Each of the customers present in the system at the time of the breakdown may become discouraged and leave with probability $1-\delta$, independently of other customers. Consequently, letting $X(t)$ denote the number of customers in the system at time t, if a breakdown occurs at time T, then

$$X(T^+) \sim \mathrm{Bin}\big(X(T), \delta\big).$$

Finally, during the breakdown/repair periods customers still arrive at Poisson rate λ, but each may become discouraged and fail to join the queue with probability $1-\theta$, independently of the state of the system. Fig. 1 shows the transitions between states for our model along with their intensities during a single working/repair cycle.

The above concept of discouragement is the feature that distinguishes our model from other queueing models with breakdowns, vacations or priority queues. A thorough review of these models may be found in [5].

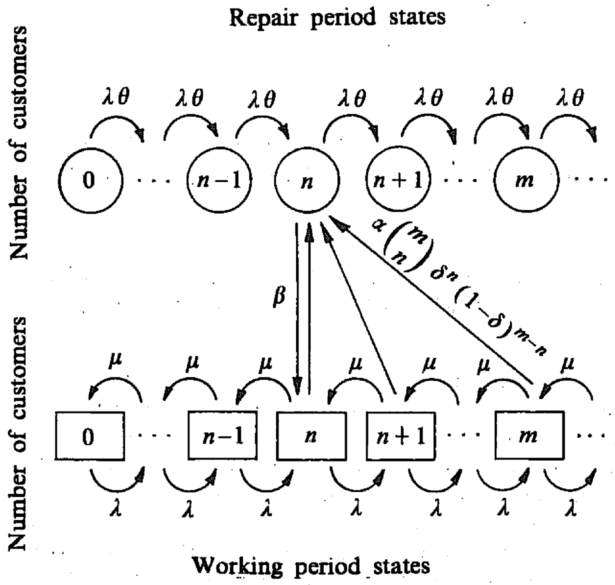

Fig. 1. Transitions between states during one cycle of the $M/M/1$ queue with breakdowns and customer discouragement

When $\delta = 1$ (no discouragement), our system may be put into the context of a queue with stochastic variation [6] or of an $M/M/1$ queue with heterogeneous arrivals and service [11]. The case $\delta = 1$ is also an example of a Markov modulated queue with arrival rates switching from λ to $\lambda\theta$ and service rates switching from μ to 0 (see, e.g., [9]). Markov modulated queues in which the service rate may periodically change have been used to model situations involving server vacations or the arrivals of high priority customers. Our model is motivated by the realistic assumption that some customers may decide to leave as a result of a drop in service rate. For example, in barber shops, auto repair shops, or medical clinics, there may be a preferred server for whom loyal customers form a queue. When the preferred server is called away, his/her customers may be offered an alternative server, causing a binomial depletion of the queue. The analytical results of this paper extend trivially to the more realistic situation where δ is a function of β, so that customers may base their decision about leaving on an estimate of the length of time the preferred server will be away. Examples of priority queues are seen in subcontracted manufacturing, computing and other areas where a high priority customer often requiring an excessive service time may interrupt the regular flow of a queue. The arrival of such a customer may cause a binomial depletion of the regular customer queue.

Finally, note the difference between customer discouragement, where customers leave when the system is not operational, and the well-developed notion of customer impatience, where customers leave after a prolonged wait regardless of the operational status of the system (see, e.g., [4] or [8]).

In this paper we shall investigate:

(a) the numbers of customers in the system at the ends of the repair periods,

(b) the stationarity of the number of customers in the system,

(c) the probability that a customer is eventually served, and

(d) the waiting time until service for a "stubborn" customer, i.e. one who refuses to become discouraged.

Some of the results will be extended to the case of $G/G/1$ queue and/or arbitrary distributions of the durations of the working and repair periods.

1. System status at the ends of repair periods. In this section, we shall consider the embedded process $\{X_n, n = 0, 1, \dots\}$ defined by

$$X_0 = X(0), \qquad X_n = X\Big(\sum_{i=1}^{n} (S_i + R_i)\Big) := X(T_n).$$

Thus, X_n is the number of customers in the system at time T_n which is the end of the n-th repair period (or the beginning of the $(n+1)$-st working period).

Since $\{X_n\}$ is a Markov chain, it is natural to start by determining the one-step transition probabilities. We begin by finding the conditional probability generating function for X_n given X_{n-1}.

THEOREM 1.

$$\mathrm{E}[z^{X_n} \mid X_{n-1} = x] = \frac{\alpha\beta}{\beta + \lambda\theta - \lambda\theta z} P_x^*(\delta z + 1 - \delta, \alpha), \tag{1}$$

where P_x^* is the Laplace transform of the probability generating function of the number of customers in the system for a regular $M/M/1$ queue which starts with x individuals.

Comment. Thus, letting $\xi(t)$ be an $M/M/1$ process,

$$(2) \quad P_x^*(u, v) = \int_0^\infty e^{-vt} \mathrm{E}[u^{\xi(t)} \mid \xi(0) = x]\, dt$$

$$= \{(1-u)[\phi(v)]^{x+1}[1-\phi(v)]^{-1} - u^{x+1}\}/\{\lambda[u-\phi(v)][u-\psi(v)]\}$$

with

$$\phi(v) = \{(\lambda+\mu+v) - \sqrt{(\lambda+\mu+v)^2 - 4\lambda\mu}\}/2\lambda$$

and

$$\psi(v) = \{(\lambda+\mu+v) + \sqrt{(\lambda+\mu+v)^2 - 4\lambda\mu}\}/2\lambda$$

(see [2]).

From (1) it may be seen that we shall need the values of the functions $\phi(\cdot)$ and $\psi(\cdot)$ only at α, so that in the sequel we let $\phi = \phi(\alpha)$ and $\psi = \psi(\alpha)$.

Proof. We condition on the durations $R_n = r$ and $S_n = s$ of the n-th repair and service periods. We have

$$\begin{aligned}
E(z^{X_n} \mid X_{n-1} = x) &= \iint \alpha\beta e^{-(\alpha s+\beta r)} E[z^{X(T_{n-1}+s+r)} \mid x, s, r]\, dr ds \\
&= \iint \alpha\beta e^{-(\alpha s+\beta r)} E[z^{X(T_{n-1}+s)} \mid x, s] e^{-\lambda\theta r(1-z)} dr ds \\
&= \iint \alpha\beta e^{-(\alpha s+\beta r)} E[(\delta z+1-\delta)^{X((T_{n-1}+s)-)} \mid x, s] e^{-\lambda\theta r(1-z)} dr ds \\
&= \int \alpha\beta e^{-r[\beta+\lambda\theta(1-z)]} dr \int e^{-\alpha s} E[(\delta z+1-\delta)^{X((T_{n-1}+s)-)} \mid x, s]\, ds \\
&= \frac{\alpha\beta}{\beta+\lambda\theta-\lambda\theta z} P_x^*(\delta z+1-\delta, \alpha).
\end{aligned}$$

The second equality comes from the Poisson arrival rate $\lambda\theta$ of customers during the repair period, while the third equality comes from the binomial discouragement at the breakdown. The last equality is due to the fact that the process is an $M/M/1$ queue during a working period. ■

This theorem can be generalized to the case where the lengths of the repair periods have arbitrary distribution with moment generating function $m_R(\cdot)$. In this case, the theorem becomes

$$E[z^{X_n} \mid X_{n-1} = x] = \alpha m_R(\lambda\theta(z-1)) P_x^*(\delta z+1-\delta, \alpha).$$

The corollary below will also be true for this generalization, provided we take $E(R) = m'_R(0) = 1/\beta$.

From the probability generating function (1) we can derive the conditional expectation:

Corollary 1.

(3) $$E(X_n \mid X_{n-1} = x) = \lambda\theta/\beta + \delta[((\lambda-\mu)/\alpha) + x + \phi^{x+1}/(1-\phi)].$$

Proof. Differentiating (1) with respect to z and substituting $z = 1$ we obtain

$$E(X_n \mid X_{n-1} = x) = \frac{\lambda\theta}{\beta} + \alpha\delta \left.\frac{\partial P_x^*(z, \alpha)}{\partial z}\right|_{z=1}.$$

The use of the fact that $(1-\phi)(1-\psi) = -\alpha/\lambda$ simplifies the algebra, which leads to (3). ■

The one-step expressions given in Theorem 1 and Corollary 1 can be exploited inductively to give corresponding multistep formulas. In order to write these formulas we introduce the following notation. Let us define

$$A_i(z) = \delta^i z + 1 - \delta^i \quad \text{for } i = 0, 1, \ldots$$

Next we put

$$B_i(z) = -\alpha\beta A_i(z)/\{\lambda[(\beta+\lambda\theta(1-A_{i-1}(z)))(A_i(z)-\phi)(A_i(z)-\psi)]\}$$

and

$$C_i(z) = \{\alpha\beta\phi[1-A_i(z)]\}/\{\lambda(1-\phi)[\beta+\lambda\theta(1-A_{i-1}(z))][A_i(z)\text{-}\phi][A_i(z)-\psi]\},$$
$$i = 1, 2, \ldots$$

In this notation, formulas (1) and (2) give

$$\text{(4)} \qquad E(z^{X_n} \mid X_{n-1}) = B_1(z)[A_1(z)]^{X_{n-1}} + C_1(z)\phi^{X_{n-1}}.$$

Let us also define D_n's recursively by

$$D_n(n, z) = 1$$

and for $i = n, \ldots, 1$

$$\text{(5)} \quad D_n(n-i, z) = \sum_{m=1}^{i-1} D_n(n-i+m, z)[\prod_{j=1}^{m-1} B_j(\phi)]C_m(\phi) + [\prod_{j=1}^{i-1} B_j(z)]C_i(z).$$

We shall now prove

THEOREM 2. *The probability generating function of X_n is given by*

$$\text{(6)} \qquad H_n(z) = E(z^{X_n} \mid X_0)$$
$$= [\prod_{j=1}^{n} B_j(z)][A_n(z)]^{X_0} + \sum_{i=0}^{n-1} D_n(i, z)[\prod_{j=1}^{i} B_j(\phi)][A_i(\phi)]^{X_0}.$$

Proof. For $n = 1$, the assertion reduces to that of Theorem 1. Assume now that (6) holds for $n \leqslant N$. We write, using (4),

$$\text{(7)} \ H_{N+1}(z) = E[E(z^{X_{N+1}} \mid X_N) \mid X_0]$$
$$= B_1(z)H_N(A_1(z)) + C_1(z)H_N(\phi)$$
$$= B_1(z)\{[\prod_{j=1}^{N} B_j(A_1(z))][A_N(A_1(z))]^{X_0}$$
$$+ \sum_{i=0}^{N-1} D_N(i, A_1(z))[\sum_{j=1}^{i} B_j(\phi)]A_i(\phi)^{X_0}\}$$
$$+ C_1(z)\{[\prod_{j=1}^{N} B_j(\phi)][A_N(\phi)]^{X_0} + \sum_{i=0}^{N-1} D_N(i, \phi)[\prod_{j=1}^{i} B_j(\phi)]A_i(\phi)^{X_0}\}.$$

To complete the proof we use the fact that $A_N(A_1(z)) = A_{N+1}(z)$, $B_j(A_1(z)) = B_{j+1}(z)$ and the definition (5) of $D_n(k, z)$ to verify that the coefficients at $[A_{N+1}(z)]^{X_0}$ and $[A_i(\phi)]^{X_0}$ for $i = 1, 2, \ldots, N$ in the formulas (7) and (6) for $n = N+1$ agree. ∎

Applying an inductive argument to Corollary 1, we arrive at two multistep formulas for the expectation of queue size at the end of the n-th repair period:

$$\text{(8)} \qquad \mathrm{E}(X_n \mid X_0) = \delta^n X_0 + \left[\frac{\lambda\theta}{\beta} + \frac{\delta(\lambda-\mu)}{\alpha}\right]\frac{1-\delta^n}{1-\delta} + \frac{\delta\phi}{1-\phi}\sum_{i=0}^{n-1} \delta^{n-i-1} H_i(\phi)$$

$$= \delta^n X_0 + \left[\frac{\lambda\theta}{\beta} + \frac{\delta(\lambda-\mu)}{\alpha}\right]\frac{1-\delta^n}{1-\delta}$$

$$+ \frac{\delta\phi}{1-\phi}\sum_{i=0}^{n-1} \delta^{n-i-1} \sum_{j=0}^{i} D_n(j, \phi)\Big[\prod_{k=1}^{j} B_k(\phi)\Big][A_j(\phi)]^{X_0}.$$

2. The stationarity of queue size. We now have the following theorem:

THEOREM 3. *The process $X(t)$ has a stationary distribution if and only if either*

(i) $\alpha = 0$ *and* $\lambda < \mu$ *or*
(ii) $\alpha > 0$, $\beta = 0$ *and* $\lambda\theta = 0$ *or*
(iii) $\delta = 1$ *and* $\alpha\lambda\theta + \beta\lambda < \beta\mu$ *or*
(iv) $\delta < 1$ *and* $\alpha\beta > 0$.

Proof. When $\alpha = 0$, our model reduces to the simple $M/M/1$ queue so that $X(t)$ has a stationary distribution if and only if $\lambda < \mu$ (see, e.g., [7]). Similarly, if $\beta = 0$, the repair period will never end, and the process is not stationary except in the trivial case of no arrivals ($\lambda\theta = 0$). Finally, when $\delta = 1$, we have an example of a Markov modulated queue of the type discussed in [10]. In that paper it is shown that

$$\frac{\alpha}{\alpha+\beta}(\lambda\theta) + \frac{\beta}{\alpha+\beta}(\lambda) < \frac{\beta}{\alpha+\beta}(\mu)$$

provides a necessary and sufficient condition for stationarity. Thus, we need only show that the condition $\delta < 1$ and $\alpha\beta > 0$ implies that the process has a stationary distribution.

We shall prove Theorem 3 in two steps. First we will show that the embedded Markov process $\{X_n\}$ is ergodic, so that it has a stationary distribution. In the next step, we shall consider the process $X(t)$ for all t, and show that the fraction of time that the process spends in a given state k converges to a limit, which is the stationary probability of state k.

Firstly, observe that $\alpha > 0$ implies $\phi = \phi(\alpha) < 1$. Consequently, for any $\delta < 1$, the expectation $\mathrm{E}(X_n \mid X_0)$ given in (8) can be bounded by a constant free of n, that is

$$\text{(9)} \qquad \sup_n \mathrm{E}(X_n \mid X_0) \leqslant K < \infty.$$

Now in the Markov chain $\{X_n\}$ every state is accessible from every other state in one step, that is, $P(X_{n+1} = j \mid X_n = i) = p_{ij} > 0$ for all (i, j). It follows that the chain is aperiodic and irreducible (see [7]). To show that $\{X_n\}$ is ergodic, it suffices to show that one state, say 0, is ergodic. Let $p_{00}^{(n)} = P\{X_n = 0 \mid X_0 = 0\}$. It suffices to show that $\liminf p_{00}^{(n)} > 0$, since then the series $\sum_n p_{00}^{(n)}$ will diverge (see [7]). From (9) it follows that if $0 < \gamma < 1$, then for some $A = A(\gamma)$ the inequality $P\{X_n \leqslant A \mid X_0 = 0\} \geqslant 1-\gamma$ holds for all n. Let

$$\tau = \min_{j \leqslant A} P\{X_1 = 0 \mid X_0 = j\}$$

so that $\tau > 0$, since every state is accessible in one step from every other state. We may now write

$$\begin{aligned} p_{00}^{(n)} &= P\{X_n = 0 \mid X_0 = 0\} \\ &= \sum_j P\{X_n = 0 \mid X_{n-1} = j\} P\{X_{n-1} = j \mid X_0 = 0\} \\ &\geqslant \sum_{j \leqslant A} P\{X_n = 0 \mid X_{n-1} = j\} P\{X_{n-1} = j \mid X_0 = 0\} \geqslant \tau(1-\gamma) > 0 \end{aligned}$$

This shows that $\{X_n\}$ is ergodic.

To complete the proof of Theorem 3, let us fix k, and let V_n be the total time spent in state k during the n-th cycle (working and repair periods). The distribution of V_n depends on n only through X_{n-1}, that is,

$$P(V_n \leqslant u \mid X_{n-1} = j) = P(V_m \leqslant u \mid X_{m-1} = j) \qquad \text{for all } j.$$

Moreover, for t in the $(n+1)$-st cycle, let q_t^{n+1} be the total time spent in state k between the beginning of this cycle and t. Consequently, for t in the $(n+1)$-st cycle, the proportion of time spent in state k is

$$\pi_n(t) = \frac{V_1 + \ldots + V_n + q_t^{n+1}}{t}.$$

We shall show that $\pi_n(t)$ converges almost surely to some limit as $t \to \infty$. We have

$$\begin{aligned} \pi_n(t) &\leqslant \frac{V_1 + \ldots + V_{n+1}}{(S_1+R_1) + \ldots + (S_n+R_n)} \\ &= \frac{V_1 + \ldots + V_n}{(S_1+R_1) + \ldots + (S_n+R_n)} + \frac{V_{n+1}}{(S_1+R_1) + \ldots + (S_n+R_n)}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \pi_n(t) &\geqslant \frac{V_1 + \ldots + V_n}{(S_1+R_1) + \ldots + (S_{n+1}+R_{n+1})} \\ &= \frac{V_1 + \ldots + V_n}{(S_1+R_1) + \ldots + (S_n+R_n)} \cdot \frac{(S_1+R_1) + \ldots + (S_n+R_n)}{(S_1+R_1) + \ldots + (S_{n+1}+R_{n+1})}. \end{aligned}$$

Standard argument shows that

$$\frac{(S_1+R_1) + \ldots + (S_n+R_n)}{(S_1+R_1) + \ldots + (S_{n+1}+R_{n+1})} \to 1 \text{ a.s.}$$

Since $V_{n+1} \leqslant S_{n+1}+R_{n+1}$, we also have

$$\frac{V_{n+1}}{(S_1+R_1)+\ldots+(S_n+R_n)} \to 0 \text{ a.s.}$$

It remains to show the a.s. convergence of the ratio

$$\frac{V_1+\ldots+V_n}{(S_1+R_1)+\ldots+(S_n+R_n)}, \tag{10}$$

that is, proportions $\pi_n(t)$ along the subsequence $\{t_i, i = 1, 2, \ldots\}$ of random times, where $t_n = S_1+R_1+\ldots+S_n+R_n$.

Writing (10) in the form $(\sum V_i/n)/[(\sum S_i/n)+(\sum R_i/n)]$ we see that the denominator converges a.s. to $1/\alpha+1/\beta$. In the numerator, we may group the terms according to the values of X_j, so that it becomes

$$(1/n)\sum_{j=0}^{\infty}\sum_{i:X_i=j} V_i = \sum_{j=0}^{\infty} N(j)/n\big[1/N(j)\sum_{i=1}^{N(j)} V_i^{(j)}\big],$$

where $V_i^{(j)}$ is the amount of time spent in state k during the i-th cycle that started with $X_i = j$ out of $N(j)$ such cycles.

$V_1^{(j)}, V_2^{(j)}, \ldots$ are now i.i.d. for each j. Since $N(j) \to \infty$ a.s. as $n \to \infty$, the term in brackets tends a.s. to $\mathrm{E}[V_1 \mid X_0 = j]$, while $N(j)/n$ converge a.s. to the stationary distribution of $\{X_n\}$. Consequently, (10) converges a.s. ∎

A rather remarkable feature of this theorem is that any genuine discouragement at the time of the breakdowns (even $\delta = 0.999$) will imply stationarity, independent of the values of any other parameters. For example, this allows the queue to grow quite rapidly between breakdowns (i.e. large λ, small μ, and θ near 1). The heuristic argument is that discouragement depletes the queue at a level proportional to its size, while other changes in the queue happen at a constant rate.

3. Probability of service and expected waiting times. In our model, the feature of discouragement makes the probability that a customer is eventually served less than one.

Let Q_n be the probability that a customer is eventually served, if he joined the queue at a time when there were n persons in the system (so that he was the $(n+1)$-st in priority to be served). The probability Q_n does not depend on the time when the customer joins the queue: if he joins the queue during a repair period, he will still have the $(n+1)$-st priority at the start of the working period. On the other hand, if he joins the queue during a working period, then the memoryless property of S makes it equivalent to joining the queue at the start of a working period, which will be assumed in the sequel.

Observe first that we have, conditioning on the duration of the first working period,

$$Q_n = \int_0^\infty \alpha e^{-\alpha t} P\{(n+1)\text{-st priority customer is eventually served} \mid S = t\}\, dt$$

$$= \int_0^\infty \alpha e^{-\alpha t}\left\{ \sum_{j=n+1}^{\infty} \frac{(\mu t)^j}{j!} e^{-\mu t} + \delta \sum_{m=0}^{n} \frac{(\mu t)^m}{m!} e^{-\mu t} \sum_{k=0}^{n-m} \binom{n-m}{k} \delta^k (1-\delta)^{n-m-k} \cdot Q_k \right\} dt.$$

In this expression, the first term represents the probability of completing the service during the first working period. In the second term, m stands for a number of persons served during the first working period, while k is the number of customers (in front of the customer in question) who decide to remain in the system at the time of the breakdown. The factor δ in front of the second term corresponds to the fact that the customer in question also has the option of leaving the queue at the beginning of the repair period.

Finally, by evaluating the integral and solving for Q_n, we get

$$(11)\qquad Q_n = \frac{1}{[\mu + \alpha(1-\delta^{n+1})]}\left(\frac{\mu}{\alpha+\mu}\right)^n \times \left\{ \mu + \alpha\delta \sum_{k=0}^{n-1} \left(\frac{\delta}{1-\delta}\right)^k Q_k \sum_{m=k}^{n} \binom{m}{k} \left[\frac{(1-\delta)(\alpha+\mu)}{\mu}\right]^m \right\}.$$

For example, taking $n = 0$ in (11) we get

$$Q_0 = \frac{\mu}{\mu + \alpha(1-\delta)},$$

and taking $n = 1$ gives

$$(12)\qquad Q_1 = \frac{\mu - \alpha\mu/(\alpha+\mu) + \alpha\delta(1-\delta)Q_0 + [\alpha\mu\delta/(\alpha+\mu)]Q_0}{\mu + \alpha(1-\delta^2)}.$$

Subsequent Q_n's may be found recursively from (11).

Let us remark that Q_n is the probability that the customer will not become discouraged, a property we call *stubbornness*. We shall now study the expected waiting time $E(W_n)$ of a stubborn customer, who joins the queue at a time when there are already n persons in the system. Once again, we will assume that the customer in question arrives at the start of the working period. In the case of an arrival during a repair period, we need to add $1/\beta$ to the formulas for $E(W_n)$ developed below. To find a recursive formula for $E(W_n)$, we simply condition on what happens first: the end of a service (with probability $\mu/(\alpha+\mu)$) or a breakdown (with probability $\alpha/(\alpha+\mu)$). Thus, defining $E(W_{-1}) = 0$, we have

$$(13)\qquad E(W_n) = \frac{\mu}{\alpha+\mu}\left[\frac{1}{\alpha+\mu} + E(W_{n-1})\right] + \frac{\alpha}{\alpha+\mu}\left[\frac{1}{\alpha+\mu} + \frac{1}{\beta} + \sum_{j=0}^{n} \binom{n}{j} \delta^j (1-\delta)^{n-j} E(W_j)\right].$$

For example, $E(W_0) = 1/[\mu\beta/(\alpha+\beta)]$ which can be seen by direct reasoning to be the expected time to remain in a system subject to breakdowns if one is currently being served (see, e.g., [1]).

A similar conditioning argument can be used to derive recursive equations for the variance of W_n.

To obtain an approximation for $E(W_n)$ valid for large n, we first apply Jensen's inequality to the sum in (13) to get

$$E(W_n) \leqslant \frac{\mu}{\alpha+\mu}[E(W_0)+E(W_{n-1})]+\frac{\alpha}{\alpha+\mu}E(W_{[n\delta]}). \tag{14}$$

Applying (14) k times, and extending the notation to non-integer indices, we obtain

$$\frac{\alpha}{\mu}[E(W_n)-E(W_{n\delta^k})]+\sum_{i=0}^{k-1}[E(W_{n\delta^i})-E(W_{n\delta^i-1})] \leqslant kE(W_0). \tag{15}$$

Taking $k = -\log(n)/\log(\delta)$, so that $n = 1/\delta^k$, and dropping the sum, we arrive at a first order approximation

$$E(W_n) \leqslant \frac{\mu}{\alpha}\left[\frac{\log(n)}{\log(1/\delta)}\right]E(W_0)+E(W_1), \tag{16}$$

where, from (13),

$$E(W_1) = \left[1+\frac{\mu}{\mu+\alpha(1-\delta)}\right]E(W_0).$$

Finally, substituting the approximation (16) back into the sum in (15), we obtain the second order approximation

$$E(W_n) \approx \frac{\mu}{\alpha\log(1/\delta)}\left[\log(n)-\frac{\mu(n-1)\delta}{\alpha n(1-\delta)}\right]E(W_0)+E(W_1). \tag{17}$$

Notice that the expressions (11)–(17) are all valid for the $G/M/1$ queue since they do not involve the arrival process.

To extend our results to a $G/G/1$ queue with general random lengths of repair and working periods, the $(n+1)$-st priority customer we follow must arrive at the beginning of a working period. Also we need to assume that the customer being served at the time of the previous breakdown was discouraged.

For example, in the integral leading to (11) we have to replace $\alpha e^{-\alpha t}$ by the density S of duration of the working period and replace $[(\mu t)^j/j!]e^{-\mu t}$ by the probability that exactly j customers are served between 0 and t (or, equivalently, $G_j(t)-G_{j+1}(t)$, where $G_k(t)$ is the cdf of the k-fold convolution of the service time distribution).

This paper has introduced the concept of customer discouragement into queueing models with breakdowns. Our process has a stationary distribution as long as any discouragement exists. This contrasts with systems lacking customer discouragement where the queue will grow without bounds if the arrival rate exceeds the service rate.

Customer discouragement can instantaneously severely deplete the queue. However, our results, together with those of an extensive simulation study [3], show that important properties of the system in the steady state are still smooth functions of the parameters.

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