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ANTI-IRREDUCIBLE PROBABILITY MEASURES

BY

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Abstract. The paper is devoted to the study of anti-irreducible probability measures associated with generalized convolutions. In particular, for convolutions other than the max-convolution it is proved that the set of anti-irreducible measures is a proper subset of the set of all infinitely divisible measures. Moreover, for a special class of convolutions containing a modification of the max-convolution it is proved that the probability measure concentrated at the origin is the only anti-irreducible measure.

1. Notation and preliminaries. For the terminology and notation used here, see [10]. In particular, V and P will stand for the set of all finite signed measures and the set of all probability measures defined on Borel subsets of the half-line $[0, \infty)$, respectively. Elements of V and P will be denoted by capitals M, N and by Greek letters μ, ν with or without subscripts, respectively. The sets V and P are endowed with the metrizable topology of weak convergence. For $M \in V$ and $a \in (0, \infty)$ we define the map $T(a)$ by setting $(T(a)M)(E) = M(a^{-1}E)$ for all Borel subsets E of $[0, \infty)$. Further, for any Borel subset A of $[0, \infty)$, $M|A$ will denote the restriction of M to A, i.e. $(M|A)(E) = M(A \cap E)$ for all Borel subsets E of $[0, \infty)$ As usual we let δ_c stand for the probability measure concentrated at the point c. Given $M \in V$ we shall denote by $s(M)$ and $\mathrm{at}(M)$ the closed support and the set of atoms of M, respectively. Throughout this paper, V_+ will stand for the subset of V consisting of non-negative measures and U_+ will stand for the subset of V_+ containing all measures M with $0 \notin \mathrm{at}(M)$. Finally, W_+ will stand for the subset of U_+ containing all measures M which do not vanish identically, have a bounded support and $0 \notin s(M)$.

A continuous commutative and associative P-valued binary operation $\circ$ on P is called a *generalized convolution* if it is distributive with respect to convex combinations and the maps $T(a)$ with $a \in (0, \infty)$, δ_0 is its unit element, and an analogue of the law of large numbers

$$T(c_n)\delta_1^{\circ n} \to \gamma \tag{1.1}$$

is fulfilled for a choice of norming constants $c_n \in (0, \infty)$ and $\gamma \neq \delta_0$. The power $\delta_1^{\circ n}$ is taken here in the sense of operation $\circ$. The limit measure γ is called a *characteristic measure* of the generalized convolution in question. By Proposition 4.4 in [10] the characteristic measure is uniquely determined up to the scale change $T(a)$ with $a \in (0, \infty)$. Moreover, by Proposition 4.5 in [10], there exists a constant $\kappa = \kappa(\circ)$ belonging to $(0, \infty]$ and called the *characteristic exponent* of $\circ$ such that

$$T(a)\gamma \circ T(b)\gamma = T\big(r(\kappa, a, b)\big)\gamma \tag{1.2}$$

for any pair $a, b \in (0, \infty)$, where

$$r(\infty, a, b) = \max(a, b) \quad \text{and} \quad r(\kappa, a, b) = (a^\kappa + b^\kappa)^{1/\kappa} \text{ if } \kappa \in (0, \infty). \tag{1.3}$$

The characteristic measure γ can be regarded as an analogue of the Gaussian measure. In the sequel we shall use the notation

$$\mathrm{Gauss}(\circ) = \{T(a)\gamma \colon a \in (0, \infty)\} \cup \{\delta_0\}.$$

It was shown in [10], Chapter 3, that every generalized convolution can be extended to a continuous operation on V by setting

$$(M \circ N)(E) = \int_0^\infty \int_0^\infty \delta_x \circ \delta_y(E) M(dx) N(dy) \tag{1.4}$$

for every Borel subset E of $[0, \infty)$ and $M, N \in V$. It is clear that the set V_+ is invariant under $\circ$.

Let m_0 be the sum of δ_0 and the Lebesgue measure on $[0, \infty)$. It has been proved in [10], Theorem 4.1 and Corollary 4.4, that each generalized convolution admits a weak characteristic function, i.e. a one-to-one correspondence $\mu \leftrightarrow \hat{\mu}$ between measures μ from P and real-valued Borel functions $\hat{\mu}$ from $L_\infty(m_0)$ such that

$$\big(c\mu + (1-c)\nu\big)\hat{} = c\hat{\mu} + (1-c)\hat{\nu}, \quad \big(T(a)\mu\big)\hat{}(t) = \hat{\mu}(at), \quad (\mu \circ \nu)\hat{} = \hat{\mu}\hat{\nu}$$

for all $c \in [0, 1]$, $a \in (0, \infty)$, and $\mu, \nu \in P$. The weak convergence $\mu_n \to \mu$ is equivalent to the convergence $\hat{\mu}_n \to \hat{\mu}$ in the $L_1(m_0)$-topology of $L_\infty(m_0)$. Moreover, if λ is absolutely continuous with respect to the measure m_0, then the function $\hat{\lambda}$ is continuous and, by Lemma 3.11, Propositions 3.3 and 3.4 and Theorem 4.1 in [10],

$$\lim_{t \to \infty} \hat{\lambda}(t) = \lambda(\{0\}). \tag{1.5}$$

Recall that the weak characteristic function is uniquely determined up to a scale change and

$$\hat{\mu}(t) = \int_0^\infty \Omega(tx) \mu(dx) \tag{1.6}$$

m_0-almost everywhere. The kernel Ω is a Borel function with $\Omega(0) = 1$ and

(1.7) $$|\Omega(t)| \leqslant 1 \quad \text{for } t \in [0, \infty).$$

Taking a suitable normalization if necessary we may always assume, by Theorem 4.2 in [10], that for $\kappa \in (0, \infty)$

(1.8) $$\hat{\gamma}(t) = \exp(-t^{\kappa})$$

m_0-almost everywhere.

Generalized convolutions admitting a continuous kernel Ω are called *regular* (see [10], p. 93). In the sequel, 1_E will denote the indicator of the set E. Further, by the *max-convolution* $*_\infty$ we mean the generalized convolution induced by the operation $\max(X, Y)$ on independent random variables X and Y. By Lemma 2.1 in [10], $\kappa(\circ) = \infty$ if and only if $\circ = *_\infty$. It is known that the max-convolution is not regular (see [8], p. 219). Throughout this paper, $\Box$ will denote the operation induced by the multiplication of independent random variables. In other words,

$$M \Box N = \int_0^\infty \int_0^\infty \delta_{xy} M(dx) N(dy) \quad \text{for } M, N \in V.$$

It is clear that

(1.9) $$M \Box N = \int_0^\infty (T(a)M) N(da),$$

where $T(0)M$ is assumed to be $M(\{0\})\delta_0$. Setting

$$\hat{M}(t) = \int_0^\infty \Omega(tx) M(dx)$$

we have the formula

(1.10) $$(M \Box N)\hat{}(t) = \int_0^\infty \hat{M}(tx) N(dx) \quad \text{for } M, N \in V.$$

We say that a probability measure μ from P is $\circ$-*infinitely divisible* if for every positive integer n there exists a measure $\mu_n \in P$ such that

(1.11) $$\mu = \mu_n^{\circ n}.$$

The set of all $\circ$-infinitely divisible probability measures will be denoted by $P_\infty(\circ)$. For regular generalized convolutions, $\circ$-infinitely divisible measures were studied in [8] and [9]. For the max-convolution we have the formula

(1.12) $$P_\infty(*_\infty) = P$$

(see [7], p. 175). Notice that, by Proposition 3.2 in [11], we may always assume that the measures μ_n in (1.11) belong to $P_\infty(\circ)$. Moreover, if $\circ \neq *_\infty$, then, by

Lemma 2.4 in [12], for every $\mu \in P_\infty(\circ)$

$$\hat{\mu}(t) > 0 \tag{1.13}$$

m_0-almost everywhere.

In the sequel we shall use the following compactness lemma:

LEMMA 1.1. *Suppose that* $\circ \neq *_\infty$ *and* $\mu = \mu_n^{\circ n}$ *with* $\mu_n \in P_\infty(\circ)$ $(n = 1, 2, \ldots)$. *Then for every* $a \in (0, \infty)$ *the sequence of restricted measures* $n\mu_n | [a, \infty)$ *is conditionally compact in* U_+. *Moreover, if in addition the weak characteristic function* $\hat{\mu}$ *is* m_0*-essentially bounded from below by a positive number, then the sequence* $n\mu_n | (0, \infty)$ *is also conditionally compact.*

Proof. Since, by (1.6), (1.7) and (1.13), $0 < \hat{\mu}_n(t) \leqslant 1$ and $\hat{\mu}(t) = \hat{\mu}_n(t)^n$ m_0-almost everywhere, we have the inequality

$$n\big(1 - \hat{\mu}_n(t)\big) \leqslant -\log \hat{\mu}(t) \tag{1.14}$$

m_0-almost everywhere. Further, by Proposition 4.2 in [10],

$$\lim_{t \to 0} t^{-1} \int_0^t \hat{\mu}(u)du = 1.$$

Consequently, for every $\varepsilon \in (0, 1)$ there exists a closed subset B of $[0, 1]$ with positive Lebesgue measure such that

$$\hat{\mu}(t) \geqslant \exp(-\varepsilon) \quad \text{for } t \in B. \tag{1.15}$$

Denoting by q^{-1} the Lebesgue measure of B and setting $\lambda(dx) = q 1_B(x)dx$, we infer that $\lambda \in P$, $s(\lambda) = B$, and the measure λ is absolutely continuous with respect to m_0. Thus, by (1.5),

$$\lim_{t \to \infty} \hat{\lambda}(t) = 0, \tag{1.16}$$

which shows that for a sufficiently large number b the inequality $\hat{\lambda}(t) < 2^{-1}$ holds whenever $t \in [b, \infty)$. Taking into account (1.14) and (1.15) we get the inequality

$$n\hat{\mu}_n([b, \infty)) \leqslant 2n \int_0^\infty \big(1 - \hat{\lambda}(t)\big)\mu_n(dt) = 2n \int_B \big(1 - \hat{\mu}_n(t)\big)\lambda(dt) \leqslant 2\varepsilon$$

for all indices n. On the other hand, by Lemma 2.5 in [12], for every $a \in (0, \infty)$ the sequence $n\mu_n([a, \infty))$ is bounded. Hence we get the first assertion of Lemma 1.1. If, in addition, $\hat{\mu}(t) \geqslant \exp(-d)$ m_0-almost everywhere for some constant $d \in (0, \infty)$, then, by (1.14),

$$n \int_0^\infty \big(1 - \hat{\lambda}(ct)\big)\mu_n(dt) = n \int_0^\infty \big(1 - \hat{\mu}_n(t)\big)\big(T(c)\lambda\big)(dt) \leqslant d$$

for all $c \in (0, \infty)$ and all indices n. Letting $c \to \infty$ and taking into account (1.16) we get, by the bounded convergence theorem, $n\mu_n((0, \infty)) \leqslant d$ for all indices n, which yields the conditional compactness of the sequence $n\mu_n|(0, \infty)$. Lemma 1.1 is thus proved.

We shall need the following characterization of the max-convolution.

LEMMA 1.2. *If $p\delta_0 + (1-p)\delta_1 \in P_\infty(\circ)$ for some $p \in (0, 1)$, then $\circ = *_\infty$.*

Proof. Suppose that $\mu_n \in P$ and

$$p\delta_0 + (1-p)\delta_1 = \mu_n^{\circ n} \quad (n = 1, 2, \ldots). \tag{1.17}$$

Observe that, by (1.4),

$$\mu_n^{\circ n} = \int_0^\infty \int_0^\infty \ldots \int_0^\infty \Big(\bigcirc_{j=1}^{n} \delta_{x_j}\Big)\mu_n(dx_1)\mu_n(dx_2)\ldots\mu_n(dx_n),$$

which yields the inclusion

$$s\Big(\bigcirc_{j=1}^{n} \delta_{x_j}\Big) \subset \{0, 1\} \tag{1.18}$$

for $(\mu_n \times \mu_n \times \ldots \times \mu_n)$-almost all n-tuples $x_1, x_2, \ldots, x_n$. By the continuity of the mapping

$$(x_1, x_2, \ldots, x_n) \to \bigcirc_{j=1}^{n} \delta_{x_j}$$

we conclude that the above inclusion holds for all $x_1, x_2, \ldots, x_n \in s(\mu_n)$. In particular, we have

$$s(\delta_x^{\circ n}) \subset \{0, 1\} \quad \text{for all } x \in s(\mu_n). \tag{1.19}$$

Suppose that the set $s(\mu_n)$ contains two positive numbers a and b. Setting $c = a/b$, by (1.19) we have

$$s(\delta_a^{\circ n}) = s\big(T(c)\delta_b^{\circ n}\big) = cs(\delta_b^{\circ n}) \subset \{0, c\}.$$

Comparing this relation with (1.19) we get the inclusion $s(\delta_a^{\circ n}) \subset \{0\}$. Thus $\delta_a^{\circ n} = \delta_0$ and, consequently, by Lemma 2.3 in [13], $\delta_a = \delta_0$ which contradicts the assumption $a > 0$. This proves the inclusion

$$s(\mu_n) \subset \{0, a_n\} \quad (n = 2, 3, \ldots) \tag{1.20}$$

for some $a_n \in (0, \infty)$. Moreover, the inequality $\mu_n \neq \delta_0$ yields

$$a_n \in s(\mu_n) \quad (n = 2, 3, \ldots). \tag{1.21}$$

Now we shall prove that

$$s(\mu_k) = \{0, a_k\} \tag{1.22}$$

for a certain index $k \geqslant 2$. Suppose the contrary. Then, by (1.20), $\mu_n = \delta_{a_n}$, which, by (1.17), yields

$$p\delta_0 + (1-p)\delta_1 = \delta_{a_n}^{\circ n} \quad (n = 2, 3, \ldots).$$

Hence the measure $p\delta_0 + (1-p)\delta_1$ is $\circ$-stable ([10], Section 2) and, consequently, by Lemma 2.2 in [10], has no atom at the origin, which is a contradiction. Relation (1.22) for a certain index $k \geqslant 2$ is thus proved. Substituting $n = k$, $x_1 = x_2 = \ldots = x_{k-1} = 0$ and $x_k = a_k$ into (1.18) we get the inclusion $s(\delta_{a_k}) \subset \{0, 1\}$, which yields $a_k = 1$. Further, setting $n = k$, $x_1 = x_2 = 1$ and $x_3 = \ldots \ldots = x_k = 0$ into (1.18) we get the inclusion $s(\delta_1 \circ \delta_1) \subset \{0, 1\}$. Consequently, $\delta_1 \circ \delta_1 = a\delta_0 + (1-a)\delta_1$ for some $a \in [0, 1]$. Hence, by standard calculations, the probability measure $a(1+a)^{-1}\delta_0 + (1+a)^{-1}\delta_1$ is an idempotent other than δ_0. Applying Theorems 4.1 and 4.2 from [13] we get the equality $\circ = *_\infty$, which completes the proof.

Given $M \in U_+$ we define the probability measure $\pi(M)$ by setting

$$\pi(M) = e^{-d}\Big(\delta_0 + \sum_{k=1}^{\infty} M^{\circ k}/k!\Big),$$

where $d = M([0, \infty))$. We record for later reference the following simple formulae:

$$\pi(M)^\wedge(t) = \exp \int_0^\infty \big(\Omega(tx) - 1\big) M(dx) \tag{1.23}$$

m_0-almost everywhere,

$$\pi(M) \circ \pi(N) = \pi(M+N), \tag{1.24}$$

$$T(a)\pi(M) = \pi\big(T(a)M\big) \tag{1.25}$$

for $M, N \in U_+$ and $a \in (0, \infty)$. Moreover, $\pi(M_n) \to \pi(M)$ whenever $M_n \to M$ in U_+. Finally, observe that the mapping $U_+ \ni M \to \pi(M)$ is one-to-one. In fact, the equality $\pi(M) = \pi(N)$ with $M, N \in U_+$ yields, by (1.23),

$$\hat{M}(t) - M([0, \infty)) = \hat{N}(t) - N([0, \infty))$$

m_0-almost everywhere or, equivalently,

$$M - M([0, \infty))\delta_0 = N - N([0, \infty))\delta_0.$$

Since both measures M and N have no atom at the origin, the last equality yields $M = N$, which completes the proof.

Throughout this paper, Poiss$(\circ)$ will stand for the set of all measures $\pi(M)$ with $M \ni U_+$. We begin with a simple characterization of this set.

LEMMA 1.3. *A measure $\mu \in$ Poiss$(\circ)$ if and only if $\mu \in P_\infty(\circ)$ and $\hat{\mu}$ is m_0-essentially bounded from below by a positive number.*

Proof. Necessity. By (1.24) we have $\pi(M) = \pi(M/n)^{\circ n}$ for all positive integers n. Hence $\mathrm{Poiss}(\circ) \subset P_\infty(\circ)$. Further, by (1.23), we have the inequality $\pi(M)^\wedge(t) \geq \exp(-2M([0, \infty)))$ m_0-almost everywhere, which completes the proof of the necessity of our conditions.

Sufficiency. First consider the case $\circ = *_\infty$. Then $\Omega = 1_{[0,1]}$ and, consequently, $\hat{\mu}(t) = \mu([0, t^{-1}])$ for $t \in [0, \infty)$. The boundedness of $\hat{\mu}(t)$ from below by a positive number yields $0 \in \mathrm{at}(\mu)$. Setting $M(\{0\}) = 0$ and $M((x, \infty)) = -\log \mu([0, x])$ for $x \in [0, \infty)$ we infer that $M \in U_+$ and $\mu = \pi(M)$.

Suppose now that $\circ \neq *_\infty$. The measure μ can be written in the form $\mu = \mu_n^{\circ n}$ for some $\mu_n \in P_\infty(\circ)$. By Lemma 1.1 in [12] there exists a subsequence $n_1 < n_2 < \ldots$ such that

$$\lim_{k \to \infty} n_k(1 - \hat{\mu}_{n_k}(t)) = -\log \hat{\mu}(t) \tag{1.26}$$

m_0-almost everywhere. Put $M_k = n_k \mu_{n_k} | (0, \infty)$ $(k = 1, 2, \ldots)$. By Lemma 1.1 the sequence M_k is conditionally compact in U_+. Passing to a subsequence if necessary we may assume without loss of generality that $M_k \to M$ for some $M \in U_+$. Consequently, $\pi(M_k) \to \pi(M)$. Since, by (1.23),

$$\pi(M_k)^\wedge(t) = \exp\left(n_k(\hat{\mu}_{n_k}(t) - 1)\right)$$

m_0-almost everywhere, we conclude, by (1.26), that $\hat{\mu}(t) = \pi(M)^\wedge(t)$ m_0-almost everywhere. Thus $\mu = \pi(M)$, which completes the proof.

2. Factorization of probability measures. Let $\mu, \nu \in P$. We say that ν is a *divisor* of μ if $\mu = \nu \circ \lambda$ for some $\lambda \in P$. The set of all divisors of μ will be denoted by $D(\circ, \mu)$. It is clear that $\{\delta_0, \mu\} \subset D(\circ, \mu)$. By Proposition 2.4 and Corollary 2.3 in [13] we have the following statement:

PROPOSITION 2.1. *For every $\mu \in P$ the set $D(\circ, \mu)$ is compact.*

By Lemma 2.3 in [13] the equation $\nu \circ \lambda = \delta_0$ yields $\nu = \lambda = \delta_0$. Hence we get the following

PROPOSITION 2.2. $D(\circ, \delta_0) = \{\delta_0\}$.

PROPOSITION 2.3. *If $\mu_1 \in D(\circ, \mu_2)$ and $\mu_2 \in D(\circ, \mu_1)$, then $\mu_1 = \mu_2$.*

Proof. Suppose that $\mu_1 = \mu_2 \circ \nu_1$ and $\mu_2 = \mu_1 \circ \nu_2$ for some $\nu_1, \nu_2 \in P$. Setting $\lambda = \nu_1 \circ \mu_2$, we have $\mu_1 = \mu_1 \circ \lambda$, which yields $\mu_1 = \mu_1 \circ \lambda^{\circ n}$ for all positive integers n. Thus $\lambda^{\circ n} \in D(\circ, \mu_1)$ and, by Proposition 2.1, the sequence $\lambda^{\circ n}$ is conditionally compact. By Theorems 4.1 and 4.2 and Corollary 3.5 in [13] we conclude that either $\circ = *_\infty$ or $\circ \neq *_\infty$ and $\lambda^{\circ n} \to \delta_0$ as $n \to \infty$. In the case of the max-convolution our assertion is obvious. In the remaining case we have, by Corollary 2.4 in [13], $\lambda = \delta_0$, which by Proposition 2.2 yields $\nu_1 = \nu_2 = \delta_0$. Consequently, $\mu_1 = \mu_2$.

LEMMA 2.1. *If $\circ \neq *_\infty$, then for every $M \in U_+$ the following inclusion is true:*

$$D(\circ, \pi(M)) \cap P_\infty(\circ) \subset \mathrm{Poiss}(\circ).$$

Proof. Suppose that $\nu \in P_\infty(\circ)$ and $\nu \circ \lambda = \pi(M)$ for some $\lambda = P$. Then $\hat{\nu}(t) \geqslant \pi(M)^\wedge(t)$ m_0-almost everywhere, which, by Lemma 1.3, yields $\nu \in \mathrm{Poiss}(\circ)$.

LEMMA 2.2. *Suppose that* $\circ \neq *_\infty$, $M \in U_+$, *and* $D(\circ, \pi(M)) \subset P_\infty(\circ)$. *Then for every* $N \in U_+$ *with* $\pi(N) \in D(\circ, \pi(M))$ *the inclusion* $s(N) \subset s(M)$ *is true.*

Proof. Applying Lemma 2.1 we conclude that there exists a measure $Q \in U_+$ fulfilling the condition $\pi(N) \circ \pi(Q) = \pi(M)$. Hence and from (1.24), by the uniqueness of the correspondence $M \leftrightarrow \pi(M)$, we get the formula $M = N + Q$ which yields the desired inclusion.

Given $\mu \in P$, by a *μ-norm* we mean a function Δ_μ from $D(\circ, \mu)$ into $[0, \infty)$ continuous at δ_0 and fulfilling the condition

$$\Delta_\mu(\nu \circ \lambda) = \Delta_\mu(\nu) + \Delta_\mu(\lambda) \tag{2.1}$$

whenever $\nu \circ \lambda \in D(\circ, \mu)$ (see [7], p. 37).

We shall need the following lemma:

LEMMA 2.3. *Suppose that* $\circ \neq *_\infty$. *Then for every* $\mu \in P$ *other than* δ_0 *there is a μ-norm* Δ_μ *with* $\Delta_\mu(\mu) > 0$.

Proof. Suppose that $\mu \in P$ and $\mu \neq \delta_0$. First we shall prove that there exists a number $b(\mu) \in (0, 1)$ such that the set

$$B(\mu) = \{t\colon\ t \in [0, 1],\ b(\mu) \leqslant |\hat{\mu}(t)| < 1\}$$

has positive Lebesgue measure. Suppose the contrary. Then, by (1.6) and (1.7), the set $\{t\colon\ t \in [0, 1],\ |\hat{\mu}(t)| \notin \{0, 1\}\}$ has the Lebesgue measure 0. Since, by Proposition 4.2 in [10],

$$\lim_{t \to 0} t^{-1} \int_0^t \hat{\mu}(u)\,du = 1,$$

we conclude that the set $\{t\colon\ t \in [0, 1],\ \hat{\mu}(t) = 1\}$ has positive Lebesgue measure. Applying Lemma 2.1 in [12] we get the equality $\mu = \delta_0$, which contradicts the assumption. The existence of the desired constant $b(\mu)$ is thus proved.

Put

$$\Delta_\mu(\nu) = -\int_{B(\mu)} \log|\hat{\nu}(t)|\,dt \qquad \text{for } \nu \in D(\circ, \mu).$$

Notice that

$$b(\mu) \leqslant |\hat{\mu}(t)| \leqslant |\hat{\nu}(t)| \tag{2.2}$$

for $\nu \in D(\circ, \mu)$ and m_0-almost all $t \in B(\mu)$. Consequently, $0 \leqslant \Delta_\mu(\nu) < \infty$ for all $\nu \in D(\circ, \mu)$. The inequality $|\hat{\mu}(t)| < 1$ on $B(\mu)$ yields $\Delta_\mu(\mu) > 0$. Formula (2.1) is evident. It remains to prove that the function Δ_μ is continuous at δ_0. Suppose that $\nu_n \in D(\circ, \mu)$ and $\nu_n \to \delta_0$. By Lemma 1.1 in [12] each subsequence of indices contains a subsequence $n_1 < n_2 < \ldots$ such that $\hat{\mu}_{n_k} \to 1$ m_0-almost everywhere. By (2.2) and the bounded convergence theorem we get the relation

$\Delta_\mu(\nu_{n_k}) \to 0 = \Delta_\mu(\delta_0)$ which proves the continuity of Δ_μ at δ_0. The lemma is thus proved.

A probability measure μ is said to be *irreducible* if $\mu \neq \delta_0$ and $D(\circ, \mu) = \{\delta_0, \mu\}$. A probability measure μ is said to be *anti-irreducible* if $D(\circ, \mu)$ contains no irreducible measure. Throughout this paper $I(\circ)$ will stand for the set of all anti-irreducible measures. The set $I(\circ)$ has drawn much attention since the inception of decomposition theory (see [7], Sections 2.8, 2.9 and 5.7). The problem of describing anti-irreducible measures for the ordinary convolution has a long history but it has not been solved yet (see [4] and [6]). It is known that for the max-convolution

$$I(*_\infty) = P_\infty(*_\infty) = P \tag{2.3}$$

(see [7], p. 175). For the Kingman convolutions, Ostrovskiĭ obtained in [5] the nice formula $I(\circ) = \text{Gauss}(\circ)$. It is of interest to clarify whether it is possible to realize the extremal case $I(\circ) = \{\delta_0\}$ for some generalized convolutions. Section 3 will be devoted to the study of this problem.

By Proposition 1.2 we have $\delta_0 \in I(\circ)$. It is clear that

$$D(\circ, \mu) \subset I(\circ) \quad \text{for } \mu \in I(\circ). \tag{2.4}$$

Bingham proved in [2] that for regular convolutions anti-irreducible measures are infinitely divisible. We shall show that this result remains true for arbitrary convolutions.

THEOREM 2.1. *For every convolution $\circ$ the inclusion $I(\circ) \subset P_\infty(\circ)$ holds.*

Proof. By (2.3) it suffices to consider the case $\circ \neq *_\infty$. By Theorem 4.2 in [13] the measure δ_0 is the only idempotent in the semigroup $(P, \circ)$. Taking into account Definitions 2.2 and 10.6 in [7] we conclude, by Propositions 2.1 and 2.3 and Lemma 2.3, that $(P, \circ)$ is a normable Hun semigroup. Thus, by Theorem 8.9 in [7], every anti-irreducible probability measure μ is infinitesimally divisible, i.e. for every neighbourhood U of δ_0 it has a representation $\mu = \mu_1 \circ \mu_2 \circ \ldots \circ \mu_k$ with $\mu_j \in U$ $(j = 1, 2, \ldots, k)$. Of course, all partial products $\mu_{i_1} \circ \mu_{i_2} \circ \ldots \circ \mu_{i_r}$ $(1 \leqslant i_1 < i_2 < \ldots < i_r \leqslant k)$ belong to $D(\circ, \mu)$. Since, by Proposition 2.1, the set $D(\circ, \mu)$ is compact, we apply Theorem 10.7 in [7] and obtain the infinite divisibility of μ, which completes the proof.

THEOREM 2.2. *$\mu \in I(\circ)$ if and only if $D(\circ, \mu) \subset P_\infty(\circ)$.*

Proof. By (2.3) it suffices to consider the case

$$\circ \neq *_\infty. \tag{2.5}$$

The necessity of the condition in question is an immediate consequence of inclusion (2.4) and Theorem 2.1. We prove its sufficiency indirectly. Suppose that $D(\circ, \mu) \subset P_\infty(\circ)$ and the set $D(\circ, \mu)$ contains an irreducible measure ν. Since $\nu \in P_\infty(\circ)$, we have $\nu = \lambda \circ \lambda$ for some $\lambda \neq \delta_0$. Of course, $\lambda \in D(\circ, \nu)$

$= \{\delta_0, \nu\}$, which yields $\lambda = \nu$. In other words, ν is an idempotent other than δ_0, which, by Theorems 4.1 and 4.2 in [13], contradicts (2.5). The theorem is thus proved.

THEOREM 2.3. *For every convolution* $\circ$ *other than* $*_\infty$, $I(\circ)$ *is a proper subset of* $P_\infty(\circ)$.

Proof. Setting $Q = \sum_{k=1}^{\infty} k^{-1}2^{-k}\delta_1^{\circ k}$ and $M = Q|(0, \infty)$ we have $M \in U_+$ and, by (1.23),

$$\pi(M)^\wedge(t) = \exp \sum_{k=1}^{\infty} k^{-1}2^{-k}(\Omega^k(t)-1) = (2-\Omega(t))^{-1}$$

m_0-almost everywhere. Introduce the notation

$$p_k = 2^{2^k}/(1+2^{2^k}) \quad \text{and} \quad \nu_k = p_k\delta_0+(1-p_k)\delta_1^{\circ 2^k} \quad (k = 0, 1, \ldots).$$

Of course, $\nu_k \in P$ and, by Lemma 1.2,

$$\nu_0 \notin P_\infty(\circ). \tag{2.6}$$

Moreover, $\hat{\nu}_k(t) = p_k+(1-p_k)\Omega^{2^k}(t)$ m_0-almost everywhere. Using the formula

$$(2-x)^{-1} = \prod_{k=0}^{\infty} (p_k+(1-p_k)x^{2^k}) \quad \text{for } x \in [-1, 1]$$

we get the equality

$$\pi(M)^\wedge(t) = \prod_{k=0}^{\infty} \hat{\nu}_k(t)$$

m_0-almost everywhere. Hence $\nu_0 \circ \nu_1 \circ \ldots \circ \nu_n \to \pi(M)$ as $n \to \infty$. By Corollary 2.3 in [13] the sequence $\nu_1 \circ \nu_2 \circ \ldots \circ \nu_n$ is conditionally compact. Taking its cluster point λ we have the equality $\nu_0 \circ \lambda = \pi(M)$. Thus $\nu_0 \in D(\circ, \pi(M))$, which, by (2.6) and Theorem 2.2, yields $\pi(M) \notin I(\circ)$. On the other hand, $\pi(M) \in P_\infty(\circ)$, which completes the proof.

We say that a generalized convolution $\circ$ has the *Cramér property* if its characteristic measure γ fulfils the condition $D(\circ, \gamma) \subset \text{Gauss}(\circ)$. Generalized convolutions with the Cramér property were studied in [14]. It is well known that the ordinary convolutions and the symmetric ones have the Cramér property. Ostrovskiĭ proved in [5] that the Kingman convolutions have also this property. Observe that, for the max-convolution $\gamma = \delta_1$, $D(*_\infty, \gamma) = \{\nu\colon s(\nu) \subset [0, 1]\}$ and $\text{Gauss}(*_\infty) = \{\delta_a\colon a \in [0, \infty)\}$, which shows that $*_\infty$ has not the Cramér property.

Now we shall give a convenient sufficient condition for a convolution to have not the Cramér property.

LEMMA 2.4. *Let* $\circ$ *be a generalized convolution with finite characteristic exponent* κ. *Suppose that there exists a probability measure* ϱ *with* $\hat{\varrho}(t)$

$=(1+ct^{\kappa})\exp(-t^{\kappa})$ *for some* $c\in(0,\infty)$. *Then* $\circ$ *does not have the Cramér property.*

Proof. Put $\nu(dx)=\kappa x^{\kappa-1}\exp(-x^{\kappa})dx$ and $\lambda=T(c^{1/\kappa})\gamma\,\Box\,\nu$. Using (1.8) and (1.10) we have $\hat{\lambda}(t)=(1+ct^{\kappa})^{-1}$, which yields $\hat{\lambda}(t)\hat{\varrho}(t)=\exp(-t^{\kappa})$ m_0-almost everywhere. Consequently, $\gamma=\lambda\circ\varrho$ and $\lambda\in D(\circ,\gamma)$. On the other hand, by (1.8), $\lambda\notin\mathrm{Gauss}(\circ)$, which shows that the convolution in question does not have the Cramér property.

Now we shall give some examples of generalized convolutions without the Cramér property. It is clear that each generalized convolution $\circ$ is uniquely determined by the expressions $\delta_a\circ\delta_b$ with $a,b\in(0,\infty)$. In our case they will be of the form

$$\delta_a\circ\delta_b(dx)=f\big(d(a,b)\big)\delta_{\max(a,b)}(dx)+g(a,b,x)dx, \tag{2.7}$$

where $d(a,b)=\min(a,b)/\max(a,b)$. The case $f\equiv 1$ and $g\equiv 0$ corresponds to the max-convolution. Therefore generalized convolutions (2.7) can be regarded as a modification of the max-convolution.

EXAMPLE 2.1. *Kendall convolutions.* The Kendall convolution depends upon a positive integer n and is defined by (2.7) with the functions $f(x)=(1-x)^n$ and

$$g(a,b,x)=\sum_{k=1}^{n}(k+1)\binom{n}{k}\binom{n}{k-1}a^{n+1-k}b^k(x-a)^{k-1}(x-b)^{n-k}x^{-1-2n}1_{[c,\infty)}(x),$$

where $c=\max(a,b)$. Here we have $\kappa=1$ and

$$\Omega(t)=(1-t)^n 1_{[0,1]}(t).$$

Put $\varrho(dx)=\big((n+1)!\big)^{-1}x^{-n-3}\exp(-1/x)dx$. By standard calculations we get the formula

$$\hat{\varrho}(t)=\big(1+(n+1)^{-1}t\big)\exp(-t)$$

which, by Lemma 2.4, shows that the Kendall convolutions do not have the Cramér property.

EXAMPLE 2.2. *$(1,p)$-convolutions.* This family of convolutions depends upon a parameter $p\in(0,1)$ and is defined by (2.7) with the functions $f(x)=1-px$,

$$g(a,b,x)=pab(2p-1)^{-1}x^{-3}\big(2p-\max(a^q,b^q)x^{-q}\big)1_{[c,\infty)}(x)\qquad\text{if } p\neq 1/2$$

and

$$g(a,b,x)=2^{-1}abx^{-3}(1+2\log x-2\log c)1_{[c,\infty)}(x)\qquad\text{if } p=1/2,$$

where $c=\max(a,b)$ and $q=(2p-1)/(1-p)$. Here we have $\kappa=1$ and

$$\Omega(t)=(1-pt)1_{[0,1]}(t).$$

Observe that this convolution in not regular. Setting $r=(1-p)^{-1}$,

$$h(x^{-1})=2^{-1}pr^2x^{1+r}\int\limits_x^{\infty}y^{-r}\big((1+x)e^{-x}-(1+y)e^{-y}\big)dy$$

for $x\in(0,\infty)$ and $\varrho(dx)=h(x)dx$, by standard calculations we get $\varrho\in P$ and

$$\hat{\varrho}(t)=(1+2^{-1}t)\exp(-t).$$

Applying Lemma 2.4 we infer that $(1,p)$-convolutions do not have the Cramér property.

EXAMPLE 2.3. *$(2,p)$-convolutions.* This family of convolutions depends upon a parameter $p\in(2,\infty)$. The functions f and g appearing in (2.7) are defined by the formulae

$$f(x)=1-p(p-1)^{-1}x+(p-1)^{-1}x^p$$

and

$$g(a,b,x)=p(p-1)^{-2}abx^{-3}\big(2(p-2)+(a^{p-1}+b^{p-1})(1+p)x^{1-p}\\-2(2p-1)a^{p-1}b^{p-1}x^{2-2p}\big)1_{[c,\infty)}(x),$$

where $c=\max(a,b)$. Here we have $\kappa=1$ and

$$\Omega(t)=\big(1-p(p-1)^{-1}t+(p-1)^{-1}t^p\big)1_{[0,1]}(t).$$

Setting

$$\varrho(dx)=(3p)^{-1}\big((p-2)x^{-3}+(p-2)x^{-4}+x^{-5}\big)\exp(-1/x)dx$$

we get a probability measure with $\hat{\varrho}(t)=(1+3^{-1}t)\exp(-t)$. Applying Lemma 2.4 we infer that the $(2,p)$-convolutions do not have the Cramér property.

EXAMPLE 2.4. *Kucharczak convolutions.* We define these convolutions for any $p\in(0,1)$ by setting in (2.7) $f\equiv 0$ and

$$g(a,b,x)\\=\pi^{-1}(\sin\pi p)a^pb^px^{-p}(2x-a-b)(x-a-b)^{-p}(x-a)^{-1}(x-b)^{-1}1_{[s,\infty)}(x),$$

where $s=(a^p+b^p)^{1/p}$. Here we have $\kappa=p$ and

$$\Omega(t)=\Gamma(p,t)/\Gamma(p),$$

where $\Gamma(p,t)$ is the incomplete gamma function, i.e. $\Gamma(p,t)=\int_t^{\infty}e^{-x}x^{p-1}dx$ for $t\in[0,\infty)$.

In the study of the Kucharczak convolutions we need the following

LEMMA 2.5. *Let $p\in(0,1)$ and $b\in(0,p(1-p))$. Then the function*

$$h(t)=(1+bt^p)\exp(-t^p)$$

is completely monotone on $[0,\infty)$.

Proof. It is clear that the function h is infinitely differentiable on $(0, \infty)$. By standard calculations we conclude that its derivatives are of the form

$$h^{(n)}(t) = (-1)^n t^{p-n} q_n(t^p) \exp(-t^p) \quad (n = 1, 2, \ldots), \tag{2.8}$$

where q_n are polynomials of degree n fulfilling the recurrence formula

$$q_{n+1}(x) = (n-p+px)q_n(x) - px\frac{d}{dx}q_n(x) \quad (n = 1, 2, \ldots). \tag{2.9}$$

Setting

$$q_n(x) = \sum_{k=0}^{n} a_{k,n} x^k \quad (n = 1, 2, \ldots), \tag{2.10}$$

we get from (2.9) the recurrence formulae

$$a_{n+1,n+1} = qa_{n,n} \quad (n = 1, 2, \ldots), \tag{2.11}$$

$$a_{0,n+1} = (n-p)a_{0,n} \quad (n = 1, 2, \ldots), \tag{2.12}$$

$$a_{k,n+1} = (n-p-pk)a_{k,n} + pa_{k-1,n} \quad (n = 1, 2, \ldots, n;\ n = 1, 2, \ldots) \tag{2.13}$$

with the initial conditions

$$a_{0,1} = p(1-b), \quad a_{1,1} = pb. \tag{2.14}$$

From (2.11) and (2.14) we get the equality

$$a_{n,n} = bp^n \quad (n = 1, 2, \ldots) \tag{2.15}$$

which, by (2.13), yields

$$a_{n,n+1} = pa_{n-1,n} + (n-p-pn)bp^n \quad (n = 1, 2, \ldots).$$

Now by induction one can easily check the formula

$$a_{n-1,n} = p^{n-1}\big(p - 8^{-1}br^2 + 8^{-1}b(2n(1-p)^{1/2} - r^2)\big) \quad (n = 1, 2, \ldots),$$

where $r = (1+p)(1-p)^{-1/2}$. Notice that

$$p - 8^{-1}br^2 \geqslant p - 8^{-1}p(1-p)r^2 > 0,$$

which yields the inequality

$$a_{n-1,n} > 0 \quad (n = 1, 2, \ldots). \tag{2.16}$$

Now we shall prove the inequality

$$a_{k,n} > 0 \quad (k = 0, 1, \ldots, n) \tag{2.17}$$

by induction with respect to n. By (2.14) our statement is obvious for $n = 1$. Suppose that inequalities (2.17) are true for $k = 0, 1, \ldots, n$. We have to prove the inequalities $a_{k,n+1} > 0$ for $k = 0, 1, \ldots, n+1$. Observe that for $k = n$ and

$k = n+1$ they are true because of (2.15) and (2.16). It remains to consider the case $0 \leqslant k \leqslant n-1$. Then $n-p-pk \geqslant n(1-p) > 0$ and, consequently, our assertion is an immediate consequence of (2.13). This completes the proof of (2.17). Taking into account (2.10) we conclude that all polynomials q_n are positive on the half-line $[0, \infty)$, which, by (2.8), yields the inequality $(-1)^n h^{(n)}(t) \geqslant 0$ for $n = 1, 2, \ldots$ and $t \in (0, \infty)$. Lemma 2.5 is thus proved.

Let us now return to the Kucharczak convolution with parameter $p \in (0, 1)$. Let $b \in (0, p(1-p))$ and $h(t) = (1+bt^p)\exp(-t^p)$. Then

$$\int_t^\infty h(u)u^{p-1}\,du = p^{-1}(1+b+bt^p)\exp(-t^p) \tag{2.18}$$

for $t \in [0, \infty)$. By Lemma 2.5 the function h is completely monotone on $[0, \infty)$. Taking the Bernstein representation

$$h(t) = \int_0^\infty e^{-tx}\nu(dx)$$

with $\nu \in V_+$ we have $0 \notin \mathrm{at}(\nu)$ and

$$\int_t^\infty h(u)u^{p-1}\,du = \int_0^\infty \Gamma(p, tx)x^{-p}\nu(dx) \qquad \text{for } t \in [0, \infty).$$

Consequently, setting $\varrho(dx) = p(1+b)^{-1}\Gamma(p)x^{-p}\nu(dx)$ and taking into account (2.18) we conclude that $\varrho \in P$ and

$$\hat{\varrho}(t) = \left(1+b(1+b)^{-1}t^p\right)\exp(-t^p),$$

which, by Lemma 2.4, shows that the Kucharczak convolutions do not have the Cramér property.

3. A class of convolutions. Given $M, N \in V_+$ we denote by $C(M, N)$ the set of all positive numbers c fulfilling the condition $M(E) \leqslant cN(E)$ for all Borel subsets E of $[0, \infty)$. Put $k(M, N) = \inf C(M, N)$, where the infimum of an empty set is assumed to be ∞. The following statements are evident:

$$k(T(a)M, T(a)N) = k(M, N) \qquad \text{for } a \in (0, \infty); \tag{3.1}$$

$$k(M, N) \leqslant k(M, Q)k(Q, N) \tag{3.2}$$

whenever $k(M, Q)$ and $k(Q, N)$ are finite;

$$k(M_1+M_2, N_1+N_2) \leqslant \max\left(k(M_1, N_1), k(M_2, N_2)\right), \tag{3.3}$$

$$k(M_1 \circ M_2, N_1 \circ N_2) \leqslant k(M_1, N_1)k(M_2, N_2) \tag{3.4}$$

whenever $k(M_1, N_1)$ and $k(M_2, N_2)$ are finite;

$$\lim_{n\to\infty} k(M_n, N_n) \geqslant k(M, N) \tag{3.5}$$

if $M_n \to M$ and $N_n \to N$;

$$s(M) \subset s(N) \tag{3.6}$$

whenever $k(M, N)$ is finite.

Throughout this paper K will stand for the set of all probability measures λ fulfilling the condition $k(T(b)\lambda, \lambda)) < \infty$ for $b \in (1, \infty)$.

LEMMA 3.1. *The measures λ from K are of the form $\lambda = p\delta_0 + (1-p)\nu$, where $p \in [0, 1]$ and the probability measure ν is equivalent to the Lebesgue measure on a half-line $[u, \infty)$ with $u \geqslant 0$.*

Proof. By (3.6) we have the inclusion $bs(\lambda) = s(T(b)\lambda) \subset s(\lambda)$ for $b \in (1, \infty)$, which shows that either $s(\lambda) = \{0\}$ or $s(\lambda \mid (0, \infty)) = [u, \infty)$ for some $u \geqslant 0$. Consequently, to prove our lemma it suffices to show that the measure $\lambda|(0, \infty)$ is absolutely continuous with respect to the Lebesgue measure on $(0, \infty)$. Taking an arbitrary Borel subset E of $(0, \infty)$ of the Lebesgue measure 0 we have the equality

$$\int_{1/2}^{\infty} 1_E(ax)a^{-2}da = 0 \quad \text{for all } x \in (0, \infty).$$

Consequently,

$$\int_{1/2}^{\infty} \lambda(a^{-1}E)a^{-2}da = \int_{0^+}^{\infty} \int_{1/2}^{\infty} 1_E(ax)a^{-2}da\lambda(dx) = 0,$$

which yields

$$\big(T(a)\lambda\big)(E) = \lambda(a^{-1}E) = 0 \quad \text{for almost all } a \in (1/2, 1). \tag{3.7}$$

Using (3.1) we conclude that the measure λ is absolutely continuous with respect to the measure $T(a)\lambda$ with $a \in (0, 1)$. Hence and from (3.7) we get the equality $\lambda(E) = 0$, which completes the proof.

Let A be a subset of $[0, \infty)$. A mapping from A into $[0, \infty)$ is said to be *locally bounded* if it is bounded on every compact subset of A.

LEMMA 3.2. *For every $\lambda \in K$ the mapping $(1, \infty) \ni b \to k\big(T(b)\lambda, \lambda\big)$ is locally bounded.*

Proof. The inequality $k\big(T(b)\lambda, \lambda\big) \geqslant 1$ for $b \in [1, \infty)$ is evident. Setting

$$F(x) = \log k\big(T(e^x)\,\lambda, \lambda\big) \quad \text{for } x \in [0, \infty),$$

by (3.1) and (3.2) we have

$$F(x+y) \leqslant \log k\big(T(e^{x+y})\lambda, T(e^y)\lambda\big) + \log k\big(T(e^y)\lambda, \lambda\big) = F(x) + F(y)$$

for $x, y \in [0, \infty)$. Thus the function F is subadditive on $[0, \infty)$. Applying Theorem 6.4.1 of [3] we infer that F is locally bounded on $(0, \infty)$, which yields the assertion of the lemma.

LEMMA 3.3. *For every $\lambda \in K$ and $M \in W_+$ with $s(M) \subset (1, \infty)$ the inequality $k(M \Box \lambda, \lambda) < \infty$ is true.*

Proof. Since $s(M)$ is a compact subset of $(1, \infty)$, we can find, by Lemma 2.3, a positive constant c such that the inequality $(T(b)\lambda)(E) \leqslant c\lambda(E)$ holds for all Borel subsets E of $[0, \infty)$ and $b \in s(M)$. Integrating both sides of the above inequality with respect to $M(db)$ and using formula (1.9) we get the inequality $k(M \Box \lambda, \lambda) \leqslant c$, which completes the proof.

We say that a generalized convolution $\circ$ is *K-majorizable* if there exists a measure $\eta \in K$ such that

$$k\big(M \circ N, (M *_\infty N) + (M *_\infty N) \Box \eta\big) < \infty \tag{3.8}$$

and

$$k\big(T(q)\eta, M \circ N\big) < \infty \tag{3.9}$$

for all $M, N \in W_+$ and some $q \in (0, \infty)$ depending on M and N. The measure η is called a *majorizing measure.*

Before taking up a more detailed study of K-majorizable convolutions we establish a very convenient sufficient condition in terms of the expressions $\delta_1 \circ \delta_a$ with $a \in (0, 1]$ for a convolution to be K-majorizable.

PROPOSITION 3.1. *Suppose that for $a \in (0, 1]$ we have a representation*

$$\delta_1 \circ \delta_a(dx) = f(a)\delta_1(dx) + h(a, x)\eta(dx), \tag{3.10}$$

where $\eta \in K$, f and h are Borel functions defined on $(0, 1]$ and $(0, 1] \times [0, \infty)$, respectively, the mapping

$$(0, 1] \ni a \to H(a) = m_0 - \operatorname{ess\,sup}\{h(a, x)\colon\ x \in s(\eta)\} \tag{3.11}$$

is locally bounded and for some $c \in (1, \infty)$

$$G(a, c) = m_0 - \operatorname{ess\,inf}\{h(a, x)\colon\ x \in cs(\eta)\} > 0. \tag{3.12}$$

Then $\circ$ is K-majorizable with η as a majorizing measure.

Proof. Observe that, by Lemma 3.1, the measure η is absolutely continuous with respect to the measure m_0. Obviously, $f(a) \in [0, 1]$, which yields the inequality

$$k(\delta_1 \circ \delta_a, \delta_1 + \eta) \leqslant \max(1, H(a)) \quad \text{for } a \in (0, 1].$$

Setting $d = d(a, b) = \min(a, b)/\max(a, b)$ for $a, b \in (0, \infty)$, by (3.1) we have

$$k\big(\delta_a \circ \delta_b, \delta_{\max(a,b)} + T(\max(a, b))\eta\big) \leqslant \max\big(1, H(d)\big). \tag{3.13}$$

Given $M, N \in W_+$ we conclude, by the local boundedness of H, that $H\big(d(a, b)\big) \leqslant r$ for some $r > 1$ and all $a \in s(M)$ and $b \in s(N)$. Consequently, by (3.13),

$$\delta_a \circ \delta_b \leqslant r\big(\delta_{\max(a,b)} + T(\max(a, b))\eta\big) \quad \text{for all } a \in s(M) \text{ and } b \in s(N).$$

Integrating both sides of the above inequality with respect to $M(da)N(db)$ we get, by virtue of (1.9), the inequality

$$M \Box N \leqslant r\big((M *_\infty N) + (M *_\infty N) \Box \eta\big),$$

which proves (3.8). Further, taking a number q fulfilling the condition $q > c\max(a, b)$ for $a \in s(M)$ and $b \in s(N)$ we have, by (3.1), the inequality

$$l(a, b) = k\big(T(q)\eta,\ T(c\max(a, b))\eta\big) < \infty. \tag{3.14}$$

Moreover, by (3.5), the function $l(a, b)$ is Borel measurable on the product $s(M) \times s(N)$. Since $cs(\eta) = s\big(T(c)\eta\big)$ and $v = k\big(T(c)\eta, \eta\big) < \infty$, we have

$$\eta\big(E \cap cs(\eta)\big) \geqslant v^{-1}\big(T(c)\eta\big)(E)$$

for all Borel subsets E of $[0, \infty)$. Hence taking into account (3.10) and (3.12) we get

$$\delta_1 \circ \delta_d \geqslant G(d, c)v^{-1}T(c)\eta \qquad \text{for all } d \in (0, 1].$$

Substituting $d = \min(a, b)/\max(a, b)$ for $a \in s(M)$ and $b \in s(N)$ and setting $u(a, b) = G(d, c)v^{-1}l^{-1}(a, b)$, from (3.14) and the last inequality we get

$$\begin{aligned} \delta_a \circ \delta_b = T\big(\max(a, b)\big)(\delta_1 \circ \delta_d) &\geqslant G(d, c)v^{-1}T\big(c\max(a, b)\big)\eta \\ &\geqslant u(a, b)T(q)\eta \end{aligned} \tag{3.15}$$

for all $a \in s(M)$ and $b \in s(N)$. It is clear that the function $u(a, b)$ is Borel measurable on the product $s(M) \times s(N)$ and maps this product into the interval $(0, 1]$. Integrating both sides of (3.15) with respect to $M(da)N(db)$ and introducing the notation

$$u_0 = \int_0^\infty \int_0^\infty u(a, b)M(da)N(db)$$

we get the inequalities $u_0 > 0$ and $u_0 T(q)\eta \leqslant M \circ N$ which imply (3.9). Proposition 3.1 is thus proved.

We shall apply Proposition 3.1 to generalized convolutions discussed in Examples 2.1–2.4. In what follows by the *Pareto measure* with parameter $\beta \in (0, \infty)$ we mean the measure $\beta x^{-1-\beta}1_{[1,\infty)}(x)dx$. We associate with $(1, 1/2)$-convolution a majorizing measure

$$\eta(dx) = (1 + 2\log x)x^{-3}1_{[1,\infty)}(x)dx.$$

In all the remaining cases as a majorizing measure η we take the Pareto measure with the following parameter β: $\beta = n+1$ for Kendall convolutions with parameter n, $\beta = \min\big(2, 1/(1-p)\big)$ for $(1, p)$-convolutions with $p \neq 1/2$, $\beta = 2$ for $(2, p)$-convolutions with $p \in (2, \infty)$, and $\beta = 2p$ for Kucharczak convolutions with $p \in (0, 1)$. Here we have $s(\eta) = [1, \infty)$. Starting from representation (2.7) and determining the function h by the formula

$$h(a, x)\eta(dx) = g(a, 1, x)dx \quad \text{for } a \in (0, 1]$$

we conclude, by standard calculations, that conditions (3.11) and (3.12) with $c = 2$ are fulfilled. Thus as an immediate consequence of Proposition 3.1 we get the following statement:

COROLLARY 3.1. *The Kendall convolutions with $n = 1, 2, \ldots$, $(1, p)$-convolutions with $p \in (0, 1)$, $(2, p)$-convolutions with $p \in (2, \infty)$, and the Kucharczak convolutions with $p \in (0, 1)$ are K-majorizable.*

Now we proceed to the study of K-majorizable convolutions.

LEMMA 3.4. *Majorizing measures of a K-majorizable convolution are equivalent to the Lebesgue measure on a half-line $[u, \infty)$ for some $u \geqslant 0$.*

Proof. Let η and γ be a majorizing measure and a characteristic measure of a K-majorizable convolution, respectively. Since, by Lemma 2.2 in [10],

$$0 \notin \operatorname{at}(\gamma), \tag{3.16}$$

we can find an interval A such that $\gamma | A \in W_+$. Setting $M = \gamma | A$, by (1.2) and (1.3) we have $M \circ M \leqslant \gamma \circ \gamma = T(c)\gamma$, where $c = r(\kappa, 1, 1) > 0$. On the other hand, by (3.9) we obtain $k(T(q)\eta, M \circ M) < \infty$, which, by (3.2), yields $k(T(q)\eta, T(c)\gamma) < \infty$. Consequently, $\operatorname{at}(\eta) \subset \operatorname{at}(T(cq^{-1})\gamma)$ and, by (3.16), $0 \notin \operatorname{at}(\eta)$. Now our assertion is an immediate consequence of Lemma 3.1.

LEMMA 3.5. *The max-convolution is not K-majorizable.*

Proof. Suppose the contrary and denote by η a majorizing measure for $*_\infty$. Since $\delta_1 \in W_+$ and $\delta_1 *_\infty \delta_1 = \delta_1$, by (3.9) we have $k(T(q)\eta, \delta_1) < \infty$ for some $q \in (0, \infty)$. Thus $T(q)\eta = \delta_1$, which contradicts Lemma 3.4.

As a consequence of our Lemma 3.5 and Lemma 2.1 in [10] we get the following statement:

COROLLARY 3.2. *The characteristic exponent of K-majorizable convolutions is finite.*

LEMMA 3.6. *Let $\circ$ be a K-majorizable convolution. Then the inclusion*

$$\operatorname{at}(M \circ N) \subset \operatorname{at}(M) \cup \operatorname{at}(N)$$

holds for all $M, N \in V_+$.

Proof. Observe that the measures M and N from V_+ can be written in the form

$$M = a\delta_0 + \sum_{k=1}^{\infty} M_k, \qquad N = b\delta_0 + \sum_{k=1}^{\infty} N_k,$$

where $a, b \in [0, \infty)$ and $M_k, N_k \in W_+$ $(k = 1, 2, \ldots)$. Starting from the formula

$$M \circ N = ab\delta_0 + a \sum_{k=1}^{\infty} N_k + b \sum_{k=1}^{\infty} M_k + \sum_{j,k=1}^{\infty} M_j \circ N_k$$

we get the inclusion

$$\operatorname{at}(M \circ N) \subset \operatorname{at}(M) \cup \operatorname{at}(N) \cup \bigcup_{j,k=1}^{\infty} \operatorname{at}(M_j \circ N_k). \tag{3.17}$$

By Lemma 3.4 the measures $(M_j *_\infty N_k) \square \eta$ are absolutely continuous with respect to the Lebesgue measure on the half-line $[0, \infty)$. Consequently, by (3.8),

$$\operatorname{at}(M_j \circ N_k) \subset \operatorname{at}(M_j *_\infty N_k) \subset \operatorname{at}(M_j) \cup \operatorname{at}(N_k) \subset \operatorname{at}(M) \cup \operatorname{at}(N),$$

which together with (3.17) yields the assertion of the lemma.

LEMMA 3.7. *The characteristic measure γ of a K-majorizable convolution has no atom.*

Proof. We argue indirectly. Suppose that $\operatorname{at}(\gamma) \neq \emptyset$. Recall that, by Corollary 3.2, the characteristic exponent κ of the convolution in question is finite. Moreover, by Lemma 2.2 in [10], $0 \notin \operatorname{at}(\gamma)$. Since the set $\operatorname{at}(\gamma)$ is at most denumerable, we can find a pair $p, q \in (0, \infty)$ such that the numbers p^κ, q^κ are linearly independent over the field generated by the numbers c^κ with $c \in \operatorname{at}(\gamma)$. By (1.2) and (1.3) we have the formula

$$T(p)\gamma \circ T(q)\gamma = T\big(r(\kappa, p, q)\big)\gamma$$

which, by Lemma 3.6, yields the inclusion

$$r(\kappa, p, q) \operatorname{at}(\gamma) \subset p \operatorname{at}(\gamma) \cup q \operatorname{at}(\gamma).$$

Consequently, for any $a \in \operatorname{at}(\gamma)$ there exists $b \in \operatorname{at}(\gamma)$ such that either $(p^\kappa + q^\kappa)a^\kappa = p^\kappa b^\kappa$ or $(p^\kappa + q^\kappa)a^\kappa = q^\kappa b^\kappa$, which contradicts the linear independence of p^κ and q^κ. The lemma is thus proved.

LEMMA 3.8. *For every K-majorizable convolution there exists a version of the kernel of the weak characteristic function which is continuous on the set $[0, w) \cup (w, \infty)$ for some $w \in (0, \infty]$.*

Proof. Given $p \in (0, \kappa)$ we denote by σ a $\circ$-stable probability measure of index p ([10], Section 2). By Corollary 4.5 in [10] the measure σ is equivalent to the Lebesgue measure on $[0, \infty)$ and, by Theorem 4.2 in [10], we may assume that

$$\hat{\sigma}(t) = \exp(-t^p) \tag{3.18}$$

m_0-almost everywhere.

Notice that, by Lemma 3.4, the majorizing measure η of the convolution in question is absolutely continuous with respect to σ. Since, by Lemma 2.2 in [10], $0 \notin \operatorname{at}(\sigma)$, we have a representation

$$\sigma = \sum_{j=1}^{\infty} M_j, \tag{3.19}$$

where $M_j \in W_+$ $(j = 1, 2, \ldots)$ and $s(M_1) \subset (1, \infty)$. Observe that

$$\delta_1 *_\infty M_1 = M_1, \quad \delta_1 *_\infty M_j \leqslant \delta_1 + \sigma, \quad (\delta_1 *_\infty M_j) \Box \eta \leqslant \eta + \sigma \Box \eta$$

$(j = 1, 2, \ldots)$. Setting

$$\lambda_j = (\delta_1 *_\infty M_j) + (\delta_1 *_\infty M_j) \Box \eta$$

we infer that the measure λ_1 is absolutely continuous with respect to σ, and the remaining measures λ_j $(j = 2, 3, \ldots)$ are absolutely continuous with respect to $\delta_1 + \sigma$. Since, by (3.8), $k(\delta_1 \circ M_j, \lambda_j) < \infty$ for $j = 1, 2, \ldots$, we conclude that the measure $\lambda_1 \circ M_1$ is absolutely continuous with respect to σ and the measures $\delta_1 \circ M_j$ $(j = 2, 3, \ldots)$ are absolutely continuous with respect to $\delta_1 + \sigma$. Hence and from (3.19) it follows that $\delta_1 \circ \sigma \neq \delta_1$ and the measure $\delta_1 \circ \sigma$ is absolutely continuous with respect to $\delta_1 + \sigma$. In other words, we have the equality

$$\delta_1 \circ \sigma = c\delta_1 + (1-c)\varrho,$$

where $c \in [0, 1)$ and the probability measure ϱ is absolutely continuous with respect to the Lebesgue measure on $[0, \infty)$. By the definition of the weak characteristic function ([10], p. 82), $\hat{\varrho}(t)$ is continuous. By (3.18) the last equality can be written in terms of the weak characteristic functions as follows:

$$\Omega(t)\exp(-t^p) = c\Omega(t) + (1-c)\hat{\varrho}(t)$$

m_0-almost everywhere. Denoting by $w \in (0, \infty]$ the only solution of the equation $\exp(-t^p) = c$ we have the formula

$$\Omega(t) = (1-c)\hat{\varrho}(t)\big(\exp(-t^p) - c\big)^{-1}$$

m_0-almost everywhere on $[0, w) \cup (w, \infty)$. Obviously, the right-hand side of the above formula is continuous on $[0, w) \cup (w, \infty)$ and can be taken as a required version of the kernel Ω, which completes the proof.

From now on it will be tacitly assumed that the kernel Ω corresponding to a K-majorizable convolution has at most one discontinuity point. We shall see that K-majorizable convolutions have many properties similar to those for regular ones. In particular, from the basic theorem on weak convergence ([1], Theorem 25.7) we get the following statement:

LEMMA 3.9. *For every K-majorizable convolution and $\mu_n, \mu \in P$ with* $\mathrm{at}(\mu) \subset \{0\}$ *the weak convergence $\mu_n \to \mu$ yields the convergence $\hat{\mu}_n \to \hat{\mu}$ uniform on every compact subset of* $[0, \infty)$.

Applying the above lemma to relation (1.1) and using (1.8) and Lemma 3.7 we get the following

COROLLARY 3.3. *For K-majorizable convolutions the relation $T(c_n)\delta_1^{\circ n} \to \gamma$ for some norming constants $c_n \in (0, \infty)$ yields $\Omega(c_n t)^n \to \exp(-t^\kappa)$ uniformly on every compact subset of* $[0, \infty)$.

LEMMA 3.10. *For every K-majorizable convolution there exists a number $v \in (0, \infty)$ such that $\Omega(t) < 1$ for $t \in (0, v)$.*

Proof. Contrary to this let us suppose that $\Omega(b_n) = 1$ for a sequence $b_n \in (0, \infty)$ tending to 0. Let c_n be a sequence of norming constants in (1.1). By Lemmas 2.6 and 2.7 in [10] we may assume without loss of generality that the sequence c_n is monotone non-increasing, $c_1 > b_1$ and $c_{n+1}/c_n \to 1$ as $n \to \infty$. Consequently, for any n there exists an index k_n such that $c_{k_n+1} \leqslant b_n < c_{k_n}$. Setting $d_n = b_n/c_{k_n}$ we have $d_n \to 1$ as $n \to \infty$. Further, by Corollary 3.3, $\Omega(c_n t)^n \to \exp(-t^\kappa)$ uniformly on every compact subset of $[0, \infty)$. Thus

$$1 = \Omega(b_n)^{k_n} = \Omega(c_{k_n} d_n)^{k_n} \to e^{-1},$$

which gives a contradiction. This proves the lemma.

Starting from Corollaries 3.2 and 3.3 and Lemma 3.10 and applying the same arguments as used in [8], Theorem 7, for regular convolutions we get the following statement:

COROLLARY 3.4. *For K-majorizable convolutions with the characteristic exponent κ the kernel Ω fulfils the condition*

$$\lim_{x \to 0} \frac{1-\Omega(tx)}{1-\Omega(x)} = t^\kappa$$

uniformly on every compact subset of $[0, \infty)$.

LEMMA 3.11. *Let $\circ$ be a K-majorizable convolution. Suppose that $\mu, \mu_n \in P_\infty(\circ)$, $\mu = \mu_n^{\circ n}$ $(n = 1, 2, \ldots)$ and $n\mu_n \to 0$ on every half-line $[b, \infty)$ with $b \in (0, \infty)$. Then $\mu \in \mathrm{Gauss}(\circ)$.*

Proof. Using the same arguments as in the proving of Lemma 1.3 we get

$$\lim_{n \to \infty} k_n(1-\hat{\mu}_{k_n}(t)) = -\log \hat{\mu}(t) \tag{3.20}$$

for a subsequence $k_1 < k_2 < \ldots$ m_0-almost everywhere. If $\hat{\mu}(t) = 1$ m_0-almost everywhere, then $\mu = \delta_0$ and, consequently, $\mu \in \mathrm{Gauss}(\circ)$. In the remaining case there exists a number $t_0 \in (0, \infty)$ such that equality (3.20) holds for $t = t_0$ and $c = -t_0^{-\kappa} \log \hat{\mu}(t_0) \in (0, \infty)$. Setting

$$u(t, b) = \sup\left\{\left|\frac{1-\Omega(tx)}{1-\Omega(x)} - t^\kappa\right| : x \in (0, b)\right\}$$

we have, by Corollary 3.4,

$$\lim_{b \to 0} u(t, b) = 0.$$

Moreover, for every $b \in (0, \infty)$

$$|k_n(1-\hat{\mu}_{k_n}(t)) - ct^\kappa| \leqslant u(tt_0^{-1}, bt_0^{-1})k_n(1-\hat{\mu}_{k_n}(t_0)) + 2(1+t^\kappa t_0^{-\kappa})\mu_{k_n}([b, \infty)),$$

which, by (3.20), yields $\hat{\mu}(t) = \exp(-ct^{\kappa})$ m_0-almost everywhere as $n \to \infty$ and $b \to 0$. Thus $\mu = T(c^{1/\kappa})\gamma$, which completes the proof.

For regular convolutions the following statement is an immediate consequence of the Lévy–Khinchin representation of $P_\infty(\circ)$ given in [8], Theorem 13.

LEMMA 3.12. *Let* $\circ$ *be a K-majorizable convolution. If* $\mu \in P_\infty(\circ)$ *and* $D(\circ, \mu) \cap \mathrm{Poiss}(\circ) = \{\delta_0\}$, *then* $\mu \in \mathrm{Gauss}(\circ)$.

Proof. We argue indirectly. Suppose that $\mu \in P_\infty(\circ) \setminus \mathrm{Gauss}(\circ)$ and

$$D(\circ, \mu) \cap \mathrm{Poiss}(\circ) = \{\delta_0\}. \tag{3.21}$$

Put $\mu = \mu_n^{\circ n}$, where $\mu_n \in P_\infty(\circ)$ $(n = 1, 2, \ldots)$. By Lemmas 1.1 and 3.11 we conclude that there exist a number $b \in (0, \infty)$, a sequence $k_1 < k_2 < \ldots$ and a measure $M \in U_+$ which does not vanish identically and $k_n \mu_{k_n} | [b, \infty) \to M$. Using the same arguments as in the proving of Lemma 1.3 we may assume that formula (3.20) is true for the same subsequence $k_1 < k_2 < \ldots$ Consequently, setting

$$M_n = k_n \mu_{k_n} | [b, \infty) \quad \text{and} \quad N_n = k_n \mu_{k_n} | (0, b),$$

we have $\pi(M_n) \to \pi(M) \neq \delta_0$ and

$$\pi(M_n)^\wedge(t)\pi(N_n)^\wedge(t) = \exp k_n\big(\hat{\mu}_{k_n}(t) - 1\big) \to \hat{\mu}(t)$$

m_0-almost everywhere, which yields the relation

$$\pi(M_n) \circ \pi(N_n) \to \mu.$$

Observing that, by Corollary 2.3 in [13], the sequence $\pi(N_n)$ is conditionally compact and denoting by λ its cluster point we get the equality $\pi(M) \circ \lambda = \mu$. Consequently, $\pi(M) \in D(\circ, \mu)$, which contradicts (3.21). The lemma is thus proved.

LEMMA 3.13. *For K-majorizable convolutions the equality* $I(\circ) \cap \mathrm{Poiss}(\circ) = \{\delta_0\}$ *is true.*

Proof. Suppose the contrary. Then we can find a measure $M \in W_+$ such that $\pi(M) \in I(\circ)$. Let η be a majorizing measure for the convolution in question. By (3.9) there exists a positive number q fulfilling the condition

$$k\big(T(q)\eta, M^{\circ 2}\big) < \infty. \tag{3.22}$$

Since, by Lemma 3.4, the measure η is equivalent to the Lebesgue measure on a half-line $[u, \infty)$ with $u \geqslant 0$, we can find an interval $A = [a, b]$ fulfilling the conditions $a > 1$, $qa > \max s(M)$ and $\eta | A \in W_+$. Setting $N = T(q)(\eta | A)$ we have $N \in W_+$,

$$s(M) \cap s(N) = \emptyset \tag{3.23}$$

and

$$M *_{\infty} N = N. \tag{3.24}$$

Note that, by (3.1), $k(N, T(q)\eta) = k(\eta \mid A, \eta) \leqslant 1$, which, by (3.2) and (3.22) yields

$$k(N, M^{\circ 2}) < \infty. \tag{3.25}$$

Since $s(\eta \mid A) \subset A \subset (1, \infty)$, we have, by (3.1) and Lemma 3.3,

$$k(N \square \eta, T(q)\eta) = k((\eta \mid A) \square \eta, \eta) < \infty,$$

which, by (3.22), yields

$$k(N \square \eta, M^{\circ 2}) < \infty. \tag{3.26}$$

Further, by (3.24),

$$(M *_{\infty} N) + (M *_{\infty} N) \square \eta = N + N \square \eta$$

and

$$(N *_{\infty} N) + (N *_{\infty} N) \square \eta \leqslant 2(N + N \square \eta),$$

which, by (3.2), (3.8), (3.25) and (3.26), yield the inequalities

$$k(M \circ N, M^{\circ 2}) < \infty \tag{3.27}$$

and

$$k(N^{\circ 2}, M^{\circ 2}) < \infty. \tag{3.28}$$

From (3.4) and (3.27) we get

$$k(M^{\circ 2} \circ N, M^{\circ 3}) < \infty. \tag{3.29}$$

Similarly, by (3.4) and (3.28), $k(N^{\circ 3}, M^{\circ 2} \circ N) < \infty$, which together with (3.2) and (3.29) yields

$$k(N^{\circ 3}, M^{\circ 3}) < \infty. \tag{3.30}$$

It is clear, by (3.23), (3.25), (3.27), (3.29) and (3.30), that setting $Q = \varepsilon N$, where ε is a sufficiently small positive number, we get a measure belonging to W_{+} and fulfilling the conditions

$$s(M) \cap s(Q) = \emptyset, \tag{3.31}$$

$$2M \circ Q + 2Q \leqslant M^{\circ 2} \tag{3.32}$$

and

$$3M^{\circ 2} \circ Q + Q^{\circ 3} \leqslant M^{\circ 3}. \tag{3.33}$$

Put $R = M-Q$. We shall prove the inequality $R^{\circ k} \geqslant 0$ for all $k \geqslant 2$. First observe that, by (3.32) and (3.33),

$$R^{\circ 2} = M^{\circ 2}-2M \circ Q+Q^{\circ 2} \geqslant 0$$

and

$$R^{\circ 3} = M^{\circ 3}-3M^{\circ 2} \circ Q+3M \circ Q^{\circ 2}-Q^{\circ 3} \geqslant 0.$$

For the remaining exponents our assertion is an immediate consequence of the equalities $R^{\circ 2n} = (R^{\circ 2})^{\circ n}$ and $R^{\circ (2n+3)} = (R^{\circ 2})^{\circ n} \circ R^{\circ 3}$ $(n = 1, 2, \ldots)$. Further, by (3.32),

$$R+2^{-1}R^{\circ 2} = M-Q+2^{-1}M^{\circ 2}-M \circ Q+2^{-1}Q^{\circ 2} \geqslant 0.$$

Hence the formula

$$\nu = e^{-r}\left(\delta_0+R+\frac{1}{2}R^{\circ 2}+\sum_{k=3}^{\infty}\frac{R^{\circ k}}{k!}\right)$$

with $r = R([0, \infty))$ defines a probability measure. Moreover, it is easy to check the equality $\pi(Q) \circ \nu = \pi(M)$. Consequently, $\pi(Q) \in D(\circ, \pi(M))$. Since, by (2.4) and Theorem 2.1, $D(\circ, \pi(M)) \subset P_\infty(\circ)$, we conclude, by Lemma 2.2, that the inclusion $s(Q) \subset s(M)$ holds. But this contradicts (3.31). The lemma is thus proved.

We are now in a position to prove the main results of this section. The following statements are an immediate consequence of (2.4) and Lemmas 3.12 and 3.13.

THEOREM 3.1. *For K-majorizable convolutions the inclusion* $I(\circ) \subset$ Gauss $(\circ)$ *is true.*

THEOREM 3.2. *Suppose that a K-majorizable convolution does not have the Cramér property. Then* $I(\circ) = \{\delta_0\}$.

Since the convolutions discussed in Examples 2.1–2.4 and Corollary 3.1 fulfil the conditions of Theorem 3.2, we have the following result:

COROLLARY 3.5. *For Kendall convolutions with* $n = 1, 2, \ldots$, $(1, p)$*-convolutions with* $p \in (0, 1)$, $(2, p)$*-convolutions with* $p \in (2, \infty)$ *and Kucharczak convolutions with* $p \in (0, 1)$, δ_0 *is the only anti-irreducible measure.*

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