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LARGE DEVIATIONS AND LAW OF THE ITERATED LOGARITHM FOR GENERALIZED DOMAINS OF ATTRACTION

BY

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Abstract. Suppose $X, X_1, X_2, \ldots$ are i.i.d. random vectors, $S_n = \sum_{i=1}^{n} X_i$ and A_n are linear operators such that $A_n S_n$ converges in law to some full random vector Y. Then we say that X belongs to the *strict generalized domain of attraction* of Y. We show that if Y has no normal component, then $(A_n S_n)$ satisfies a large deviation principle. This large deviation result is used to show that a law of the iterated logarithm for $(A_n S_n)$ holds, which gives the precise growth behavior of the sample paths of the random walk (S_n).

1. Introduction. Suppose that $X, X_1, X_2, \ldots$ are independent random vectors on $\mathbf{R}^d$ with common distribution μ and Y is a full random vector on $\mathbf{R}^d$ with distribution ν. If there exist linear operators A_n on $\mathbf{R}^d$ and constants $b_n \in \mathbf{R}^d$ such that for $S_n = \sum_{i=1}^{n} X_i$ we have

$$A_n S_n - b_n \Rightarrow Y, \tag{1.1}$$

then we say that μ belongs to the *generalized domain of attraction* of ν and we write $\mu \in \mathrm{GDOA}(\nu)$. Here $\Rightarrow$ denotes convergence in distribution. The class of all possible limit laws in (1.1) is called the *operator stable laws.*

Operator stable laws were characterized by Sharpe [14]. He showed that an operator stable law is infinitely divisible and satisfies

$$\nu^t = t^A \nu * \delta(a(t)) \tag{1.2}$$

for all $t > 0$, where $t^A = \exp(A \log t)$ is defined in terms of the exponential operator $\exp(B) = \sum B^k/k!$. The linear operator A in (1.2) is called an *exponent* of ν. Generalized domains of attraction have been examined in a number of papers, including Hahn and Klass [5] and Meerschaert [12].

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It is well known that central limit behavior of sums of i.i.d. random vectors can imply strong limit theorems, in particular, laws of the iterated logarithm, for these sums. If ν is a pure Gaussian measure, Weiner [17] gave necessary and sufficient conditions on the distribution of X such that a law of the iterated logarithm holds. We will prove, using a large deviation result for the sums S_n proved in Section 2, that if ν is nonnormal, a law of the iterated logarithm also holds. Since we can decompose ν into a normal and a nonnormal part and since, using the spectral decomposition of Meerschaert [11], this decomposition carries over to the norming operators A_n, we get a precise knowledge of the almost sure behavior of S_n.

In the following let ν be a symmetric full nonnormal operator stable law on $\mathbf{R}^d$ and let $\mu \in \mathrm{GDOA}(\nu)$ be symmetric such that

$$A_n S_n \Rightarrow Y. \tag{1.3}$$

Note that in view of the symmetry no centering in (1.3) is required.

2. Large deviations. Fix any unit vector $\theta \in \mathbf{R}^d$. Using (1.3) we get asymptotic information about $P\{|\langle A_n S_n, \theta\rangle| \leqslant x_n\}$ if $x_n = O(1)$, and thus only trivial information in the case where $x_n \to \infty$ as $n \to \infty$. However, as will be seen in Section 3 below, we often require information on $P\{|\langle A_n S_n, \theta\rangle| > x_n\}$ under these circumstances. This type of problem is called a *problem of the probability of large deviations.* In the one-dimensional situation $d = 1$, Heyde [6] and [7] proved that $P\{|A_n S_n| > x_n\}$ is asymptotically equal to $nP\{|A_n X_1| > x_n\}$ as $n \to \infty$, using the theory of regular variation. However, in the multidimensional setting of generalized domains of attraction the tail functions

$$t \mapsto \mu\{x \in \mathbf{R}^d\colon\ |\langle x, \theta\rangle| > t\}$$

are no longer regular varying, but only R–O varying. (See Seneta [13] for a definition of R–O variation.) Using the theory of multivariable regular variation developed by Meerschaert [10] we will show that the ratio between $P\{|\langle A_n S_n, \theta\rangle| > x_n\}$ and $nP\{|\langle A_n X_1, \theta\rangle| > x_n\}$ remains bounded from zero and infinity. It turns out that this is sufficiently sharp to prove a law of the iterated logarithm.

THEOREM 2.1. *For every compact subset $K \subset \Gamma = \mathbf{R}^d \setminus \{0\}$ there exist positive constants C_1 and C_2 such that for all $\theta \in K$ and every nondecreasing sequence (x_n) of real numbers tending to infinity we have*

$$C_1 \leqslant \liminf_{n\to\infty} \frac{P\{|\langle A_n S_n, \theta\rangle| > x_n\}}{nP\{|\langle A_n X_1, \theta\rangle| > x_n\}} \leqslant \limsup_{n\to\infty} \frac{P\{|\langle A_n S_n, \theta\rangle| > x_n\}}{nP\{|\langle A_n X_1, \theta\rangle| > x_n\}} \leqslant C_2. \tag{2.1}$$

Remark 2.2. Theorem 2.1 gives information about how fast the tails of $A_n S_n$ decrease in any radial direction, whereas from (1.3) we only get information about $P\{|\langle A_n S_n, \theta\rangle| > x_n\}$ if $x_n = O(1)$.

Proof of Theorem 2.1. First we will prove the lower bound in (2.1). For $\varepsilon > 0$ and $1 \leqslant i \leqslant n$ let

$$B_i^{(n)} = \{|\langle A_n \sum_{j=1, j\neq i}^{n} X_j, \theta\rangle| < \varepsilon x_n\} \quad \text{and} \quad D_i^{(n)} = \{|\langle A_n X_i, \theta\rangle| > (1+\varepsilon) x_n\}.$$

Then we have

$$\{|\langle A_n S_n, \theta\rangle| > x_n\} \supset \bigcup_{i=1}^{n} (D_i^{(n)} \cap B_i^{(n)}),$$

and hence by the i.i.d. assumption on the X_i we get

$$(2.2)\qquad P\{|\langle A_n S_n, \theta\rangle| > x_n\} \geqslant P\big(\bigcup_{i=1}^{n} (D_i^{(n)} \cap B_i^{(n)})\big)$$

$$= \sum_{i=1}^{n} P\big((D_i^{(n)} \cap B_i^{(n)}) \cap \bigcap_{j=1}^{i-1} (D_j^{(n)} \cap B_j^{(n)})^c\big) \geqslant \sum_{i=1}^{n} P\big((D_i^{(n)} \cap B_i^{(n)}) \setminus \bigcup_{j=1}^{i-1} (D_j^{(n)} \cap D_i^{(n)})\big)$$

$$\geqslant \sum_{i=1}^{n} \big[P(D_i^{(n)}) P(B_i^{(n)}) - \sum_{j=1}^{i-1} P(D_j^{(n)}) P(D_i^{(n)})\big] \geqslant nP(D_1^{(n)}) \big[P(B_1^{(n)}) - nP(D_1^{(n)})\big].$$

From a standard convergence of types argument we know that $\{(A_n A_{n-1}^{-1})^*\}$ is relatively compact in $\mathrm{GL}(\mathbb{R}^d)$, so $\{(A_n A_{n-1}^{-1})^* \theta\colon\ n \geqslant 1,\ \theta \in K\}$ is compactly contained in Γ. Therefore, from (1.3) we get

$$P(B_1^{(n)}) = P\{|\langle A_n S_{n-1}, \theta\rangle| < \varepsilon x_n\}$$
$$= P\{|\langle A_{n-1} S_{n-1}, (A_n A_{n-1}^{-1})^* \theta\rangle| < \varepsilon x_n\} \to 1$$

as $n \to \infty$ uniformly in $\theta \in K$. Hence for any $0 < \delta < 1$ there exists a number N_1 such that

$$(2.3)\qquad P(B_1^{(n)}) > 1 - \delta$$

for all $n \geqslant N_1$ uniformly in $\theta \in K$. On the other hand, it follows from (1.3) and standard convergence criteria for triangular arrays (see e.g. Araujo and Giné [1]) that $n(A_n \mu) \to \phi$, where ϕ is the Lévy measure of ν. Hence, since $x_n \to \infty$, we easily get

$$nP\{|\langle A_n X_1, \theta\rangle| > (1+\varepsilon) x_n\} \to 0$$

as $n \to \infty$ uniformly in $\theta \in K$. Therefore there exists a number N_2 such that

$$(2.4)\qquad nP(D_1^{(n)}) < \delta$$

for all $n \geqslant N_2$ and all $\theta \in K$. Then, using (2.2)–(2.4), we infer for all $n \geqslant \max(N_1, N_2)$ and all $\theta \in K$ that

$$P\{|\langle A_n S_n, \theta\rangle| > x_n\} \geqslant (1-2\delta) nP\{|\langle A_n X_1, \theta\rangle| > (1+\varepsilon) x_n\}. \tag{2.5}$$

Writing $A_n^* \theta = r_n \theta_n$ with $\|\theta_n\| = 1$ and $r_n = r_n(\theta) > 0$, from (1.3) we get $\|A_n^*\| \to 0$, and hence $r_n \to 0$ as $n \to \infty$ uniformly in $\theta \in K$. Let

$$V_0(r, \theta) = \mu\{x \in \mathbb{R}^d \colon |\langle x, \theta\rangle| > r\} \tag{2.6}$$

and for $b > 0$

$$U_b(r, \theta) = \int_{|\langle x, \theta\rangle| \leqslant r} |\langle x, \theta\rangle|^b \mu\{dx\} \tag{2.7}$$

denote the tail and the truncated moment functions of μ. Then for large n, using (2.5) and Lemma 2 in Meerschaert [12], we obtain

$$\frac{P\{|\langle A_n S_n, \theta\rangle| > x_n\}}{nP\{|\langle A_n X_1, \theta\rangle| > x_n\}} \geqslant (1-2\delta) \frac{V_0(r_n^{-1}(1+\varepsilon) x_n, \theta_n)}{V_0(r_n^{-1} x_n, \theta_n)} \geqslant C(1-2\delta)(1+\varepsilon)^{-1/m-\alpha}$$

uniformly in $\|\theta_n\| = 1$, where $m = \min\{\mathrm{Re}(\lambda)\}$, λ is an eigenvalue of A, and $\alpha > 0$ is arbitrarily small. This concludes the proof of the lower bound in (2.1).

Now we will prove the upper bound in (2.1). For $1 \leqslant k \leqslant n$ and $\theta \in K$ let

$$X_{k,n}^\theta = X_k I(|\langle A_n X_k, \theta\rangle| \leqslant x_n)$$

and write $S_{n,n}^\theta = \sum_{k=1}^n X_{k,n}^\theta$. Define

$$E_n = \{|\langle A_n X_k, \theta\rangle| > x_n \text{ for at least one } k \leqslant n\}$$

and

$$G_n = \{|\langle A_n S_{n,n}^\theta, \theta\rangle| > x_n\}.$$

Then a simple calculation shows that $\{|\langle A_n S_n, \theta\rangle| > x_n\} \subset E_n \cup G_n$. Therefore we infer for every $\theta \in K$ that

$$P\{|\langle A_n S_n, \theta\rangle| > x_n\} \leqslant P(E_n) + P(G_n) \leqslant nP\{|\langle A_n X_1, \theta\rangle| > x_n\} + P(G_n). \tag{2.8}$$

Using Tschebyscheff's inequality we get

$$P(G_n) \leqslant \frac{1}{x_n^2} E(\langle A_n S_{n,n}^\theta, \theta\rangle^2). \tag{2.9}$$

Since the X_i are i.i.d., we get

$$E(\langle A_n S_{n,n}^\theta, \theta\rangle^2) = nE(\langle A_n X_{1,n}^\theta, \theta\rangle^2) + n(n-1)(E\langle A_n X_{1,n}^\theta, \theta\rangle)^2.$$

Therefore from (2.8) and (2.9) we infer for all $\theta \in K$ that

$$\frac{P\{|\langle A_n S_n, \theta\rangle| > x_n\}}{nP\{|\langle A_n X_1, \theta\rangle| > x_n\}} \tag{2.10}$$

$$\leqslant 1 + \frac{E(\langle A_n X_{1,n}^\theta, \theta\rangle^2)}{x_n^2 P\{|\langle A_n X_1, \theta\rangle| > x_n\}} + \frac{n(E\langle A_n X_{1,n}^\theta, \theta\rangle)^2}{x_n^2 P\{|\langle A_n X_1, \theta\rangle| > x_n\}}.$$

Using Lemma 2 of Meerschaert [12] we know that V_0 is uniform R–O varying, and hence by a uniform version of Feller [4], p. 289, for every $b > 0$ there exists a constant $M = M_b$ such that $U_b(r, \theta) \leqslant Mr^b V_0(r, \theta)$ for all $r \geqslant r_0$ and all $\theta \in K$. Then writing $A_n^* \theta = r_n \theta_n$ again, we get uniformly in $\theta \in K$

$$E(\langle A_n X_{1,n}^\theta, \theta\rangle^2) = r_n^2 U_2(x_n r_n^{-1}, \theta_n) \leqslant M_2 x_n^2 V_0(x_n r_n^{-1}, \theta_n)$$
$$= M_2 x_n^2 P\{|\langle A_n X_1, \theta\rangle| > x_n\},$$

and hence the first fraction on the right-hand side of (2.10) is bounded by M_2.

For the second fraction of (2.10) we use

$$|E\langle A_n X_{1,n}^\theta, \theta\rangle| \leqslant E|\langle A_n X_{1,n}^\theta, \theta\rangle| = r_n U_1(x_n r_n^{-1}, \theta_n).$$

Therefore

$$(E\langle A_n X_{1,n}^\theta, \theta\rangle)^2 \leqslant r_n^2 (U_1(x_n r_n^{-1}, \theta_n))^2 \leqslant r_n^2 (M_1 x_n r_n^{-1} V_0(x_n r_n^{-1}, \theta_n))^2$$
$$= M_1^2 x_n^2 (P\{|\langle A_n X_1, \theta\rangle| > x_n\})^2.$$

Finally, this gives for all $\theta \in K$

$$\frac{n(E\langle A_n X_{1,n}^\theta, \theta\rangle)^2}{x_n^2 P\{|\langle A_n X_1, \theta\rangle| > x_n\}} \leqslant M_1^2 nP\{|\langle A_n X_1, \theta\rangle| > x_n\},$$

where $nP\{|\langle A_n X_1, \theta\rangle| > x_n\} \to 0$ as $n \to \infty$. This completes the proof.

3. Law of the iterated logarithm. In the one-dimensional situation $d = 1$ it was shown by Chover [3] that if $\mu = \nu$ and $A_n = n^{-\alpha}$, where $1/\alpha$ is the index of the stable law ν, the following law of the iterated logarithm holds:

$$\limsup_{n\to\infty} |n^{-\alpha} S_n|^{1/\log\log n} = e^\alpha \text{ almost surely.}$$

Later, Vasudeva [15] showed that this is also true if μ is only in the domain of attraction of ν, i.e. (1.3) holds. Furthermore, Weiner [16] showed that a slightly different law of iterated logarithm holds on R^d if $\mu = \nu$ and $A_n = n^{-A}$. An extension of this result to the case of domains of normal attraction was considered by Khokhlov [9]. We will show that Chover's type of law of iterated logarithm also holds for measures in the generalized domain of attraction of ν. Additionally, we will prove that every point in a certain interval is almost surely a cluster point of the random sequence $(\|A_n S_n\|^{1/\log\log n})$. Our method of proof also shows that some results of law of the iterated logarithm type are strongly related to the large deviation result proved in Section 2.

Since the formulation and the proof of our result depend strongly on the spectral decomposition derived in Meerschaert [11], we will first introduce some notation. Factor the minimal polynomial of A into $f_1(x)\ldots f_p(x)$ such that all roots of $f_i(x)$ have real part equal a_i and $a_j < a_i$ for $j < i$. Sharpe [14] showed that if ν has no normal component, the set $\{a_1, \ldots, a_p\}$ is contained in the interval $(1/2, \infty)$. If we define $V_i = \operatorname{Ker}(f_i(A))$, then $V_1 \oplus \ldots \oplus V_p$ is a direct sum decomposition of $\mathbb{R}^d$ into A-invariant subspaces. We will call this the *spectral decomposition* of $\mathbb{R}^d$ relative to A. Now let $\mu \in \mathrm{GDOA}(\nu)$ be such that (1.3) holds. Using Theorem 4.2 of Meerschaert [11] we can assume without loss of generality that μ is spectrally compatible to ν, i.e. the spaces V_i are A_n-invariant for all n. Given any random vector X we write $X = X^{(1)} + \ldots + X^{(p)}$ with respect to the spectral decomposition and for $1 \leqslant i \leqslant p$ we set $X^{(1,\ldots,i)} = X^{(1)} + \ldots + X^{(i)}$.

Using the above assumptions and notation we will prove the following law of the iterated logarithm:

THEOREM 3.1. *For any* $1 \leqslant i \leqslant p$

$$\limsup_{n\to\infty} \|A_n S_n^{(1,\ldots,i)}\|^{1/\log\log n} = e^{a_i} \quad \textit{almost surely.} \tag{3.1}$$

Remark 3.2. Theorem 3.1 not only shows that the maximal growth rate of $\|A_n S_n\|$ is of order $(\log n)^{a_p}$, but it also shows that if $(A_n S_n)$ is restricted to the lower dimensional subspaces $V_1 \oplus \ldots \oplus V_i$ of $\mathbb{R}^d$ for some $1 \leqslant i \leqslant p$, then the different growth rate $(\log n)^{a_i}$ is obtained. Furthermore, since from Hudson et al. [8] we know sharp bounds on the norm of the norming operators A_n, it is easy to see that the maximal growth rate of the random walk (S_n) restricted to a subspace V_i is of order $(n \log n)^{a_i}$.

The structure of the proof of Theorem 3.1 is as follows. First we will show that (3.1) is true if ν is spectrally simple, i.e. $\operatorname{Re}(\lambda) = a > 1/2$ for all eigenvalues λ of A. This will be done in Proposition 3.3. Then we show that this special result implies the general case.

In the following let ν be a symmetric full operator stable law without Gaussian component on the finite-dimensional vector space V such that $\operatorname{Re}(\lambda) = a$ for all eigenvalues λ of A, where A is any exponent of ν. Furthermore, let ϕ denote the Lévy measure of ν and let $\mu \in \mathrm{GDOA}(\nu)$ be symmetric such that if $X_1, X_2, \ldots$ are i.i.d. random variables distributed according to μ, we infer for $S_n = \sum_{i=1}^n X_i$ and some sequence (A_n) of linear operators on V that $A_n S_n \Rightarrow \nu$. In this case the following law of the iterated logarithm holds:

PROPOSITION 3.3.

$$\limsup_{n\to\infty} \|A_n S_n\|^{1/\log\log n} = e^{a} \quad \textit{almost surely.} \tag{3.2}$$

Proof. Due to the nature of the power in (3.2) it suffices to show that for any $0 < \varepsilon < 1$ with probability one we have

$$\|A_n S_n\| > (\log n)^{(1+\varepsilon)a} \quad \text{for at most finitely many } n \tag{3.3}$$

and

$$\|A_n S_n\| > (\log n)^{(1-\varepsilon)a} \quad \text{for infinitely many } n. \tag{3.4}$$

We will first show (3.3). To do this for $n \geqslant 1$ define the event $D_n = \{\|A_n S_n\| > (\log n)^{(1+\varepsilon)a}\}$. Let $n_k = 2^k$ and let

(3.5) $$C = (\sup_{k \geqslant 1} \sup_{n_k \leqslant n < n_{k+1}} \|A_n A_{n_k}^{-1}\|)^{-1}.$$

Then an application of Theorem 3.1 of Meerschaert [11] shows that C is finite. Furthermore, let

$$B_k = \{\max_{n_k \leqslant n < n_{k+1}} \|A_{n_k} S_n\| > C(\log n_k)^{(1+\varepsilon)a}\}.$$

If $n_k \leqslant n < n_{k+1}$, then

$$\|A_n S_n\| \leqslant \|A_n A_{n_k}^{-1}\| \, \|A_{n_k} S_n\| \leqslant C^{-1} \|A_{n_k} S_n\|,$$

so the monotonicity of $\log t$ implies that $D_n \subset B_k$. Hence

$$\limsup_{n \to \infty} D_n \subset \limsup_{k \to \infty} B_k.$$

We next establish that for some number k_0 and all $k \geqslant k_0$

(3.6) $$\max_{n_k \leqslant n < n_{k+1}} P\left\{\|A_{n_k} \sum_{j=n+1}^{n_{k+1}} X_j\| > \frac{C}{2}(\log n_k)^{(1+\varepsilon)a}\right\} \leqslant D < 1$$

is valid. Let

$$M = \sup_{k \geqslant 1} \sup_{n_k \leqslant n < n_{k+1}} \|A_{n_k} A_{n_{k+1}-n}^{-1}\|.$$

Then using Theorem 3.1 of Meerschaert [11] again, we see that M is finite. In view of (1.3) we know that the laws of $A_n S_n$ are uniformly tight, and hence for every $0 < D < 1$ there exists a number k_0 such that

$$P\{\|A_l S_l\| > (C/2M)(\log n_{k_0})^{(1+\varepsilon)a}\} < D \quad \text{for all } l.$$

Hence for $k \geqslant k_0$ and $n_k \leqslant n < n_{k+1}$ we get

$$P\left\{\|A_{n_k} \sum_{j=n+1}^{n_{k+1}} X_j\| > \frac{C}{2}(\log n_k)^{(1+\varepsilon)a}\right\} \leqslant P\left\{\|A_{n_k} S_{n_{k+1}-n}\| > \frac{C}{2}(\log n_{k_0})^{(1+\varepsilon)a}\right\}$$
$$\leqslant P\left\{\|A_{n_{k+1}-n} S_{n_{k+1}-n}\| > \frac{C}{2M}(\log n_{k_0})^{(1+\varepsilon)a}\right\} < D,$$

which proves (3.6).

Now let $\{\theta^{(1)}, \ldots, \theta^{(m)}\}$, $m = \dim V$, be an orthonormal basis of V. Then it is easy to see that if Z is any random vector with values in V, we have for some positive real constant C_1

(3.7) $$P\{\|Z\| > t\} \leqslant \sum_{j=1}^{m} P\{|\langle Z, \theta^{(j)}\rangle| > C_1 t\}$$

for all $t > 0$. Then an application of Ottaviani's maximum inequality (see e.g. Breiman [2], p. 45) along with (3.6), (3.7) and the definition of the constant C in (3.5) gives, for some constant $E > 0$,

$$\begin{aligned} P(B_k) &\leqslant \frac{1}{1-D} P\left\{\|A_{n_k} S_{n_{k+1}}\| > \frac{C}{2}(\log n_k)^{(1+\varepsilon)a}\right\} \\ &\leqslant \frac{1}{1-D} P\left\{\|A_{n_{k+1}} S_{n_{k+1}}\| > \frac{1}{2}(\log n_k)^{(1+\varepsilon)a}\right\} \\ &\leqslant \frac{1}{1-D} \sum_{j=1}^{m} P\{|\langle A_{n_{k+1}} S_{n_{k+1}}, \theta^{(j)}\rangle| > Ek^{(1+\varepsilon)a}\}. \end{aligned}$$

But in view of Theorem 2.1, for some constant $D_1 > 0$ we obtain

$$P\{|\langle A_{n_{k+1}} S_{n_{k+1}}, \theta^{(j)}\rangle| > Ek^{(1+\varepsilon)a}\} \leqslant D_1 n_{k+1} P\{|\langle A_{n_{k+1}} X_1, \theta^{(j)}\rangle| > Ek^{(1+\varepsilon)a}\}.$$

Writing $A^*_{n_{k+1}}\theta^{(j)} = r_{k+1}\theta^{(j)}_{k+1}$ with $\|\theta^{(j)}_{k+1}\| = 1$ and $r_{k+1} > 0$ again and recalling the definition of V_0 in (2.6), we see that the right-hand side of the last inequality is equal to

$$D_1 \frac{V_0(r_{k+1}^{-1} Ek^{(1+\varepsilon)a}, \theta^{(j)}_{k+1})}{V_0(r_{k+1}^{-1}, \theta^{(j)}_{k+1})} n_{k+1} P\{|\langle A_{n_{k+1}} X_1, \theta^{(j)}\rangle| > 1\}. \tag{3.8}$$

In view of Lemma 2 of Meerschaert [12] and a uniform version of Feller [4], p. 289, for some positive constant E_1 and every $\delta > 0$ we have

$$\frac{V_0(t\lambda, \theta)}{V_0(t, \theta)} \leqslant E_1 \lambda^{-1/a+\delta} \tag{3.9}$$

for all $t \geqslant t_0$, $\lambda \geqslant 1$ and $\|\theta\| = 1$. Furthermore, by the standard convergence criteria for triangular arrays, (1.3) implies that

$$n_{k+1} P\{|\langle A_{n_{k+1}} X_1, \theta^{(j)}\rangle| > 1\} \to \phi\{x \in V\colon |\langle x, \theta^{(j)}\rangle| > 1\} < \infty.$$

Hence, if $\delta > 0$ is small enough, (3.8) is bounded above for all large k by

$$D_2 (k^{(1+\varepsilon)a})^{-1/a+\delta} = D_2 k^{-(1+\varepsilon_1)}$$

for some constant $D_2 > 0$ and some $\varepsilon_1 > 0$. Consequently, $P(B_k) \leqslant Bk^{-(1+\varepsilon_1)}$ for all large k and some positive real constant B. Finally, an application of the Borel–Cantelli lemma gives

$$P(\limsup_{n\to\infty} D_n) \leqslant P(\limsup_{k\to\infty} B_k) = 0,$$

so (3.3) is valid.

Now we will prove (3.4). From a convergence of types argument we know that $K = \sup_{n\geqslant 2} \|A_n A_{n-1}^{-1}\|$ is finite. Enlarge K if necessary to have $K \geqslant 1$.

Then, using the inequality $\|A_n X_n\| \leqslant \|A_n S_n\| + \|A_n S_{n-1}\|$, we get

$$\{\|A_n X_n\| > 2K(\log n)^{(1-\varepsilon)a} \text{ i.o.}\} \subset \{\|A_n S_n\| > K(\log n)^{(1-\varepsilon)a} \text{ i.o.}\} \cup \{\|A_n S_{n-1}\| > K(\log n)^{(1-\varepsilon)a} \text{ i.o.}\}.$$

But

$$\{\|A_n S_{n-1}\| > K(\log n)^{(1-\varepsilon)a} \text{ i.o.}\} \subset \{\|A_n S_n\| > (\log n)^{(1-\varepsilon)a} \text{ i.o.}\},$$

and so

$$\{\|A_n X_n\| > 2K(\log n)^{(1-\varepsilon)a} \text{ i.o.}\} \subset \{\|A_n S_n\| > (\log n)^{(1-\varepsilon)a} \text{ i.o.}\}. \tag{3.10}$$

Hence it is enough to show that the probability of the left-hand side of (3.10) is one. Therefore, by the independence part of the Borel–Cantelli lemma we have to show that

$$\sum_{n=1}^{\infty} P\{\|A_n X_n\| > 2K(\log n)^{(1-\varepsilon)a}\} = \infty. \tag{3.11}$$

But for any $\|\theta\| = 1$ we have, writing $A_n^* \theta = r_n \theta_n$ again,

$$P\{\|A_n X_n\| > 2K(\log n)^{(1-\varepsilon)a}\} \geqslant P\{|\langle A_n X_n, \theta\rangle| > 2K(\log n)^{(1-\varepsilon)a}\}$$
$$= \frac{1}{n} \frac{V_0(r_n^{-1} 2K(\log n)^{(1-\varepsilon)a}, \theta_n)}{V_0(r_n^{-1}, \theta_n)} nP\{|\langle A_n X_1, \theta\rangle| > 1\}.$$

By the standard convergence criteria for triangular arrays of random variables we know that

$$nP\{|\langle A_n X_1, \theta\rangle| > 1\} \to \phi\{x \in V\colon |\langle x, \theta\rangle| > 1\} > 0.$$

Furthermore, we infer from Lemma 2 of Meerschaert [12] that for every sufficiently small $\delta > 0$ there exists a constant $C > 0$ such that for all large n we have

$$\frac{V_0(r_n^{-1} 2K(\log n)^{(1-\varepsilon)a}, \theta_n)}{V_0(r_n^{-1}, \theta_n)} \geqslant C\big(2K(\log n)^{(1-\varepsilon)a}\big)^{-1/a-\delta} \geqslant C_1 \frac{1}{\log n},$$

where C_1 is a positive constant. Hence there exists a $C_2 > 0$ such that for all large n we have

$$P\{\|A_n X_n\| > 2K(\log n)^{(1-\varepsilon)a}\} \geqslant C_2 \frac{1}{n \log n},$$

which yields (3.11). This completes the proof of Proposition 3.3.

Proof of Theorem 3.1. Fix any $1 \leqslant i \leqslant p$. As in the proof of Proposition 3.3 it suffices to show that for any sufficiently small $\varepsilon > 0$ and some constant $C = C(\varepsilon)$ we have with probability one

$$\|A_n S_n^{(1,\ldots,i)}\| \leqslant C(\log n)^{(1+\varepsilon)a_i} \quad \text{for almost all } n \tag{3.12}$$

and

(3.13) $\|A_n S_n^{(1,\dots,i)}\| > C(\log n)^{(1-\varepsilon)a_i}$ for infinitely many n.

By Proposition 3.3 we infer for every $1 \leqslant j \leqslant i$ that for almost all sample points and almost all n the inequality $\|A_n S_n^{(j)}\| \leqslant (\log n)^{(1+\varepsilon)a_j}$ holds. Since $a_j \leqslant a_i$ for all $1 \leqslant j \leqslant i$, we have

$$\|A_n S_n^{(1,\dots,i)}\| \leqslant \sum_{j=1}^{i} \|A_n S_n^{(j)}\| \leqslant i(\log n)^{(1+\varepsilon)a_i}$$

for almost all n almost surely, and hence (3.12) holds.

For the proof of (3.13), let $\varepsilon < (a_i - a_{i-1})/(a_{i-1} + a_i)$. From Proposition 3.3 we get with probability one $\|A_n S_n^{(i)}\| > (\log n)^{(1-\varepsilon/2)a_i}$ for infinitely many n. Furthermore, for $1 \leqslant j \leqslant i-1$, $\|A_n S_n^{(j)}\| \leqslant (\log n)^{(1+\varepsilon)a_j}$ for almost all n almost surely. Hence, with probability one, for infinitely many n

$$\|A_n S_n^{(1,\dots,i)}\| \geqslant \|A_n S_n^{(i)}\| - \sum_{j=1}^{i-1} \|A_n S_n^{(j)}\|$$

$$> (\log n)^{(1-\varepsilon/2)a_i} - \sum_{j=1}^{i-1} (\log n)^{(1+\varepsilon)a_j} \geqslant (\log n)^{(1-\varepsilon/2)a_i} - (i-1)(\log n)^{(1+\varepsilon)a_{i-1}}.$$

But by the choice of ε the last difference is greater than $(\log n)^{(1-\varepsilon)a_i}$ for all large n, which gives (3.13). This completes the proof of Theorem 3.1.

In addition to Theorem 3.1 we can prove the following clustering statement which gives additional information about the path behavior of the random walk (S_n).

COROLLARY 3.5. *Under the assumptions of Theorem* 3.1, *for any* $1 \leqslant i \leqslant p$, *with probability one every point in the interval* $(1, e^{a_i}]$ *is a cluster point of the sequence*

$$\{\|A_n S_n^{(1,\dots,i)}\|:\ n \geqslant 1\}.$$

Proof. For $1 \leqslant i \leqslant p$ and $0 < \lambda \leqslant a_i$, let $\delta = a_i/\lambda$ and let $n_k = [2^{k^\delta}]$, where $[x]$ denotes the integer part of x. We will show that

(3.14) $$\limsup_{k\to\infty} \|A_{n_k} S_{n_k}^{(1,\dots,i)}\|^{1/\log\log n_k} = e^{\lambda} \text{ almost surely.}$$

We will show that for any small $\varepsilon > 0$ and for almost all sample points we have

(3.15) $\|A_{n_k} S_{n_k}^{(1,\dots,i)}\| > (\log n_k)^{(1+\varepsilon)\lambda}$ for at most finitely many k

and

(3.16) $\|A_{n_k} S_{n_k}^{(1,\dots,i)}\| > (\log n_k)^{(1-\varepsilon)\lambda}$ for infinitely many k.

For $1 \leqslant l \leqslant p$ let $\{\theta^{(l,1)}, \ldots, \theta^{(l,m_l)}\}$, $m_l = \dim V_l$, be an orthonormal basis of V_l. Then for some constant $C > 0$ we obtain

$$P\{\|A_{n_k} S_{n_k}^{(1,\ldots,i)}\| > (\log n_k)^{(1+\varepsilon)\lambda}\} \leqslant \sum_{l=1}^{i} \sum_{j=1}^{m_l} P\{|\langle A_{n_k} S_{n_k}, \theta^{(l,j)}\rangle| > C(\log n_k)^{(1+\varepsilon)\lambda}\}.$$

The inequality (3.15) now follows upon arguing just as in the proof of (3.3).

The proof of (3.16) is more involved than the proof of (3.4). First we will show that with probability one we have

$$(3.17)\quad \|A_{n_k} \sum_{j=n_{k-1}+1}^{n_k} X_j^{(1,\ldots,i)}\| > (\log n_k)^{(1-\varepsilon/2)\lambda} \quad \text{for infinitely many } k.$$

Note that these are independent events. Let

$$M = \sup_{k \geqslant 1} \|A_{n_k - n_{k-1}} A_{n_k}^{-1}\|.$$

Since $\delta \geqslant 1$, Theorem 3.1 of Meerschaert [11] shows that M is finite. Fix any unit vector $\theta \in V_i$ and set $A^*_{n_k - n_{k-1}} \theta = r_k \theta_k$ with $r_k > 0$ and $\|\theta_k\| = 1$. Then from Theorem 2.1, for some positive real constant K, we infer that

$$\begin{aligned} P\{\|A_{n_k} &\sum_{j=n_{k-1}+1}^{n_k} X_j^{(1,\ldots,i)}\| > (\log n_k)^{(1-\varepsilon/2)\lambda}\} \\ &= P\{\|A_{n_k} S_{n_k - n_{k-1}}^{(1,\ldots,i)}\| > (\log n_k)^{(1-\varepsilon/2)\lambda}\} \\ &\geqslant P\{\|A_{n_k - n_{k-1}} S_{n_k - n_{k-1}}^{(1,\ldots,i)}\| > M(\log n_k)^{(1-\varepsilon/2)\lambda}\} \\ &\geqslant P\{|\langle A_{n_k - n_{k-1}} S_{n_k - n_{k-1}}, \theta\rangle| > M(\log n_k)^{(1-\varepsilon/2)\lambda}\} \\ &\geqslant K(n_k - n_{k-1}) P\{|\langle A_{n_k - n_{k-1}} X_1, \theta\rangle| > M(\log n_k)^{(1-\varepsilon/2)\lambda}\} \\ &= K\frac{V_0(r_k^{-1} M(\log n_k)^{(1-\varepsilon/2)\lambda}, \theta_k)}{V_0(r_k^{-1}, \theta_k)}(n_k - n_{k-1}) P\{|\langle A_{n_k - n_{k-1}} X_1, \theta\rangle| > 1\}. \end{aligned}$$

Since $nP\{|\langle A_n X_1, \theta\rangle| > 1\} \to \phi\{|\langle x, \theta\rangle| > 1\} > 0$, we get from Lemma 2 of Meerschaert [12] that the last expression above is bounded from below for all large k by some positive constant times $k^{-(1-\varepsilon_1)}$ for some $\varepsilon_1 > 0$. Now the independence part of the Borel–Cantelli lemma gives (3.17).

Finally, suppose that (3.16) does not hold on a set of positive probability. Then for almost all points in this set we have

$$\begin{aligned} (\log n_k)^{(1-\varepsilon)\lambda} \geqslant \|A_{n_k} S_{n_k}^{(1,\ldots,i)}\| &\geqslant \|A_{n_k} \sum_{j=n_{k-1}+1}^{n_k} X_j^{(1,\ldots,i)}\| - \|A_{n_k} S_{n_{k-1}}^{(1,\ldots,i)}\| \\ &> (\log n_k)^{(1-\varepsilon/2)\lambda} - \|A_{n_k} A_{n_{k-1}}^{-1}\| (\log n_{k-1})^{(1-\varepsilon)\lambda} \end{aligned}$$

for infinitely many k using (3.17) and the assumption. But since $\|A_{n_k} A_{n_{k-1}}^{-1}\| \leqslant C$ for all k and some constant $C > 0$, the last difference is greater than $(\log n_k)^{(1-\varepsilon)\lambda}$ for all large k, which is a contradiction, and hence (3.16) holds. This completes the proof.

4. Concluding remarks. Since for every unit vector $\theta \in V_1 \oplus \ldots \oplus V_i \setminus V_1 \oplus \ldots \oplus V_{i-1}$ we have $|\langle A_n S_n, \theta\rangle| \leqslant \|A_n S_n^{(1,\ldots,i)}\|$, Theorem 3.1 implies that

$$\limsup_{n\to\infty} |\langle A_n S_n, \theta\rangle|^{1/\log\log n} \leqslant e^{a_i} \text{ almost surely.}$$

Furthermore, the methods of our proof actually show that for any $1 \leqslant i \leqslant p$ there exists at least one unit vector $\theta \in V_1 \oplus \ldots \oplus V_i \setminus V_1 \oplus \ldots \oplus V_{i-1}$ such that

$$\limsup_{n\to\infty} |\langle A_n S_n, \theta\rangle|^{1/\log\log n} = e^{a_i} \text{ almost surely.} \tag{4.1}$$

Though it seems that the law of the iterated logarithm depends only on the tail behavior of the random variable $\langle X_1, \theta\rangle$ and this tail behavior is uniform in $\|\theta\| = 1$, we were unable to prove (4.1) for all unit vectors θ. It might require a different method of proof or additional information about the norming operators A_n, i.e. a sharper spectral decomposition, which decomposes every V_i further.

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