PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 17, Fasc. 1 (1997), pp. 79–93

LEVEL CROSSINGS OF STOCHASTIC PROCESSES WITH STATIONARY BOUNDED VARIATIONS AND CONTINUOUS DECREASING COMPONENTS

BY

MASAKIYO MIYAZAWA* (CHIBA) AND VOLKER SCHMIDT (ULM)

Abstract. Formulas for level crossing probabilities, ladder height distributions and related characteristics of a general class of processes with stationary bounded variations and continuous decreasing components are derived under certain mild conditions. Results for a risk process with a constant premium rate and with a claim process generated by a stationary marked point process are generalized to the case where the premium rate itself can be a stochastic process and the claim arrival process can have both jumps and continuous components. The case of infinitely many jumps in finite intervals is not excluded. The main tool for investigating this more general class of stochastic models in an exchange formula for Palm probabilities of stationary random measures. Our results can be used to derive a formula for the ascending ladder height distribution of the time-stationary workload process in single-server queues.

1. Introduction. We consider a stochastic process $\{X_0(t)\}_{t\geq 0}$ with the property that

$$X_0(t) = A(t) - D(t) \quad \text{for every } t \geq 0, \tag{1.1}$$

where $\{A(t)\}_{t\geq 0}$ and $\{D(t)\}_{t\geq 0}$ are stochastic processes with nondecreasing trajectories such that $A(0) = D(0) = 0$. Moreover, we assume:

(i-a) $\{A(t)\}$ and $\{D(t)\}$ have jointly stationary increments;

(ii-a) with probability 1, there exist disjoint (random) Borel sets I_A and I_D on $\boldsymbol{R}_+ = [0, +\infty)$ such that, for all $t \in \boldsymbol{R}_+$,

$$A(t) = \int_0^t \mathbf{1}_{I_A}(u)\, dA(u) \quad \text{and} \quad D(t) = \int_0^t \mathbf{1}_{I_D}(u)\, dD(u),$$

where $\mathbf{1}_C$ denotes the indicator function of the set C;

* The work of this author was partially supported by NEC C&C Research Laboratories.

(iii-a) the trajectories of $\{D(t)\}$ are continuous functions in t.

In the literature (see, e.g., Miyazawa [11]), (ii-a) is referred to as $\{A(t)\}$ and $\{D(t)\}$ being mutually singular. Note that (ii-a) is always satisfied under (iii-a), provided that $\{A(t)\}$ is a pure jump process. For the above process $\{X_0(t)\}$, we consider the time τ when the process gets into above level zero as defined by $\tau = \inf\{u \geqslant 0\colon X_0(u) > 0\}$ and we consider the ladder heights just after and before this time, which are defined by $Z^+ = X_0(\tau+)$ and $Z^- = -X_0(\tau-)$, respectively. τ is referred to as the level-zero crossing time.

In insurance mathematics, the process $\{X_0(t)\}$ is called a *claim surplus process* with constant premium rate (equal to 1) and with claims U_n arriving at claim epochs T_n if $\{A(t)\}$ and $\{D(t)\}$ satisfy the following additional conditions:

(ii-b) $\{A(t)\}$ is a locally finite jump process, i.e.

$$A(t) = \sum_{\{i:0<T_i\leqslant t\}} U_i, \tag{1.2}$$

where $\{(T_n, U_n)\}_{n\in \mathbf{Z}}$ is a stationary marked point process with nonnegative marks ($\mathbf{Z} = \{\ldots, -1, 0, 1, \ldots\}$);

(iii-b) $D(t) = t$ for every $t \geqslant 0$.

Sometimes the dual process $\{-X_0(t)\}$ is considered, which is called an *insurance risk process*. For insurance risk processes, the level-zero crossing probability $P(\tau < \infty)$ is called a *ruin probability*, while the ladder height Z^+ is called the *severity of ruin*. Thus the model of $\{X_0(t)\}$ given by (1.1) and (i-a)–(iii-a) can be considered as a claim surplus process governed by the more general claim and premium processes $\{A(t)\}$ and $\{D(t)\}$, respectively. Furthermore, the process $\{X_0(t)\}$ can also describe the excursions of workload processes in queueing systems, as shown in Section 4.

In the classical compound Poisson risk model given by (ii-b) and (iii-b), $\{(T_n, U_n)\}$ is assumed to be an independently marked Poisson process with intensity $\lambda = E\max\{n\colon T_n \leqslant 1\}$. For this model it is well known (see, e.g., Feller [5]) that the (defective) distribution function $G(x) = P(Z^+ \leqslant x)$ is given by

$$G(x) = \lambda \int_0^x P(U_0 > u)\,du \quad \text{for every } x \geqslant 0, \tag{1.3}$$

provided that

(iv-a) $\lambda E U_0 < 1$.

In a series of papers (see Asmussen and Schmidt [2], Frenz and Schmidt [7], Miyazawa and Schmidt [15]) the assumptions on the claim arrival process $\{(T_n, U_n)\}$ are relaxed step by step. They showed that formula (1.3) remained valid in a general point-process set-up where instead of $P(U_0 > u)$ a corresponding Palm probability is considered. In Miyazawa [11] and in Asmussen and Schmidt [3] a similar formula is obtained for the joint distribution of (Z^-, Z^+), and in [3] a more general claim arrival process $\{(T_n, U_n, M_n)\}$ has been considered, where the additional mark component M_n can contain further information on the claim arriving at time T_n, e.g. its type etc. This extension turned out to be very useful for investigating ruin probabilities of risk processes governed by a Markov-modulated Poisson claim arrival process (see Asmussen et al. [1]).

Asmussen and Schmidt [3] showed that the joint distribution of (Z^-, Z^+, M^+) can be described under conditions (ii-b) and (iii-b) as follows, where $M^+ = M_\eta$ and η is an integer-valued random variable such that $T_\eta = \tau$, i.e. M^+ is the type of the claim arriving at ruin time τ. Let $\varrho = \lambda E_0 U_0 \leqslant 1$, where E_0 denotes the expectation with respect to the Palm distribution of the stationary (and ergodic) marked point process $\{(T_n, U_n, M_n)\}$. Then the equality

$$E(\phi(Z^+, Z^-, M^+); \tau < \infty) = \lambda E_0\Big(\int_0^{U_0} \phi(U_0-u, u, M_0)\, du\Big) \tag{1.4}$$

holds for every nonnegative measurable function ϕ, where, instead of (iv-a), the somewhat weaker condition

(iv-b) $\sup_{t<0}(A(t)-D(t)) = \infty$ with probability 1

has been used. Here, $\{A(t)\}_{t\in\mathbf{R}}$ and $\{D(t)\}_{t\in\mathbf{R}}$ denote the stationary extensions of $\{A(t)\}_{t\geqslant 0}$ and $\{D(t)\}_{t\geqslant 0}$ to the whole real line $\mathbf{R}$, see also (2.1).

The purpose of the present paper is to generalize (1.4) to the model described by conditions (i-a)–(iii-a). This generalization reveals that the distribution of ladder heights still has an insensitive structure like (1.4), with respect to the form of the distribution of the sequence $\{T_n\}$ of arrival epochs when the arrival intensity λ is fixed, and where the origin is randomly chosen from the decreasing period of $\{X_0(t)\}$. In Section 2, we introduce notation and give some basic facts on Palm distributions and approximation of (not necessarily purely atomic) random measures. In Section 3 we first consider the process $\{X_0(t)\}$ under the additional condition that

(ii-c) $\{A(t)\}$ is a pure jump process.

In Section 4, we show how our general results can be applied in order to determine the ascending ladder height distribution of the time-stationary workload process in single-server queues.

Although we assume that condition (iii-b) may not be fulfilled and, consequently, the premium process $\{D(t)\}$ itself may be random, the approach of Asmussen and Schmidt [3] still works in the special case that $\{D(t)\}$ is deterministic and linear and that $\{A(t)\}$ is induced by a marked point process. However, instead of using Campbell's formula for stationary marked point processes as in [3], our main tool is an exchange formula for Palm measures of stationary random measures (see Lemma 2.1). We then remove the additional condition (ii-c) approximating the claim process $\{A(t)\}$ by a sequence of processes $\{A_n(t)\}$ that satisfy (ii-c), and applying a limiting argument to the result obtained for the case when the claim process is a pure jump process.

2. Stationary framework and preliminary results. We begin with the formal description of an appropriate stochastic model for the processes introduced in Section 1. This is an important step in our approach because we will consider several probability measures on a common measurable space.

Let $(\Omega, \mathscr{F}, P)$ be a probability space, and $\{\theta_t\}_{t \in \boldsymbol{R}}$ a measurable operator group on $(\Omega, \mathscr{F})$. Let A and D be random measures on $(\boldsymbol{R}, \mathscr{B}(\boldsymbol{R}))$, where $\mathscr{B}(\boldsymbol{R})$ is the Borel σ-field on $\boldsymbol{R}$. Our basic assumption is that

(i-b) these two random measures are consistent with $\{\theta_t\}$, i.e.

$$A(B) \circ \theta_t = A(B+t) \quad \text{and} \quad D(B) \circ \theta_t = D(B+t) \quad \text{for all } B \in \mathscr{B}(\boldsymbol{R}),\ t \in \boldsymbol{R},$$

where $f \circ \theta_t(\omega) = f(\theta_t(\omega))$ and $B+t = \{u+t:\ u \in B\}$. The consistency conditions (i-b) just mean that the measures A and D have a common time axis. For these random measures, we define nondecreasing random functions $\{A(t)\}$ and $\{D(t)\}$ by

$$A(t) = \begin{cases} A((0, t]) & \text{if } t \geqslant 0, \\ -A((t, 0]) & \text{if } t < 0, \end{cases} \qquad D(t) = \begin{cases} D((0, t]) & \text{if } t \geqslant 0, \\ -D((t, 0]) & \text{if } t < 0. \end{cases} \tag{2.1}$$

We distinguish these functions from the corresponding measures by their arguments. Then, condition (i-a) is satisfied if

(i-c) $\{\theta_t\}$ is stationary with respect to P, i.e. $P(\theta_t^{-1}(C)) = P(C)$ for every $C \in \mathscr{F}$ and $t \in \boldsymbol{R}$.

In particular, we see that because of (i-b) and (i-c) the random measures A and D are stationary, i.e. their distributions are invariant with respect to the shift operators θ_t. We further assume that

(i-d) A and D have finite and positive intensities $\lambda_A (= E(A(1)))$ and $\lambda_D (= E(D(1)))$, respectively.

Similarly to Asmussen and Schmidt [3], we consider a continuous-time process $\{J(t)\}$ as a background process for $\{X_0(t)\}$. We assume that the random variables $\{J(t)\}$ take values in a Polish space, i.e. a complete, separable metric space K with the Borel σ-field $\mathscr{B}(K)$. Furthermore, we assume that

(i-e) $\{J(t)\}$ is consistent with $\{\theta_t\}$, i.e. $J(s) \circ \theta_t = J(s+t)$ for all $s, t \in \boldsymbol{R}$.

This means that, under P, the background process $\{J(t)\}$ is jointly stationary with the increments of the random measures A and D. Unless stated differently, we assume that all processes considered in this paper are continuous from the right and have left-hand limits, CORLOL for short. Throughout this paper we additionally assume that (i-b), (i-c), (i-d) and (i-e) are satisfied. For simplicity reasons, we denote this set of conditions by (i).

A further basic notion which we will use in order to describe our results is that of a Palm probability measure induced by a stationary random measure (see, e.g., Mecke [10], Daley and Vere-Jones [4], König and Schmidt [9]).

DEFINITION 2.1. Under conditions (i-b), (i-c) and (i-d), let the probability measure P_A on $(\Omega, \mathscr{F})$ be given by

$$(2.2)\qquad P_A(C) = \lambda_A^{-1} E\left(\int_0^1 \mathbf{1}_C \circ \theta_u A(du)\right) \quad \text{for every } C \in \mathscr{F},$$

where the integration is taken over the interval (0, 1] open to the left. P_A is called the *Palm probability measure* of P with respect to A. The Palm measure P_D with respect to the stationary random measure D is defined analogously. The expectation under the Palm measures $P_A, P_D, \ldots$ is denoted by $E_A, E_D, \ldots$, respectively.

Our main tool in this paper will be the following exchange formula for Palm measures of stationary random measures (see, e.g., Miyazawa [14], Schmidt and Serfozo [17]).

LEMMA 2.1. *For each nonnegative measurable function $f\colon \boldsymbol{R} \times \Omega \to [0, \infty)$ the equality*

$$(2.3)\qquad \lambda_D E_D\left(\int_{-\infty}^{+\infty} f(u, \theta_u) A(du)\right) = \lambda_A E_A\left(\int_{-\infty}^{+\infty} f(-u, \theta_0) D(du)\right)$$

holds.

A simple consequence of Lemma 2.1 is the following result:

COROLLARY 2.1. *If $\{A(t)\}$ is a locally finite jump process, i.e. the jump points T_n of $\{A(t)\}$ induce a locally finite counting measure N_A with $N_A(B) = \#\{n\colon T_n \in B\}$, then*

$$(2.4)\qquad P_A(C) = \frac{\lambda_{N_A}}{\lambda_A} E_{N_A}(\Delta A(0)\mathbf{1}_C),$$

where $\Delta A(u) = A(u) - A(u-)$ and $\lambda_{N_A} = E(N_A((0, 1]))$.

Note, however, that (2.4) can directly be obtained from definition (2.2). Namely,

$$P_A(C) = \lambda_A^{-1} E\left(\int_0^1 (\Delta A(0)\mathbf{1}_C) \circ \theta_u \frac{1}{\Delta A(u)} A(du)\right)$$

$$= \lambda_A^{-1} E\left(\int_0^1 (\Delta A(0)\mathbf{1}_C) \circ \theta_u N_A(du)\right) = \frac{\lambda_{N_A}}{\lambda_A} E_{N_A}(\Delta A(0)\mathbf{1}_C).$$

Another auxiliary result is an approximation of random measures in terms of pure jump processes. For $n = 1, 2, \ldots$, let V_n be a random variable which is uniformly distributed on $(0, 1/n]$, independent of A and D, and which satisfies

$$(2.5)\qquad V_n \circ \theta_t = V_n + t \ (\mathrm{mod}\, 1/n) \quad \text{for every } t \in \boldsymbol{R}.$$

We can easily construct a sequence of such V_n by extending the probability space $(\Omega, \mathscr{F}, P)$ to $\left(\Omega \times \boldsymbol{R}^\infty, \mathscr{F} \times \mathscr{B}(\boldsymbol{R}^\infty), P \otimes \prod_{n=1}^{\infty} P_n\right)$ and by redefining θ_t

according to (2.5), where P_n is the uniform distribution on $(0, 1/n]$, and $\otimes$ denotes the product of measures. Note that the new operator group $\{\theta_t\}$ is invariant with respect to $P \otimes \prod_{n=1}^{\infty} P_n$. For every $n = 1, 2, \ldots$ we define the process $\{A_n(t)\}$ by

$$(2.6)\qquad A_n(t) = A\left(\frac{[n(t+V_n)]}{n} - V_n\right) - A\left(\frac{[nV_n]}{n} - V_n\right) \quad \text{for every } t \in \mathbf{R},$$

where $[a]$ denotes the largest integer not greater than a. Then, $A_n(0) = 0$, and

$$\begin{aligned} A_n(t) \circ \theta_s &= A\left(\frac{[n(t+s+V_n)]}{n} - V_n\right) - A\left(\frac{[n(s+V_n)]}{n} - V_n\right) \\ &= A_n(t+s) - A_n(s). \end{aligned}$$

Hence, the random measure induced by $\{A_n(t)\}$ is consistent with $\{\theta_t\}$. Moreover,

$$A(t-2/n) \leqslant A_n(t) \leqslant A(t) - A(-1/n).$$

Thus, we get the following result:

LEMMA 2.2. *In the above setting, $\{A_n(t)\}$ is a locally finite jump process with finite intensity $\lambda_{A_n} \in ((1-2/n)\lambda_A, \lambda_A]$, and A_n and D are jointly stationary. Furthermore, if $\{A(t)\}$ is continuous for all t, then $A_n(t)$ converges to $A(t)$ uniformly on each finite interval as n tends to infinity.*

Finally, we introduce the notion of a generalized Lévy measure, see also Miyazawa [13]. This measure will be used in case $\{A(t)\}$ admits infinitely many jumps in a finite interval.

DEFINITION 2.2. Let μ_A be the probability measure on $\mathscr{B}(\mathbf{R})$ given by $\mu_A(B) = P_A(\Delta A(0) \in B)$, and ν_A the σ-finite measure on $\mathscr{B}((0, +\infty))$ given by

$$\nu_A((x, +\infty)) = \int_x^{+\infty} \frac{\lambda_A}{y} \mu_A(dy).$$

Then ν_A is called the *generalized Lévy measure* of A.

Note that $\nu_A((0, +\infty))$ may be infinite, but $\int_0^{\infty} x \nu_A(dx) = \lambda_A < \infty$. Moreover, if $\{A(t)\}$ is a nondecreasing and pure jump Lévy process, i.e. a pure jump process with independent increments, then ν_A agrees with the Lévy measure of $\{A(t)\}$ (see, e.g., p. 27 of Protter [16]).

3. Level crossing probabilities and ladder height distributions. Throughout this section, we assume that conditions (i), (ii-a), (iii-a), and (iv-b) are satisfied. First, we consider the level crossing behavior of $\{X_0(t)\}$ under the additional condition (ii-c) that the claim process $\{A(t)\}$ is a pure jump process. Note, however, that we do not make any additional assumptions on the premium process $\{D(t)\}$. In particular we do not assume that (iii-b) holds. Nevertheless we can still proceed similarly to Asmussen and Schmidt [3] provided that we use the exchange formula given in Lemma 2.1 instead of the standard Campbell formula.

In addition to the claim surplus process $\{X_0(t)\}$, we define two processes $\{X_u(v)\}_{v \geq u}$ and $\{X_u^*(v)\}_{v \geq u}$ for any $u \geq 0$. Namely, let

$$X_u(v) = X_0(v-u) \circ \theta_u \quad \text{for } u \leq v,$$

and

$$X_u^*(v) = -X_{-v}(-u) \quad \text{for } u \leq v.$$

The process $\{-X_u^*(v)\}_{v \geq u}$ is dual to the process $\{X_u(v)\}_{v \geq u}$ in the sense that $X_u^{**}(v) = X_u(v)$, see also Asmussen and Schmidt [2], [3], and Miyazawa [12]. Note that the dual process $\{-X_u^*(v)\}_{v \geq u}$ is not CORLOL, but COLLOR, i.e. continuous to the left and having right-hand limits. Furthermore, we put

$$V(u) = \sup_{0<s<u} \big(X_{0+}^*(s)\big).$$

Since

$$X_{0+}^*(s) = -X_{-s}(0-) = -X_0(s-) \circ \theta_{-s} \tag{3.1}$$

$$= -A(0-)+A(-s)+D(0-)-D(-s),$$

we have

$$V(u) = \sup_{0<s<u} \big(A(-s)-D(-s)\big) - A(0-)+D(0-) \tag{3.2}$$

$$= \sup_{0<s<u} \big(A(s-u)-D(s-u)\big) - A(0-)+D(0-).$$

Note that, because of (ii-a) and (iii-a), $\{V(u)\}$ increases if and only if $V(u) = X_{0+}^*(u)$, and we have $V(du) = -D(-du)$ at these points of increase.

Given these notations we are now in a position to derive a representation formula for the joint distribution of $(Z^+, Z^-, J(\tau))$ under the Palm measure P_D.

THEOREM 3.1. *Under conditions* (i), (ii-c), (iii-a) *and* (iv-b), *the equation*

$$E_D\big(\phi(Z^+, Z^-, J(\tau));\ \tau < \infty\big) \tag{3.3}$$

$$= \frac{\lambda_A}{\lambda_D} E_A\left(\frac{1}{\Delta A(0)} \int_0^{\Delta A(0)} \phi\big(\Delta A(0)-u,\ u,\ J(0)\big)\, du\right)$$

holds for every nonnegative measurable function ϕ: $\mathbf{R}^2 \times K \to [0, \infty)$.

Proof. We apply Lemma 2.1 similarly to how Campbell's formula has been used in [3]. Note that the assumption that A is induced by a pure jump process leads to

$$\phi(Z^+, Z^-, J(\tau))\mathbf{1}_{\{\tau<\infty\}} = \int_{-\infty}^{+\infty} \phi(X_0(u), -X_0(u-), J(u))\mathbf{1}_{\{\tau=u\}}\frac{\mathbf{1}_{\{u>0\}}}{\Delta A(u)}A(du)$$

$$= \int_{-\infty}^{+\infty}\left(\phi(X_{-u}(0), -X_{-u}(0-), J(0))\mathbf{1}_{\{\tau\circ\theta_{-u}=u\}}\frac{\mathbf{1}_{\{u>0\}}}{\Delta A(0)}\right)\circ\theta_u A(du)$$

$$= \int_{-\infty}^{+\infty}\left(\phi(\Delta A(0)-X^*_{0+}(u), X^*_{0+}(u), J(0))\mathbf{1}_{\{\tau\circ\theta_{-u}=u\}}\frac{\mathbf{1}_{\{u>0\}}}{\Delta A(0)}\right)\circ\theta_u A(du).$$

Thus, Lemma 2.1 yields

(3.4) $\lambda_D E_D(\phi(Z^+, Z^-, J(\tau)); \tau<\infty)$

$$= \lambda_A E_A\left(\frac{-1}{\Delta A(0)}\int_0^{+\infty}\phi(\Delta A(0)-X^*_{0+}(u), X^*_{0+}(u), J(0))\mathbf{1}_{\{\tau\circ\theta_{-u}=u\}}D(-du)\right).$$

Since

$$\tau\circ\theta_{-u} = \inf\{s>0:\ A(s-u)-A(-u)-(D(s-u)-D(-u))>0\},$$

equalities (3.1) and (3.2) imply that the event $\{\tau\circ\theta_{-u}=u\}$ is equal to

$$\{A(s-u)-D(s-u)\leqslant A(-u)-D(-u)\ \ \forall s\in(0,u)\}\cap\{-A(-u)+D(-u)>0\}$$

$$= \{\sup_{0<s<u}(A(s-u)-D(s-u))\leqslant A(-u)-D(-u)\}\cap\{-A(-u)+D(-u)>0\}$$

$$= \{V(u)+A(0-)-D(0-)\leqslant A(-u)-D(-u)\}$$

$$\cap\{-A(0-)+D(0-)-X^*_{0+}(u)>0\}$$

$$= \{V(u)\leqslant X^*_{0+}(u) < -A(0-)+D(0-)\}$$

$$= \{V(u)=X^*_{0+}(u)<\Delta A(0)+D(0-)\},$$

where the last equation follows from the fact that $V(u)\geqslant X^*_{0+}(u)$ by the definition of $V(u)$. Further note that (iv-b) implies $\lim_{u\to\infty}V(u)=+\infty$, and that $D(0-)=0$ because of (iii-a). Hence, by using the fact that $V(du)=-D(-du)$ for the increasing points of $\{V(u)\}$, we have

$$-\int_0^{\infty}\phi(\Delta A(0)-X^*_{0+}(u), X^*_{0+}(u), J(0))\mathbf{1}_{\{\tau\circ\theta_{-u}=u\}}D(-du)$$

$$= \int_0^{\infty}\phi(\Delta A(0)-V(u), V(u), J(0))\mathbf{1}_{\{V(u)<\Delta A(0)\}}dV(u)$$

$$= \int_0^{\Delta A(0)}\phi(\Delta A(0)-v, v, J(0))\,dv,$$

where the last equation has been obtained by substituting $v=V(u)$. Thus, (3.4) implies (3.3). ■

COROLLARY 3.1. *If, in addition to the assumptions of Theorem* 3.1, *the jump process* $\{A(t)\}$ *is locally finite, then*

$$E_D\big(\phi(Z^+, Z^-, J(\tau)); \tau < \infty\big) = \frac{\lambda_{N_A}}{\lambda_D} E_{N_A}\Big(\int_0^{\Delta A(0)} \phi(\Delta A(0)-u, u, J(0))\,du\Big). \tag{3.5}$$

The proof of (3.5) follows directly from Theorem 3.1 and Corollary 2.1. Moreover, since $P_D = P$ and $\lambda_D = 1$ if $D(t) \equiv t$, Corollary 3.1 generalizes Theorem 1 and Corollary 1 of Asmussen and Schmidt [3] (see (1.4)).

Remark 3.1. Note that in the proof of Theorem 3.1, conditions (ii-a) and (iii-a) have been exploited essentially, since $V(du) = -D(-du)$ does not hold otherwise.

Remark 3.2. If the random measures A and D are ergodic, then the inequality $\lambda_A < \lambda_D$ implies (iv-b). The proof of Theorem 3.1 shows that, even if condition (iv-b) does not hold, we still get a certain expression for $E_D\big(\phi(Z^+, Z^-, J(\tau)); \tau < \infty\big)$. That is, putting $V_\infty = \sup_{u\geq 0} V(u)$, we have

$$\begin{aligned} &E_D\big(\phi(Z^+, Z^-, J(\tau)); \tau < \infty\big) \\ &\qquad = \frac{\lambda_A}{\lambda_D} E_A\Big(\frac{1}{\Delta A(0)} \int_0^{\min(\Delta A(0), V_\infty)} \phi(\Delta A(0)-u, u, J(0))\,du\Big). \end{aligned} \tag{3.6}$$

We further note that if $D'(0)$ exists with probability 1, then the left-hand sides of (3.3) and (3.5) can be written as $E\big(\phi(Z^+, Z^-, J(\tau))D'(0); \tau < \infty\big)$. Thus, a possible interpretation of the expectation under P_D is that the starting point of the process $\{X_0(t)\}$ is chosen at the arrival of a typical (infinitesimally small) unit of increase of the process $\{D(t)\}$. Moreover, the right-hand sides of (3.3) and (3.5) depend on D only through its intensity, provided that D is independent of $\Delta A(0)$ and $J(0)$. This insensitivity property will also hold for all our results stated in the remaining part of this paper.

We now remove condition (ii-c). For this purpose, we decompose A into two components. Define the nondecreasing processes $\{A_d(t)\}$ and $\{A_c(t)\}$ by

$$A_d(t) = \sum_{0<u\leq t} \Delta A(u), \qquad A_c(t) = A(t) - A_d(t) \qquad \text{for every } t \in \mathbf{R}.$$

Clearly, $\{A_d(t)\}$ and $\{A_c(t)\}$ have stationary increments under P.

THEOREM 3.2. *Under conditions* (i), (ii-a), (iii-a) *and* (iv-b), *the equality*

$$\begin{aligned} &E_D\big(\phi(Z^+, Z^-, J(\tau)); \tau < \infty\big) \\ &= \frac{\lambda_{A_d}}{\lambda_D} E_{A_d}\Big(\frac{1}{\Delta A_d(0)} \int_0^{\Delta A_d(0)} \phi(\Delta A_d(0)-u, u, J(0))\,du\Big) + \frac{\lambda_{A_c}}{\lambda_D} E_{A_c}\big(\phi(0, 0, J(0))\big) \end{aligned} \tag{3.7}$$

holds for every bounded measurable function $\phi\colon \mathbf{R}^2 \times K \to \mathbf{R}$.

Remark 3.3. By using the generalized Lévy measure ν_{A_d}, (3.7) can be rewritten in the form

$$(3.8)\quad E_D\big(\phi(Z^+, Z^-, J(\tau)); \tau < \infty\big) = \frac{1}{\lambda_D}\int_0^\infty\Big(\int_0^x E_{A_d}\big(\phi(x-u, u, J(0)) \mid \Delta A_d(0) = x\big)\,du\Big)\nu_{A_d}(dx) + \frac{\lambda_{A_c}}{\lambda_D}E_{A_c}\big(\phi(0, 0, J(0))\big).$$

Proof of Theorem 3.2. First note that it is enough to prove (3.7) for a bounded continuous function ϕ. For each positive integer n, we define $A_{c,n}$ for A_c in the same way as (2.5). We also define

$$A_n(t) = A_d(t) + A_{c,n}(t).$$

Then, by Lemma 2.2, $A_n(t)$ has stationary increments and satisfies (i-c), (i-d) and (iv-b). In accordance with the definitions of τ, Z^+ and Z^- we gave above for A, we define τ_n, Z_n^+ and Z_n^- for A_n, respectively. Hence, the definition of Palm probabilities and Theorem 3.1 imply

$$(3.9)\quad E_D\big(\phi(Z_n^+, Z_n^-, J(\tau_n)); \tau_n < \infty\big) = \frac{\lambda_{A_d}}{\lambda_d}E_{A_d}\left(\frac{1}{\Delta A_d(0)}\int_0^{\Delta A_d(0)}\phi(\Delta A_d(0)-u, u, J(0))\,du\right) + \frac{\lambda_{A_{c,n}}}{\lambda_D}E_{A_{c,n}}\left(\frac{1}{\Delta A_{c,n}(0)}\int_0^{\Delta A_{c,n}(0)}\phi(\Delta A_{c,n}(0)-u, u, J(0))\,du\right).$$

The second term on the right-hand side of (3.9) can be rewritten as

$$\lambda_D^{-1}E\left(\int_0^1\frac{1}{\Delta A_{c,n}(t)}\int_0^{\Delta A_{c,n}(t)}\phi(\Delta A_{c,n}(t)-u, u, J(t))\,du A_{c,n}(dt)\right).$$

Hence, by Lemma 2.2, this term converges to the second term on the right-hand side of (3.7), since $A_{c,n}(t)$ converges to $A_c(t)$ uniformly on finite intervals as n tends to infinity and

$$\inf_{0<u,v\leqslant \Delta A_{c,n}(t)}\phi(v, u, J(t)) \leqslant \frac{1}{\Delta A_{c,n}(t)}\int_0^{\Delta A_{c,n}(t)}\phi(\Delta A_{c,n}(t)-u, u, J(t))\,du \leqslant \sup_{0<u,v\leqslant \Delta A_{c,n}(t)}\phi(v, u, J(t)),$$

where the difference of the upper and lower bound tends to zero uniformly in t as $n\to\infty$. On the other hand, the left-hand side of (3.9) equals

$$\lambda_D^{-1}E\Big(\int_0^1\phi(Z_n^+, Z_n^-, J(\tau_n))\,1_{\{\tau_n<\infty\}}\circ\theta_u D(du)\Big).$$

Since, for each fixed $\omega \in \Omega$, the ruin time τ can attain at most countably many values, Lemma 2.2 yields

$$\lim_{n\to\infty} \phi(Z_n^+, Z_n^-, J(\tau_n))\mathbf{1}_{\{\tau_n<\infty\}}\circ\theta_u = \phi(Z^+, Z^-, J(\tau))\mathbf{1}_{\{\tau<\infty\}}\circ\theta_u,$$

except for possibly countably many u's for each $\omega \in \Omega$. Hence, the left-hand side of (3.9) converges to the one of (3.7). ■

An immediate consequence of Theorem 3.2 is the following result:

COROLLARY 3.2. *Given the conditions of Theorem* 3.2, *the equality*

$$(3.10)\qquad E_D\big(\phi(Z^+, Z^-, J(\tau)); \tau<\infty\big) = \frac{1}{\lambda_D} E\Big(\int_0^1 \phi(0, 0, J(u))\,A(du)\Big)$$

holds, provided that $\{A(t)\}$ *is continuous in* t *with probability* 1.

4. Ladder heights of stationary workload. Note that in Section 3 we considered the stochastic processes $\{X_0(t)\}$ and $\{J(t)\}$ not on the basic probability space $(\Omega, \mathscr{F}, P)$, but we replaced the probability measure P by the Palm probability P_D. This means, in particular, that the claim surplus process $\{X_0(t)\}$ is started at time $t = 0$ under the condition that it is decreasing at this time. Furthermore, from the definition of P_D it follows that, within the decreasing periods of $\{X_0(t)\}$, the starting point is chosen with a probability proportional to $D(du)$.

Let us briefly discuss what can be concluded from the above results if the process $\{X_0(t)\}$ is considered under P. Since the general case seems to be very difficult, we will consider a rather simple example. That is, we assume that $D'(u) = 0$ or $D'(u) = c$ for a constant $c > 0$, i.e. locally $\{D(u)\}$ is either constant or increases with a constant rate c during its increasing periods. This means that, for every $C \in \mathscr{F}$, we have

$$(4.1)\qquad P(C) = E\Big(\int_0^1 \mathbf{1}_C\circ\theta_u\Big(\mathbf{1}_{\{D'(u)=0\}}\,du + \frac{1}{c}D(du)\Big)\Big)$$
$$= P\big(C; D'(0) = 0\big) + \frac{\lambda_D}{c} P_D(C).$$

Choosing $C = \Omega$ in (4.1) gives

$$P\big(D'(0) = 0\big) = 1 - \lambda_D/c.$$

Hence, Theorem 3.2 yields the following result:

THEOREM 4.1. *In addition to the conditions of Theorem* 3.2, *assume that* $D'(t) = c$ *for every increasing point of* $\{D(t)\}$, *where* $c > 0$ *is a certain constant.*

Then

$$(4.2)\quad E\big(\phi(Z^+, Z^-, J(\tau)); \tau < \infty\big)$$

$$= \left(1-\frac{\lambda_D}{c}\right) E\big(\phi(Z^+, Z^-, J(\tau)); \tau < \infty \mid D'(0) = 0\big)$$

$$+\frac{\lambda_{A_d}}{c} E_{A_d}\left(\frac{1}{\Delta A_d(0)} \int_0^{\Delta A_d(0)} \phi(\Delta A_d(0)-u, u, J(0))\, du\right) + \frac{1}{c} E\big(\int_0^1 \phi(0, 0, J(u)) A_c(du)\big).$$

Note that (4.2) extends Theorem 2 of Asmussen and Schmidt [3], where the probability $P(J(\tau) \in F, \tau < \infty)$ is considered under the assumption that $\{A(t)\}$ has a density $r(J(t))$ and that the increasing and decreasing periods alternate one by one as well as that their starting times form a (locally finite) point process. In this case,

$$(4.3)\quad E\big(\phi(Z^+, Z^-, J(\tau)); \tau < \infty \mid D'(0) = 0\big) = E\big(\phi(0, 0, J(0)) \mid D'(0) = 0\big),$$

since $A(t)$ increases at $t = 0$ when $D'(0) = 0$. On the other hand, the second term on the right-hand side of (4.2) disappears, and the third term, multiplied by c, becomes

$$E\big(\int_0^1 \phi(0, 0, J(u)) r(J(u))\, du\big) = E\Big(\sum_{T_n^+ \in (0,1]} \int_{T_n^+}^{T_n^-} \phi(0, 0, J(u)) r(J(u))\, du\Big)$$

$$+E\Big(\int_0^{T_1^-} \phi(0, 0, J(u)) r(J(u))\, du\Big) - E\Big(\int_1^{T_{N^-((0,1))+1}^-} \phi(0, 0, J(u)) r(J(u))\, du\Big)$$

$$= \lambda_{N^+} E_{N^+}\Big(\int_0^{T_1^-} \phi(0, 0, J(u)) r(J(u))\, du\Big),$$

where T_n^+ and T_n^- are the n-th switching times for up and down periods, respectively, and N^+ and N^- are the counting measures generated by $\{T_n^+\}$ and $\{T_n^-\}$, respectively. This gives formula (18) of Asmussen and Schmidt [3], i.e.

$$(4.4)\quad E\big(\phi(J(\tau)) \mid D'(0) > 0, 0 < \tau < \infty\big) = \frac{E_{N^+}\Big(\int_0^{T_1^-} \phi(J(u)) r(J(u))\, du\Big)}{E_{N^+}\Big(\int_0^{T_1^-} r(J(u))\, du\Big)}$$

for every bounded measurable function $\phi\colon K \to \mathbf{R}$.

Now we give an example of how Theorem 4.1 can be used in order to determine the distribution of the first ascending ladder height of the stationary workload process in single-server queues. Another interesting relationship between queueing characteristics and ladder heights of risk processes is discuss-ed in Sigman [18]. For the purpose of this section we interpret process $\{A(t)\}$ as the input of the queue which can either be a locally finite jump process, a common assumption in queueing theory, or an arbitrary pure jump process. We assume that $\{A(t)\}$ induces a stationary ergodic random measure with intensity $\lambda_A < 1$. In case $\{A(t)\}$ is a locally finite jump process, the famous ergodic theorem of Loynes says (see, e.g., Franken et al. [6]) that, under these assumptions, a unique stationary workload process $\{W(t)\}$ exists which is consistent with the shift operators θ_t. For a general (ergodic) jump input $\{A(t)\}$ with $\lambda_A < 1$, the existence of such a stationary workload process $\{W(t)\}$ can be proved analogously to the proof of the locally finite jump process case. Thus, the stochastic process $\{D(t)\}$ given by

$$D(t) = \int_0^t \mathbf{1}_{\{W(u)>0\}}\, du \quad \text{for every } t \geqslant 0 \tag{4.5}$$

is also consistent with $\{\theta_t\}$. It is easy to see that

$$W(t) = X_0(t) + W(0) \quad \text{for every } t \geqslant 0, \tag{4.6}$$

where $X_0(t)$ is given by (1.1). Moreover, $\lambda_A = \lambda_D$. This means, in particular, that in many cases interesting from a practitioner's point of view, condition (iv-b) is fulfilled, e.g. when $\{A(t)\}$ is induced by an independently marked Poisson process, a Markov-modulated Poisson process or a Lévy process, respectively. Thus, from Theorem 4.1 we get the following result, where W^+ denotes the first ascending ladder height of the stationary workload process $\{W(t)\}$ as given by

$$W^+ = \begin{cases} W(\tau) - W(0) & \text{if } \tau < \infty, \\ 0 & \text{otherwise} \end{cases}$$

with $\tau = \inf\{t > 0\colon\ W(t) > W(0)\}$.

THEOREM 4.2. *Assume that* (iv-b) *holds. Then*

$$E\phi(W^+) = (1-\lambda_A)\, E\big(\phi(A(\tau_A)) \mid W(0) = 0\big) + \lambda_A E_A\left(\frac{1}{\Delta A(0)} \int_0^{\Delta A(0)} \phi(u)\, du\right) \tag{4.7}$$

for any bounded measurable function $\phi\colon \mathbf{R} \to \mathbf{R}$, *where* $\tau_A = \min\{t\colon\ A(t) > 0\}$.

Proof. Because of $\lambda_A = \lambda_D$, putting $\phi = 1$ we get $P(\tau < \infty) = 1$ from (4.2). Moreover, the third term on the right-hand side of (4.2) vanishes because we assume that $\{A(t)\}$ is a pure jump process. Thus, (4.7) follows from (4.2). ∎

Note that the random variable W^+ considered in Theorem 4.2 is the overshoot when, for the first time after time zero, the time-stationary workload process $\{W(t)\}_{t>0}$ crosses the *random* level $W(0)$ which this process had at $t = 0$. A related but still different problem is the study of excursions of the workload process above a given (deterministic) level (see, e.g., [8]). However, although the results obtained by Guillemin and Mazumdar [8] are different from Theorem 4.2, their method of proof is similar to our approach, because they use a special case of exchange formula (2.3).

A formula similar to (4.7) can be proved for the ladder height distribution of the stationary workload process in a single-server queue with fluid input provided that for the continuous component $\{A_c(t)\}$ of $\{A(t)\}$ we have

$$\frac{d}{dt}A_c(t) = 0 \text{ or } > 1 \quad \text{with probability 1.}$$

On the other hand, the case of multiserver queues seems to be much more complicated. Here we see, similar to (4.1), that

$$E(\mathbb{1}_C \cdot Q) = \lambda_D P_D(C),$$

where $Q = -(d/dt)D(t)|_{t=0}$ denotes the number of busy servers at time zero. Thus, instead of obtaining (4.7), we would get a representation formula for the covariance $E(\phi(W^+)\cdot Q)$ as a consequence of Theorem 4.1.

REFERENCES

[1] S. Asmussen, A. Frey, T. Rolski and V. Schmidt, *Does Markov-modulation increase the risk?*, ASTIN Bull. 25 (1995), pp. 49–66.

[2] S. Asmussen and V. Schmidt, *The ascending ladder height distribution for a class of dependent random walks*, Statist. Neerlandica 47 (1993), pp. 269–277.

[3] – *Ladder height distributions with marks*, Stochastic Process. Appl. 58 (1995), pp. 105–119.

[4] D. J. Daley and D. Vere-Jones, *An Introduction to the Theory of Point Processes*, Springer, New York 1988.

[5] W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. 2, 2nd edition, Wiley, New York 1971.

[6] P. Franken, D. König, U. Arndt and V. Schmidt, *Queues and Point Processes*, Wiley, Chichester 1982.

[7] M. Frenz and V. Schmidt, *An insensitivity property of ladder height distributions*, J. Appl. Probab. 29 (1992), pp. 616–624.

[8] F. M. Guillemin and R. R. Mazumdar, *Excursions of the workload process in G/GI/1 queues*, Stochastic Process. Appl. 58 (1995), pp. 139–154.

[9] D. König and V. Schmidt, *Zufällige Punktprozesse*, Teubner, Stuttgart 1992.

[10] J. Mecke, *Stationäre zufällige Masse auf lokalkompakten Abelschen Gruppen*, Z. Wahrsch. verw. Gebiete 9 (1967), pp. 36–58.

[11] M. Miyazawa, *Rate conservation laws: a survey*, Queueing Systems 15 (1994), pp. 1–58.

[12] – *Time-dependent rate conservation laws for a process defined with a stationary marked point process and their applications*, J. Appl. Probab. 31 (1994), pp. 114–129.

[13] – *Palm calculus for a process with a stationary random measure and its applications to fluid queues*, Queueing Systems 17 (1994), pp. 183–211.

[14] – *Note on generalizations of Mecke's formula and extensions of $H = \lambda G$*, J. Appl. Probab. 32 (1995), pp. 105–122.

[15] – and V. Schmidt, *On ladder height distributions of general risk processes*, Ann. Appl. Probab. 3 (1993), pp. 763–776.

[16] P. Protter, *Stochastic Integration and Differential Equations*, 2nd edition, Springer, New York 1992.

[17] V. Schmidt and R. Serfozo, *Campbell's formula and applications to queueing*, in: *Advances in Queueing: Theory, Methods and Open Problems*, J. H. Dshalalow (Ed.), CRC Press, Boca Raton 1995, pp. 225–242.

[18] K. Sigman, *Queues under preemptive LIFO and ladder height distributions for risk processes: a duality*, Stochastic Models 12 (1996), pp. 725–735.

Masakiyo Miyazawa
Dept. of Information Sci.
Science University of Tokyo
Noda City, Chiba 278, Japan

Volker Schmidt
Institute of Stochastics
University of Ulm
89069 Ulm, Germany

Received on 6.11.1995;
revised version on 10.6.1996