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PROPERTIES OF GREEN FUNCTION OF SYMMETRIC STABLE PROCESSES

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Abstract. We study the Green function $G_D(x, y)$ of symmetric α-stable processes in $\boldsymbol{R}^d$ for an open set D $(0 < \alpha < 2, d \geqslant 3)$. Our main result gives the upper and the lower bound estimates of $G_D(x, y)$ for a bounded open set D with a $C^{1,1}$ boundary. We also get a more direct formula for the Green function for a ball. As a simple conclusion we obtain "3G Theorem" and estimates of $E^x(\tau_D)$, where τ_D is the exit time of D.

1. Introduction. The aim of this work is to study the Green function of symmetric α-stable processes in $\boldsymbol{R}^d$ (or of Riesz potentials of order α), where $0 < \alpha < 2$ and $d \geqslant 3$. The main result of this paper may be stated as follows:

THEOREM. *Let $\alpha \in (0, 2)$ and $d \geqslant 3$. Let $D \subset \boldsymbol{R}^d$ be a bounded open set with a $C^{1,1}$ boundary and let $G_D(x, y)$ be the Green function of symmetric α-stable processes for D. Then there exist constants $C_1, C_2 > 0$ depending only on D, d, α such that for any $x, y \in D$*

$$C_1 \min\left(\frac{1}{|x-y|^{d-\alpha}}, \frac{\delta^{\alpha/2}(x)\,\delta^{\alpha/2}(y)}{|x-y|^d}\right) \leqslant \frac{G_D(x, y)}{A_{d,\alpha}}$$

$$\leqslant \min\left(\frac{1}{|x-y|^{d-\alpha}}, C_2 \frac{\delta^{\alpha/2}(x)\,\delta^{\alpha/2}(y)}{|x-y|^d}\right),$$

where $\delta(x) = \operatorname{dist}(x, \partial D)$ *and* $A_{d,\alpha} = 2^{-\alpha}\pi^{-d/2}\Gamma\big((d-\alpha)/2\big)\big(\Gamma(\alpha/2)\big)^{-1}$.

As an immediate application of these estimates one can get the so-called "3G Theorem" (cf. [5] or [6]). We also use our main result to obtain estimates of $E^x(\tau_D)$, where τ_D is the exit time of D, $\tau_D = \inf\{t > 0\colon X_t \notin D\}$ (Proposition 4.9).

These results extend the classical theory, related to Brownian motion, to the case of symmetric α-stable processes, $\alpha \in (0, 2)$. The class of α-harmonic functions, $\alpha \in (0, 2)$, has simple homogeneity properties analogous to those of the classical harmonic functions $(\alpha = 2)$. Also, the potential theory for $0 < \alpha < 2$ in $\boldsymbol{R}^d$ $(d \geqslant 2)$ enjoys the explicit formulation in terms of M. Riesz kernels

$|x-y|^{\alpha-d}$ similar to the classical theory based on Newtonian kernel $|x-y|^{2-d}$ (see [8]). The main difference and difficulty in the theory of α-harmonic functions is that the support of the harmonic measure for an open set is not contained in its boundary, as in the classical case, but in its complement. This exhibits the fact that paths of symmetric α-stable process fail to be continuous.

Estimates of the classical Green function attracted attention for a long time because of important consequences in potential theory, harmonic analysis and differential equations. They are important tools in studying the Schrödinger equation and Feynman–Kac gauge theory. Our work follows rather recent papers by Chung and Zhao (see [5] and [9]). Nevertheless, some similar estimates in the classical case were obtained e.g. by K.-O. Widman in the sixties and by M. V. Keldyš and M. A. Lavrent'ev in the thirties.

Section 2 sets up notation and collects together some standard facts for further use. We introduce the Green function of symmetric α-stable processes and state its basic properties. The idea of studying properties of Green function using methods of stochastic processes can be traced to the work of Hunt [7].

Section 3 is devoted to the study of the Green function of symmetric α-stable processes for a ball. In our investigations we were inspired by the paper by Chung [5], concerning the classical Green function for a ball. By arguments from [8] we express the potential of harmonic measure for a ball by the potential I_r of the equilibrium measure for a ball. Thus we obtain a more direct formula for the Green function. It may be interesting that for $\alpha = 1$ and $d = 4$ this function can be expressed by elementary functions. Using estimates of I_r and some ideas from [5] we prove the main inequalities for the Green function.

In Section 4 we extend the results obtained in Section 3 to an arbitrary bounded open set with a $C^{1,1}$ boundary. These results are analogous to the ones proved in [9] for the classical Green function. The proof of the lower bound estimate follows [9]. The main difference is caused by the fact that in our case the support of the Poisson kernel $P_r(x, \cdot)$ is not contained in the boundary of a ball but in its complement. In the proof of the upper bound estimate we exploit direct estimates of the kernel $P_r(x, \cdot)$. At the end of this section we obtain some applications of our main results. We also point out some counterexamples.

2. Preliminaries. The notation $C = C(x, y, z)$, frequent in this paper, means that the constant C depends only on x, y, z. "Constants" are always numbers in $(0, \infty)$, so that we can freely multiply and divide them to get other constants.

For $x \in \mathbf{R}^d$, $r > 0$ we put

$$B(x, r) = \{y \in \mathbf{R}^d\colon\ |y-x| < r\} \quad \text{and} \quad S(x, r) = \{y \in \mathbf{R}^d\colon\ |y-x| = r\}.$$

The surface area of the $(d-1)$-dimensional sphere $S(0, 1) \subset \mathbf{R}^d$ will be denoted by $\omega_d = 2\pi^{d/2}/\Gamma(d/2)$. Let ε_x be a unit mass at x. For any subset $A \subset \mathbf{R}^d$, we denote its complement by $A^c = \mathbf{R}^d \setminus A$, its closure by $\overline{A}$, and its boundary by $\partial A = \overline{A} \cap \overline{A^c}$. Furthermore, we put

$$\operatorname{dist}(A, B) = \inf\{|x-y|:\ x \in A,\ y \in B\}, \qquad \operatorname{diam}(A) = \sup\{|x-y|:\ x,\ y \in A\}$$

for $A, B \subset \mathbf{R}^d$. We write $m(A)$ for the d-dimensional Lebesgue measure of the set $A \subset \mathbf{R}^d$. Let $\mathscr{B}(\mathbf{R}^d)$ denote the Borel σ-field of $\mathbf{R}^d$.

For the rest of the paper let $\alpha \in (0, 2)$ and $d \geqslant 3$. By (X_t, P^x) we denote the standard rotation invariant ("symmetric") α-stable, $\mathbf{R}^d$-valued Lévy process (i.e. homogeneous, with independent increments), with index of stability α and the characteristic function of the form

$$E^0 \exp\{i\xi X_t\} = \exp\{-t|\xi|^\alpha\}, \qquad \xi \in \mathbf{R}^d,\ t \geqslant 0.$$

As usual, E^x denotes the expectation with respect to the distribution P^x of the process starting from $x \in \mathbf{R}^d$. We always assume that sample paths of X_t are right-continuous and have left-hand limits almost surely. (X_t, P^x) is a Markov process with transition probabilities given by $P_t(x, A) = P^x(X_t \in A)$ and is strong Markov with respect to the so-called "standard filtration" and quasi--left-continuous on $[0, \infty)$ (see e.g. [3]). For the sake of brevity we will refer to this process as to "symmetric α-stable".

The distribution of X_t with respect to P^0 has the continuous and bounded density h_t $(t > 0)$. According to [10] we have

$$h_t(x) = t^{-d/\alpha} h_1(t^{-1/\alpha} x) \quad \text{and} \quad h_t(x) \leqslant ct|x|^{-d-\alpha}, \qquad x \in \mathbf{R}^d,$$

where $c = c(d, \alpha)$.

For $A \in \mathscr{B}(\mathbf{R}^d)$, we define $T_A = \inf\{t > 0:\ X_t \in A\}$, the first hitting time of A. The first hitting time of A^c is called the *exit time* from A and denoted by $\tau_A = \inf\{t > 0:\ X_t \in A^c\}$.

Let $f \geqslant 0$ be a Borel measurable function on $\mathbf{R}^d$. We say that f is *α-harmonic* in an open set $D \subset \mathbf{R}^d$ if

$$f(x) = E^x f(X_{\tau_A}), \qquad x \in A,$$

for every bounded open set A with the closure $\overline{A}$ contained in D.

We define the harmonic measure ω_D^x (for D, in x, with respect to X) by the formula $\omega_D^x(A) = P^x(X_{\tau_D} \in A)$, where $x \in \mathbf{R}^d$; $A, D \in \mathscr{B}(\mathbf{R}^d)$. It is clear that $\operatorname{supp}(\omega_D^x) \subset \overline{D^c}$.

If $A \in \mathscr{B}(\mathbf{R}^d)$, then, for each $x \in \mathbf{R}^d$, $P^x(T_A = 0)$ is either zero or one according to the Blumenthal zero-one law. A point $x \in \mathbf{R}^d$ is called *regular* for $A \in \mathscr{B}(\mathbf{R}^d)$ if $P^x(T_A = 0) = 1$, and x is called *irregular* for A if $P^x(T_A = 0) = 0$. We denote by A^r the set of all points which are regular for the set A.

Now, we will give a condition concerning regularity (the outer cone condition). Let $B \in \mathscr{B}(\mathbf{R}^d)$. Suppose there exists a cone V with vertex $y \in \partial B$ such that $V \cap B(y, r) \subset B^c$ for some $r > 0$. Then y is regular for B^c.

For $f \geqslant 0$ and Borel measurable we define the potential operator of X_t by

$$Uf(x) = E^x \int_0^\infty f(X_t)\,dt.$$

According to [3] we have

$$Uf(x) = \int A_{d,\alpha} |x-y|^{\alpha-d} f(y)\,dy,$$

$$\text{where } A_{d,\alpha} = 2^{-\alpha}\pi^{-d/2}\Gamma((d-\alpha)/2)(\Gamma(\alpha/2))^{-1}.$$

U is called the *Riesz potential* and $A_{d,\alpha}|x-y|^{\alpha-d}$ is called the *Riesz kernel of order* α. We will write $u(x, y) = A_{d,\alpha}|x-y|^{\alpha-d}$.

Rewriting Theorem 1.16 in Chapter VI in [3] for symmetric α-stable processes we get the important technical fact:

PROPOSITION 2.1. *Let $B \in \mathscr{B}(\mathbf{R}^d)$. Then*

$$\int u(z, y)\,d\omega_B^x(z) = \int u(x, z)\,d\omega_B^y(z).$$

We point out that in this work we consider only non-negative σ-finite measures. We define the potential $U\mu$ of a measure μ by

$$U\mu(x) = \int u(x, y)\,d\mu(y).$$

According to Proposition 2.1 and [3] we have the following fact:

PROPOSITION 2.2. *Let μ be a measure on $\mathbf{R}^d$ and $B \in \mathscr{B}(\mathbf{R}^d)$. We have*

(i) $$U\mu(x) \geqslant \int U\mu(z)\,d\omega_B^x(z), \qquad x \in \mathbf{R}^d;$$

(ii) *if* $\operatorname{supp}(\mu) \subset (B^c)^r$, *then*

$$U\mu(x) = \int U\mu(z)\,d\omega_B^x(z), \qquad x \in \mathbf{R}^d.$$

The crucial fact of the potential theory of symmetric α-stable processes is that the density of the harmonic measure for a ball is given by an explicit formula (see e.g. [2] or [8]). Let $x \in B(0, r)$. The harmonic measure $\omega_{B(0,r)}^x$ for $B(0, r)$ has the density function $P_r(x, \cdot)$ (with respect to the Lebesgue measure) given by the formula

$$P_r(x, y) = \begin{cases} c_\alpha^d \left(\dfrac{r^2-|x|^2}{|y|^2-r^2}\right)^{\alpha/2} |x-y|^{-d} & \text{for } |y| > r, \\ 0 & \text{for } |y| \leqslant r, \end{cases} \tag{2.1}$$

where $c_\alpha^d = \Gamma(d/2)\pi^{-d/2-1}\sin(\pi\alpha/2)$.

Now we will introduce the Green function.

DEFINITION 2.3. If $B \in \mathscr{B}(\mathbf{R}^d)$, define

$$G_B(x, y) = u(x, y) - E^x u(X(\tau_B), y) \quad \text{for } x, y \in \mathbf{R}^d,\ x \neq y.$$

We put

$$G_B(x, x) = 0 \text{ for } x \in (B^c)^r \quad \text{and} \quad G_B(x, x) = \infty \text{ for } x \notin (B^c)^r.$$

$G_B(x, y)$ is called the *Green function* for B.

It follows easily from the definition that if $x \in (B^c)^r$, then $G_B(x, y) = 0$ for all $y \in \boldsymbol{R}^d$. Using Proposition 2.2 for $\mu = \varepsilon_x$ we get $0 \leqslant G_B(x, y) \leqslant u(x, y)$. By Proposition 2.1 we have $G_B(x, y) = G_B(y, x)$ for all $x, y \in \boldsymbol{R}^d$. Notice also that if D is an open set and $x \in D$, then $G_D(x, x) = \infty$ equals $u(x, x) - E^x u(X(\tau_D), x)$.

According to [3], if $B \in \mathscr{B}(\boldsymbol{R}^d)$, then $B \backslash B^r$ is polar. Using this and the strong Markov property we can obtain the following technical lemma:

LEMMA 2.4. *Let f be a Borel measurable function on $\boldsymbol{R}^d$, $f \geqslant 0$ and let $D_1, D_2 \subset \boldsymbol{R}^d$ be open sets such that $D_1 \subset D_2$. Then for every $x \in \boldsymbol{R}^d$ we have*

$$E^x f\big(X(\tau_{D_2})\big) = E^x \big[E^{X(\tau_{D_1})} f\big(X(\tau_{D_2})\big)\big].$$

By Proposition 2.2 and Lemma 2.4 we get the following fact:

PROPOSITION 2.5. *Let D be an open set in $\boldsymbol{R}^d$, $B(x_1, r) \subset D$, $x \in B(x_1, r)$, and $y \in \boldsymbol{R}^d$. Then*

(i) $$G_D(x, y) \geqslant \int\limits_{B(x_1, r)^c} G_D(u, y) P_r(x - x_1, u - x_1)\, du;$$

(ii) *if in addition $y \notin \overline{B(x_1, r)}$, we have*

$$G_D(x, y) = \int\limits_{B(x_1, r)^c} G_D(u, y) P_r(x - x_1, u - x_1)\, du.$$

Of course, this proposition remains valid if we replace $B(x_1, r)$ by an open set A $(A \subset D)$ and $P_r(x - x_1, u - x_1)\, du$ by $d\omega_A^x(u)$. In particular, $G_D(\cdot, y)$ is α-harmonic in $D \backslash \{y\}$.

Let $D_1 \subset D_2$ be open sets in $\boldsymbol{R}^d$. As a simple conclusion of Lemma 2.4 we get $G_{D_1}(x, y) \leqslant G_{D_2}(x, y)$ for every $x, y \in \boldsymbol{R}^d$.

Let $D \subset \boldsymbol{R}^d$ be an open set. We will present some facts concerning continuity of $G_D(x, y)$. The proofs of these properties are almost the same as in the classical case, taking into account properties of h_t, so the reader is referred to [1] and [6].

$G_D(\cdot, \cdot)$ is continuous in the extended sense as a mapping from $D \times D$ into $[0, \infty]$. This follows from the proof in [6], Theorems 2.4 and 2.6. By similar arguments to those in [6], Theorem 1.23, we can obtain $\lim_{x \to z} G_D(x, y) = 0$ for each $y \in D$ and $z \in \partial D \cap (D^c)^r$. If in addition D is bounded and $\partial D \subset (D^c)^r$, we have

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} G(x, y) = G(x_0, y_0) \quad \text{for } x_0, y_0 \in \boldsymbol{R}^d,\ x_0 \neq y_0.$$

This follows from the proof in [1] (Chapter II, Proposition 4.7) and analogous arguments to those in [6], Theorem 2.6.

By Proposition 2.5 (ii) and the extended continuity of $G_D(\cdot, \cdot)$ it is easy to notice that if $D \subset \boldsymbol{R}^d$ is an open set, then $G_D(x, y) > 0$ for each $x, y \in D$.

3. Green function for a ball. In this section we abbreviate $B(0, r)$ to B_r and G_{B_r} to G_r. By Definition 2.3 we have $G_r(x, y) = u(x, y) - U\omega_{B_r}^x(y)$. The density of the measure $\omega_{B_r}^x$ is given by (2.1). However, we will not use this formula but the equivalent characterization given by the following proposition:

PROPOSITION 3.1. *Let $x \in B_r$. Suppose ν is a measure on $\mathbf{R}^d$ satisfying the following conditions*:

(i) $\operatorname{supp}(\nu) \subset B_r^c$;

(ii) $U\nu(y) = u(x, y)$ *for* $y \in B_r^c$.

Then $\nu = \omega_{B_r}^x$.

Proof. Of course, $\operatorname{supp}(\omega_{B_r}^x) \subset B_r^c$. By Proposition 2.1 it follows that $\omega_{B_r}^x$ satisfies (ii). On the other hand, if μ satisfies (i) and (ii), then by Proposition 2.2 (ii) and again Proposition 2.1 we get $U\mu(y) = U\omega_{B_r}^x(y)$ for all $y \in \mathbf{R}^d$. Then $\mu = \omega_{B_r}^x$ by Theorem 1.12 in [8].

We will adapt some arguments and methods from [8] to get the expression for $U\omega_{B_r}^x$ which will be convenient for our purposes.

Let $r > 0$ and $x \notin \partial B_r$. Define a mapping $z \to z^*$ by

$$z \to z^* = x-(z-x)\frac{r^2-|x|^2}{|z-x|^2}. \tag{3.1}$$

If $x \in (\overline{B_r})^c$, we call this mapping *inversion* with center at x and radius $R = (|x|^2-r^2)^{1/2}$; if $x \in B_r$, we call it *imaginary inversion* with center at x and radius $R = (r^2-|x|^2)^{1/2}$.

In the sequel we collect some well-known properties of these mappings for further reference (see the Appendix in [8]).

PROPOSITION 3.2. *If $z \to z^*$ is defined as above, we have*

(i)
$$|z^*-x|\,|z-x| = R^2;$$

(ii)
$$|z_1^*-z_2^*| = \frac{|z_1-z_2|\,R^2}{|z_2-x|\,|z_1-x|};$$

(iii)
$$|z^*|^2 = r^2+\frac{(r^2-|x|^2)(r^2-|z|^2)}{|z-x|^2};$$

(iv) *the inversion maps B_r onto itself, $S(0, r)$ onto itself, and $(\overline{B_r})^c$ onto itself; the imaginary inversion maps B_r onto $(\overline{B_r})^c$, $S(0, r)$ onto itself, and $(\overline{B_r})^c$ onto B_r*;

(v)
$$\frac{dz^*}{|z^*-x|^d} = \frac{dz}{|z-x|^d}.$$

Let $z \to z^*$ be an imaginary inversion with center at $x \in B_r$ and radius $R = (r^2-|x|^2)^{1/2}$. With each measure ν which does not have an atom at the point x we associate another measure ν^* by the formula

$$d\nu^*(z^*) = \left(\frac{|z-x|}{R}\right)^{\alpha-d} d\nu(z).$$

ν^* is called the *Kelvin transform* of ν.

By Proposition 3.2 (ii) we have

$$(3.2)\quad U\nu^*(y^*) = \int u(y^*, z^*) \frac{|z-x|^{\alpha-d}}{R^{\alpha-d}} d\nu(z)$$

$$= \int A_{d,\alpha} \frac{|y-x|^{d-\alpha}|z-x|^{d-\alpha}}{|y-z|^{d-\alpha}R^{2d-2\alpha}} \frac{|z-x|^{\alpha-d}}{R^{\alpha-d}} d\nu(z) = \left(\frac{|y-x|}{R}\right)^{d-\alpha} U\nu(y).$$

Now, let us introduce the measure λ having a density $\lambda(x)$ given by the formula

$$\lambda(x) = \begin{cases} (r^2-|x|^2)^{-\alpha/2} & \text{for } |x| < r, \\ 0 & \text{for } |x| \geqslant r. \end{cases}$$

Denote the potential of the measure λ by $I_r = U\lambda$. According to the Appendix in [8] we have $I_r(y) = C_{d,\alpha}^{-1}$ for $|y| \leqslant r$, where

$$(3.3)\quad C_{d,\alpha}^{-1} = \frac{A_{d,\alpha}\omega_d}{2} \int_0^\infty \frac{1}{b^{\alpha/2}(1+b)} db = \frac{A_{d,\alpha}\pi^{(d/2)+1}}{\Gamma(d/2)\sin(\pi\alpha/2)}.$$

Let λ_x^* be the Kelvin transform of the measure λ with respect to imaginary inversion* with center at $x \in B_r$ and radius $R = (r^2-|x|^2)^{1/2}$. By (3.2) we have

$$U\lambda_x^*(y^*) = \left(\frac{|y-x|}{R}\right)^{d-\alpha} I_r(y).$$

It follows that $U\lambda_x^*(y^*) = C_{d,\alpha}^{-1} R^{\alpha-d}|y-x|^{d-\alpha}$ for $|y| \leqslant r$. By Proposition 3.2 (i) and (iv) we get

$$U\lambda_x^*(y^*) = C_{d,\alpha}^{-1} R^{d-\alpha}|y^*-x|^{\alpha-d} \quad \text{for } |y^*| \geqslant r.$$

Since $\operatorname{supp}(\lambda) = \overline{B_r}$, it follows from Proposition 3.2 (iv) that $\operatorname{supp}(\lambda_x^*) = B_r^c$. Thus $\mu = A_{d,\alpha} C_{d,\alpha} R^{\alpha-d} \lambda_x^*$ satisfies conditions (i) and (ii) in Proposition 3.1. Hence $\omega_{B_r}^x = \mu$. Consequently,

$$U\omega_{B_r}^x(y^*) = \frac{A_{d,\alpha}C_{d,\alpha}}{R^{d-\alpha}} U\lambda_x^*(y^*) = A_{d,\alpha}C_{d,\alpha}\left(\frac{|y-x|}{R^2}\right)^{d-\alpha} I_r(y) = \frac{A_{d,\alpha}C_{d,\alpha}}{|y^*-x|^{d-\alpha}} I_r(y).$$

Finally,

$$(3.4)\quad U\omega_{B_r}^x(y) = C_{d,\alpha} u(x, y) I_r(y^*), \quad x \in B_r,\ y \in \mathbf{R}^d.$$

Furthermore, if $|y| < r$, then $|y^*| > r$.

We are interested in deriving a more direct expression for $U\omega_{B_r}^x(y)$, $y \in B_r$. Having (3.4), we can see that it remains to compute $I_r(y)$ for $|y| > r$.

After the inversion $x \to x^*$ with center at $y \in B_r^c$ and radius $R = (|y|^2-r^2)^{1/2}$, for the integral $I_r(y)$ we obtain the expression

$$I_r(y) = \int_{|x| \leqslant r} \frac{A_{d,\alpha}\,dx}{(r^2-|x|^2)^{\alpha/2}|x-y|^{d-\alpha}} = (|y|^2-r^2)^{\alpha/2} \int_{|x^*| \leqslant r} \frac{A_{d,\alpha}\,dx^*}{(r^2-|x^*|^2)^{\alpha/2}|x^*-y|^{d}}.$$

We introduce spherical coordinates $(x_1^*, \ldots, x_d^*) \to (\varrho, \phi_1, \ldots, \phi_{d-1})$ with origin 0 and principal axis $0y$. Then

$$|x^*-y|^d = (\varrho^2 - 2|y|\varrho\cos\phi_1 + |y|^2)^{d/2}.$$

Hence

$$I_r(y) = A_{d,\alpha}(|y|^2-r^2)^{\alpha/2}\, 2\pi \prod_{k=1}^{d-3}\int_0^{\pi} \sin^k\theta\, d\theta$$

$$\times \int_0^r \frac{\varrho^{d-1}}{(r^2-\varrho^2)^{\alpha/2}} \int_0^{\pi} \frac{\sin^{d-2}\phi_1\, d\phi_1}{(\varrho^2-2|y|\varrho\cos\phi_1+|y|^2)^{d/2}}\, d\varrho.$$

We will put

$$c = 2\pi \prod_{k=1}^{d-3}\int_0^{\pi}\sin^k\theta\, d\theta = \frac{\omega_d}{\int_0^{\pi}\sin^{d-2}\phi_1\, d\phi_1}.$$

For $a > 1$ we have

$$\int_0^{\pi} \frac{\sin^{d-2}\phi_1\, d\phi_1}{(1-2a\cos\phi_1+a^2)^{d/2}} = \frac{\omega_d}{c}\, \frac{1}{a^{d-2}(a^2-1)}.$$

This relation is verified in the Appendix in [8]. Consequently,

$$I_r(y) = A_{d,\alpha}(|y|^2-r^2)^{\alpha/2} c \int_0^r \frac{\varrho^{d-1}}{(r^2-\varrho^2)^{\alpha/2}}\, \frac{1}{\varrho^d} \int_0^{\pi} \frac{\sin^{d-2}\phi_1\, d\phi_1}{\left(1-2(|y|/\varrho)\cos\phi_1+(|y|^2/\varrho^2)\right)^{d/2}}\, d\varrho$$

$$= A_{d,\alpha}(|y|^2-r^2)^{\alpha/2} c \int_0^r \frac{\varrho^{d-1}}{(r^2-\varrho^2)^{\alpha/2}}\, \frac{1}{\varrho^d}\, \frac{\omega_d}{c}\, \frac{1}{(|y|^{d-2}/\varrho^{d-2})\left((|y|^2/\varrho^2)-1\right)}\, d\varrho.$$

Finally, we obtain

$$I_r(y) = A_{d,\alpha}\,\omega_d \frac{(|y|^2-r^2)^{\alpha/2}}{|y|^{d-2}} \int_0^r \frac{\varrho^{d-1}}{(r^2-\varrho^2)^{\alpha/2}(|y|^2-\varrho^2)}\, d\varrho, \tag{3.5}$$

where $|y| > r$.

Now, set $a = |y|^2 - r^2$ and put $b = (r^2-\varrho^2)/a$ in (3.5). Hence

$$I_r(y) = \frac{A_{d,\alpha}\,\omega_d}{2}\, \frac{1}{|y|^{d-2}} \int_0^{r^2/a} \frac{(r^2-ab)^{(d-2)/2}}{b^{\alpha/2}(1+b)}\, db, \tag{3.6}$$

where $|y| > r$.

By the definition of I_r we easily get $I_1(z/r) = I_r(z)$, $z \in \mathbf{R}^d$. This permits us to concentrate on the case $r = 1$. Now we are able to prove some technical estimates of the integral I_1.

LEMMA 3.3. *Let us put* $I(y) = I_1(y)$ *and* $a = |y|^2 - 1$. *For* $|y| \geqslant 1$ *we have*

$$I(y) \leqslant \frac{1}{C_{d,\alpha}|y|^{d-2}}. \tag{3.7}$$

If in addition $a \leqslant 1$, then

$$\frac{1}{C_{d,\alpha}|y|^{d-2}} - C_1 \frac{a^{\alpha/2}}{|y|^{d-2}} \leqslant I(y) \leqslant \frac{1}{C_{d,\alpha}|y|^{d-2}} - C_2 \frac{a^{\alpha/2}}{|y|^{d-2}}, \tag{3.8}$$

where $C_1 = C_1(d, \alpha)$ and $C_2 = C_2(d, \alpha)$ are the constants.

Proof. From (3.6) we have

$$\begin{aligned} I(y) &= \frac{A_{d,\alpha}\omega_d}{2|y|^{d-2}} \int_0^\infty \frac{1}{b^{\alpha/2}(b+1)} db - \frac{A_{d,\alpha}\omega_d}{2|y|^{d-2}} \int_{1/a}^\infty \frac{1}{b^{\alpha/2}(b+1)} db \\ &\quad - \frac{A_{d,\alpha}\omega_d}{2|y|^{d-2}} \int_0^{1/a} \frac{1-(1-ab)^{(d-2)/2}}{b^{\alpha/2}(b+1)} db = J_1 - J_2 - J_3. \end{aligned} \tag{3.9}$$

Notice that J_1, J_2, J_3 are positive. Inequality (3.7) follows immediately from (3.3) and (3.9).

Let $a \leqslant 1$. We have

$$I(y) \leqslant J_1 - J_2 \leqslant J_1 - \frac{A_{d,\alpha}\omega_d}{2|y|^{d-2}} \int_{1/a}^\infty \frac{1}{2b^{\alpha/2+1}} db = \frac{1}{C_{d,\alpha}|y|^{d-2}} - \frac{A_{d,\alpha}\omega_d}{2\alpha} \frac{a^{\alpha/2}}{|y|^{d-2}}.$$

On the other hand,

$$J_2 \leqslant \frac{A_{d,\alpha}\omega_d}{2|y|^{d-2}} \int_{1/a}^\infty \frac{1}{b^{\alpha/2+1}} db = \frac{A_{d,\alpha}\omega_d}{\alpha} \frac{a^{\alpha/2}}{|y|^{d-2}}.$$

It is easy to check that

$$(1-ab)^{(d-2)/2} \geqslant 1-(d-1)ab/2 \quad \text{for } b \in [0, 1/a].$$

Hence

$$\begin{aligned} J_3 &\leqslant \frac{A_{d,\alpha}\omega_d}{2|y|^{d-2}} \frac{d-1}{2} \int_0^{1/a} \frac{ab}{b^{\alpha/2}(b+1)} db \leqslant \frac{A_{d,\alpha}\omega_d(d-1)}{4|y|^{d-2}} \int_0^{1/a} \frac{a}{b^{\alpha/2}} db \\ &= \frac{A_{d,\alpha}\omega_d(d-1)}{4-2\alpha} \frac{a^{\alpha/2}}{|y|^{d-2}}. \end{aligned}$$

Consequently, using (3.9) we obtain the left-hand inequality in (3.8).

Let us write

$$f_r(x, y) = \left(r^2|x-y|^2 + (r^2-|x|^2)(r^2-|y|^2)\right)^{1/2}$$

and

$$g_r(x, y) = \frac{f_r(x, y)}{|x-y|} = \left(r^2 + \frac{(r^2-|x|^2)(r^2-|y|^2)}{|x-y|^2}\right)^{1/2}, \tag{3.10}$$

where $x, y \in \mathbf{R}^d$.

The function $I_r(y)\colon \mathbf{R}^d \to \mathbf{R}$ depends only on $|y|$. To simplify the notation, we will use the same letter I_r to denote the function $I_r\colon [0, \infty] \to \mathbf{R}$, defined by $I_r(|y|) = I_r(y)$ for $y \in \mathbf{R}^d$.

By the definition of the Green function and (3.4) we have

$$G_r(x, y) = u(x, y) - U\omega_{B_r}^x(y) = u(x, y) - C_{d,\alpha} u(x, y) I_r(|y^*|), \qquad x, y \in B_r,$$

where $|y^*| = g_r(x, y)$ by Proposition 3.2 (iii). Hence

$$G_r(x, y) = \frac{A_{d,\alpha}}{|x-y|^{d-\alpha}} \left[1 - C_{d,\alpha} I_r(g_r(x, y))\right], \qquad x, y \in B_r, \tag{3.11}$$

which with (3.3), (3.6) and (3.10) gives a more direct formula for the Green function for a ball.

In particular, from (3.11) it follows directly that G_r is symmetric.

We can now formulate our main theorem for the Green function for a ball. The analogous theorem for the classical Green function is proved in [5]. Also the proof of our theorem follows that in [5].

THEOREM 3.4. *Let us put $\delta(x) = r - |x|$ for $x \in B_r$. There exist constants $A_1 = A_1(d, \alpha)$ and $A_2 = A_2(d, \alpha)$ such that*

$$A_1 \min\left(\frac{1}{|x-y|^{d-\alpha}}, \frac{\delta^{\alpha/2}(x)\,\delta^{\alpha/2}(y)}{|x-y|^d}\right) \leqslant \frac{G_r(x, y)}{A_{d,\alpha}} \leqslant \min\left(\frac{1}{|x-y|^{d-\alpha}}, \frac{A_2\,\delta^{\alpha/2}(x)\,\delta^{\alpha/2}(y)}{|x-y|^d}\right)$$

for all $x, y \in B_r$.

Proof. It is a simple matter to check that $g_1(x/r, y/r) = r^{-1} g_r(x, y)$. Since $I_1(z/r) = I_r(z)$, $z \in \mathbf{R}^d$, it follows that $G_r(x, y) = r^{\alpha-d} G_1(x/r, y/r)$. Hence it is not difficult to observe that it is sufficient to prove our theorem for $r = 1$.

Let $r = 1$. We put $G(x, y) = G_1(x, y)$, $I(y) = I_1(y)$, $f(x, y) = f_1(x, y)$, $g = g_1(x, y)$ and $a = g^2 - 1$.

Let us first prove the right-hand inequality. The inequality with the first term under the min is obvious. So, to get the right-hand inequality it is sufficient to consider the case:

$$\frac{A_2\,\delta^{\alpha/2}(x)\,\delta^{\alpha/2}(y)}{|x-y|^\alpha} \leqslant 1.$$

Set $A_2 \geqslant 4^{\alpha/2}$. Consequently, we have

$$\frac{4\delta(x)\,\delta(y)}{|x-y|^2} \leqslant 1.$$

Hence

$$a = g^2 - 1 = \frac{(1-|x|^2)(1-|y|^2)}{|x-y|^2} \leqslant \frac{4\delta(x)\,\delta(y)}{|x-y|^2} \leqslant 1.$$

By (3.11) and Lemma 3.3 we obtain

$$\frac{G(x, y)}{A_{d,\alpha}} \leqslant \frac{1}{|x-y|^{d-\alpha}} \left(1 - \frac{1}{|g|^{d-2}} + C_{d,\alpha} C_1 \frac{a^{\alpha/2}}{|g|^{d-2}}\right). \tag{3.12}$$

We have

$$1-\frac{1}{|g|^{d-2}}=\frac{f(x,y)^{d-2}-|x-y|^{d-2}}{f(x,y)^{d-2}}. \tag{3.13}$$

Since $f(x, y) \geqslant |x-y|$, the numerator in (3.13) is less than

$$(d-2)f(x,y)^{d-3}\big(f(x,y)-|x-y|\big) \leqslant (d-2)f(x,y)^{d-4}\big(f(x,y)^2-|x-y|^2\big)$$
$$\leqslant 4(d-2)f(x,y)^{d-4}\delta(x)\delta(y).$$

Substituting this into (3.13) and using $f(x, y) \geqslant |x-y|$ again we obtain

$$1-\frac{1}{|g|^{d-2}} \leqslant \frac{4(d-2)\delta(x)\delta(y)}{|x-y|^2}=\frac{4(d-2)\delta^{\alpha/2}(x)\delta^{\alpha/2}(y)}{|x-y|^{\alpha}}\frac{\delta^{1-\alpha/2}(x)\delta^{1-\alpha/2}(y)}{|x-y|^{2-\alpha}}$$
$$\leqslant \frac{4(d-2)\delta^{\alpha/2}(x)\delta^{\alpha/2}(y)}{|x-y|^{\alpha}4^{1-\alpha/2}}.$$

It remains to estimate the third term in the brackets in (3.12). We have

$$C_{d,\alpha}C_1\frac{a^{\alpha/2}}{|g|^{d-2}} \leqslant C_{d,\alpha}C_1 a^{\alpha/2} \leqslant C_{d,\alpha}C_1 2^{\alpha}\frac{\delta^{\alpha/2}(x)\delta^{\alpha/2}(y)}{|x-y|^{\alpha}}.$$

By (3.12) we get

$$\frac{G(x,y)}{A_{d,\alpha}} \leqslant \big(2^{\alpha}(d-2)+2^{\alpha}C_{d,\alpha}C_1\big)\frac{\delta^{\alpha/2}(x)\delta^{\alpha/2}(y)}{|x-y|^{d}},$$

keeping in mind that we assumed at the beginning that $A_2 \geqslant 4^{\alpha/2}$.

We now turn to the left-hand inequality. We will consider two cases: $a > 1$ and $a \leqslant 1$.

Let $a > 1$. Then, since $a = g^2-1$, we get $g \geqslant \sqrt{2}$. By Lemma 3.3 we have $I(g) \leqslant C_{d,\alpha}^{-1}|g|^{2-d}$. Therefore

$$G(x,y) \geqslant \frac{A_{d,\alpha}}{|x-y|^{d-\alpha}}\left(1-\frac{1}{|g|^{d-2}}\right) \geqslant \frac{A_{d,\alpha}}{|x-y|^{d-\alpha}}\left(1-\frac{1}{\sqrt{2}^{d-2}}\right).$$

Let $a \leqslant 1$. Then, since $a = g^2-1$, we get $g \leqslant \sqrt{2}$. From Lemma 3.3 we have

$$I(g) \leqslant \frac{1}{C_{d,\alpha}|g|^{d-2}}-C_2\frac{a^{\alpha/2}}{|g|^{d-2}}.$$

Consequently,

$$G(x,y) \geqslant \frac{A_{d,\alpha}}{|x-y|^{d-\alpha}}\left(1-\frac{1}{|g|^{d-2}}+C_2 C_{d,\alpha}\frac{a^{\alpha/2}}{|g|^{d-2}}\right)$$
$$\geqslant \frac{A_{d,\alpha}}{|x-y|^{d-\alpha}}C_2 C_{d,\alpha}\frac{a^{\alpha/2}}{|g|^{d-2}}$$
$$= C_2 C_{d,\alpha}\frac{1}{\sqrt{2}^{d-2}}\frac{A_{d,\alpha}}{|x-y|^{d-\alpha}}\frac{(1-|x|^2)^{\alpha/2}(1-|y|^2)^{\alpha/2}}{|x-y|^{\alpha}}.$$

Since $1-|x|^2 \geqslant 1-|x| = \delta(x)$, we finally obtain

$$G(x, y) \geqslant C_2 C_{d,\alpha} \frac{1}{\sqrt{2^{d-2}}} \frac{A_{d,\alpha}}{|x-y|^d} \delta^{\alpha/2}(x) \delta^{\alpha/2}(y).$$

4. Green function for $C^{1,1}$ open sets. In this section we will use the following notation. D denotes a bounded open set with a $C^{1,1}$ boundary, $G(x, y)$ the Green function for D, and $\delta(x) = \operatorname{dist}(x, \partial D)$.

A function $F: \boldsymbol{R}^d \to \boldsymbol{R}$ is called $C^{1,1}$ if it has a first derivative F' and there exists a constant λ such that for all $x, y \in \boldsymbol{R}^d$ we have $|F'(x)-F'(y)| \leqslant \lambda |x-y|$.

We say that a bounded open set $D \subset \boldsymbol{R}^d$ has a $C^{1,1}$ *boundary* if for each $x \in \partial D$ there are: a $C^{1,1}$ function $F_x: \boldsymbol{R}^{d-1} \to \boldsymbol{R}$ (with a constant $\lambda = \lambda(D)$), an orthonormal coordinate system CS_x and a constant $\eta = \eta(D)$ such that if $y = (y_1, \ldots, y_n)$ in CS_x coordinates, then

$$D \cap B(x, \eta) = \{y: y_n > F_x(y_1, \ldots, y_{n-1})\} \cap B(x, \eta).$$

We point out that the set D is not necessarily connected. However, it follows by the definition above that if D_1 and D_2 are two different connected components of D, then $\operatorname{dist}(D_1, D_2) \geqslant \eta$.

Now we will present some important properties of a bounded open set D with a $C^{1,1}$ boundary. They may be found in [9].

A normal internal vector n_z exists at each point $z \in \partial D$. It is also known that there exist positive constants $s_0 = s_0(D)$ and $r_0 = r_0(D)$ such that for any $z, w \in \partial D$, $|n_z - n_w| \leqslant s_0 |z-w|$ and for any $z \in \partial D$, $0 < r \leqslant r_0$, there exist two balls B_1^z and B_2^z of radius r such that

$$B_1^z \subset D, \quad B_2^z \subset \boldsymbol{R}^d \backslash \overline{D}, \quad \text{and} \quad \{z\} = \partial B_1^z \cap \partial B_2^z.$$

In particular, by the outer cone condition, all points of ∂D are regular for D^c.

The main results of this section are Theorems 4.3 and 4.5 — the lower and the upper bound estimates of the Green function for D.

At first we prove the lower bound estimate. We follow the approach designed in [9] for the classical Green function. However, there are major changes in proofs.

For the sake of brevity we set a constant $A' = A_1 A_{d,\alpha}$, where A_1 is such as in Theorem 3.4.

LEMMA 4.1. *There exists a constant $C_1 = C_1(d, \alpha, D)$ such that for any $x, y \in D$ we have*

$$\frac{G(x, y)}{\delta^{\alpha/2}(x) \delta^{\alpha/2}(y)} > C_1.$$

Proof. Set $r = r_0/5$. Let x^* and y^* be the points on ∂D such that $|x-x^*| = \delta(x)$ and $|y-y^*| = \delta(y)$. If $\delta(x) < r$, set $B(a, r) = B_1^{x^*}$, and if $\delta(x) \geqslant r$, set $B(a, r) = B(x, r)$. If $\delta(y) < r$, set $B(b, r) = B_1^{y^*}$, and if $\delta(y) \geqslant r$, set

$B(b, r) = B(y, r)$. Notice that $x \in B(a, r) \subset D$ and $y \in B(b, r) \subset D$. We also have

$$\frac{r^2 - |x-a|^2}{\delta(x)} \geqslant \min\left(r, \frac{r^2}{\operatorname{diam}(D)}\right) \quad \text{and} \quad \frac{r^2 - |y-b|^2}{\delta(y)} \geqslant \min\left(r, \frac{r^2}{\operatorname{diam}(D)}\right).$$

By Proposition 2.5 (i) we get

$$G(x, y) \geqslant \int_{B(a,r)^c} G(u, y) P_r(x-a, u-a)\, du.$$

Using the symmetry of $G(u, y)$ and again Proposition 2.5 (i) we get

$$G(x, y) \geqslant \int_{B(a,r)^c} \int_{B(b,r)^c} G(v, u) P_r(y-b, v-b) P_r(x-a, u-a)\, dv\, du.$$

For $u \in D$ we have

$$\frac{P_r(x-a, u-a)}{\delta^{\alpha/2}(x)} = \frac{(r^2 - |x-a|^2)^{\alpha/2}}{(|u-a|^2 - r^2)^{\alpha/2} |u-x|^d \delta^{\alpha/2}(x)}$$
$$\geqslant \frac{1}{(\operatorname{diam}(D))^{d+\alpha}} \min\left(r^{\alpha/2}, \frac{r^\alpha}{(\operatorname{diam}(D))^{\alpha/2}}\right) = c.$$

Similarly, if $v \in D$, we have

$$\frac{P_r(y-b, v-b)}{\delta^{\alpha/2}(y)} \geqslant c.$$

Since $r = r_0/5$, it is not difficult to notice that there exists a ball $B(z, r)$ such that

$$B(z, r) \subset D \cap B(a, r)^c \cap B(b, r)^c.$$

Let us write $B = B(z, r)$ and $\delta_B(u) = \operatorname{dist}(u, B^c)$. If $u, v \in B(z, r/2)$, we have

$$\delta_B(u) \geqslant |u-v|/2 \quad \text{and} \quad \delta_B(v) \geqslant |u-v|/2.$$

By Theorem 3.4 we obtain

$$G(v, u) \geqslant G_B(v, u) \geqslant A' \min\left(\frac{1}{|v-u|^{d-\alpha}}, \frac{\delta_B^{\alpha/2}(v)\, \delta_B^{\alpha/2}(u)}{|v-u|^d}\right) \geqslant \frac{A'}{2^\alpha |v-u|^{d-\alpha}}$$

for all $v, u \in B(z, r/2)$. Hence

$$\frac{G(x, y)}{\delta^{\alpha/2}(x)\, \delta^{\alpha/2}(y)} \geqslant c^2 \int_B \int_B G(v, u)\, dv\, du \geqslant \frac{c^2 A'}{2^\alpha} \int_{B(z,r/2)} \int_{B(z,r/2)} \frac{1}{|u-v|^{d-\alpha}}\, dv\, du.$$

The last integral is positive and depends only on d, α, r.

LEMMA 4.2. *Let $x, y \in D$ satisfy the inequalities*

$$\max(\delta(x), \delta(y)) \leqslant 2|x-y| \quad \textit{and} \quad |x-y| \leqslant \frac{r_0}{10(1 + r_0 s_0)}.$$

Then the inequality

$$G(x, y) \geqslant C_2 \frac{\delta^{\alpha/2}(x)\,\delta^{\alpha/2}(y)}{|x-y|^d}$$

holds for a constant $C_2 = C_2(d, \alpha, D)$.

Proof. Let x^* and y^* be the points on ∂D such that $|x-x^*| = \delta(x)$ and $|y-y^*| = \delta(y)$. Set $r = r_0$, $B_x = B_1^{x^*} = B(o_x, r)$, $B_y = B_1^{y^*} = B(o_y, r)$. Thus,

$$|x^*-y^*| \leqslant |x-y|+\delta(x)+\delta(y) \leqslant 5|x-y|,$$

$$\begin{aligned} |o_x-o_y| &\leqslant |x^*-y^*|+r\,|n_{x^*}-n_{y^*}| \leqslant (1+r_0 s_0)\,|x^*-y^*| \\ &\leqslant 5(1+r_0 s_0)\,|x-y| \leqslant r/2. \end{aligned}$$

Since $\delta(x) \leqslant r/5$ and o_x, x, x^* lie on the same line, we have $B(x, \delta(x)) \subset B_x$. Similarly, $B(y, \delta(y)) \subset B_y$.

Set $h = \operatorname{dist}(y, \partial B_x)$. There are three kinds of situations.

Case 1. $y \in B_x$, $h \geqslant \delta(y)/4$.

Let us write $\delta_{B_x}(u) = \operatorname{dist}(u, \partial B_x)$. By Theorem 3.4 we obtain

$$G(x, y) \geqslant G_{B_x}(x, y) \geqslant A' \min\left(\frac{1}{|x-y|^{d-\alpha}}, \frac{\delta_{B_x}^{\alpha/2}(x)\,\delta_{B_x}^{\alpha/2}(y)}{|x-y|^d}\right). \tag{4.1}$$

But $\delta_{B_x}(x) = \delta(x)$ and $\delta_{B_x}(y) = h \geqslant \delta(y)/4$. We also have $|x-y| \geqslant \delta(x)/2$ and $|x-y| \geqslant \delta(y)/2$. Therefore the right-hand side of (4.1) is greater than or equal to

$$A' 2^{-\alpha} \frac{\delta^{\alpha/2}(x)\,\delta^{\alpha/2}(y)}{|x-y|^d}.$$

Case 2. $y \notin B_x$, $h \geqslant \delta(y)/4$.

Set $P = \{u \in \mathbf{R}^d\colon h/2 < |u-y| < h\}$. Of course, $P \subset B_x^c$. By Theorem 2.5 (i) for $x \in B_x \subset D$ we get

$$\begin{aligned} G(x, y) &\geqslant \int_{B_x^c} P_r(x-o_x, u-o_x)\,G(u, y)\,du \\ &\geqslant \int_{P \cap B_y} P_r(x-o_x, u-o_x)\,G_{B_y}(u, y)\,du \\ &= c_\alpha^d \int_{P \cap B_y} \frac{(r^2-|x-o_x|^2)^{\alpha/2}}{(|u-o_x|^2-r^2)^{\alpha/2}} \frac{1}{|x-u|^d} G_{B_y}(u, y)\,du. \end{aligned} \tag{4.2}$$

Let $u \in P \cap B_y$ and write $\delta_{B_y}(u) = \operatorname{dist}(u, \partial B_y)$. We have

$$|u-y| > h/2 \geqslant \delta(y)/8 = \delta_{B_y}(y)/8,$$

$$\delta_{B_y}(u) \leqslant |u-y|+\delta_{B_y}(y) < 9\,|u-y|.$$

By Theorem 3.4 we obtain

$$G_{B_y}(u, y) \geqslant A' \min\left(\frac{1}{|u-y|^{d-\alpha}}, \frac{\delta_{B_y}^{\alpha/2}(u)\,\delta_{B_y}^{\alpha/2}(y)}{|u-y|^d}\right) \geqslant \frac{A'}{72^{\alpha/2}} \frac{\delta_{B_y}^{\alpha/2}(u)\,\delta^{\alpha/2}(y)}{|u-y|^d}.$$

Notice that $r^2-|x-o_x|^2 \geqslant r\delta(x)$. Since $x \in B_x$, it follows that $h < |x-y| \leqslant r/10$. If $u \in P \cap B_y$, we also have

$$|x-u| \leqslant |x-y|+|y-u| < |x-y|+h < 2|x-y|,$$

$$|u-o_x|-r \leqslant |u-y|+|y-o_x|-r = |u-y|+r+h-r < 2h,$$

$$|u-o_x|+r < 2h+2r < 3r, \qquad |u-o_x|^2-r^2 < 6rh.$$

Thus the right-hand side of (4.2) is greater than or equal to

$$c_\alpha^d \int_{P\cap B_y} \frac{r^{\alpha/2}\delta^{\alpha/2}(x)}{6^{\alpha/2} r^{\alpha/2} h^{\alpha/2}} \frac{1}{2^d |x-y|^d} \frac{A' \delta_{B_y}^{\alpha/2}(u) \delta^{\alpha/2}(y)}{72^{\alpha/2}|u-y|^d} du$$

$$= \frac{c_\alpha^d A'}{6^{\alpha/2} 72^{\alpha/2} 2^d} \frac{\delta^{\alpha/2}(x)\delta^{\alpha/2}(y)}{|x-y|^d} \frac{1}{h^{\alpha/2}} \int_{P\cap B_y} \frac{\delta_{B_y}^{\alpha/2}(u)}{|u-y|^d} du.$$

It remains to show that there exists a constant $c = c(d, \alpha)$ such that

$$\frac{1}{h^{\alpha/2}} \int_{P\cap B_y} \frac{\delta_{B_y}^{\alpha/2}(u)}{|u-y|^d} du \geqslant c. \tag{4.3}$$

To do this, introduce spherical coordinates $(\varrho, \varphi_1, \ldots, \varphi_{d-1})$ with origin y and principal axis yo_y. Let us consider

$$S = \{u = (\varrho, \varphi_1, \ldots, \varphi_{d-1}) \colon h/2 < \varrho < h,\ 0 \leqslant \varphi_1 \leqslant \pi/6\}.$$

Obviously, $S \subset P$.

Let $b \in \partial B(y, h)$ be the point which lies on the line yo_y between y and o_y. Since $h < r/10$ and $|y-o_y| = r-\delta(y) \geqslant r-r/5$, such a point exists. If $u = (\varrho, \varphi_1, \ldots, \varphi_{d-1}) \in S$, we have

$$|u-b|^2 = |y-b|^2+|u-y|^2-2|u-y||y-b| \cos\varphi_1 = h^2+\varrho^2-2h\varrho\cos\varphi_1$$

$$\leqslant h^2+\varrho^2-2h\varrho\cos(\pi/6) = h^2+\varrho^2-\sqrt{3}h\varrho.$$

It is easy to notice that the function $f(\varrho) = h^2+\varrho^2-\sqrt{3}h\varrho$ defined for $\varrho \in [h/2, h]$ takes its maximum at the point $h/2$. We have $f(h/2) = c_1 h^2$, where $c_1 = 5/4-\sqrt{3}/2$. Notice that $0 < c_1 < 1$. Hence $|u-b|^2 \leqslant c_1 h^2$. Consequently,

$$|u-o_y| \leqslant |u-b|+|b-o_y| = |u-b|+r-h-\delta(y)$$

$$\leqslant h\sqrt{c_1}+r-h = r-h(1-\sqrt{c_1}).$$

Therefore

$$r-|u-o_y| \geqslant r-\big(r-h(1-\sqrt{c_1})\big) = h(1-\sqrt{c_1}) > 0.$$

Hence we infer that $S \subset B_y$ and $\delta_{B_y}(u) = r-|u-o_y| \geqslant h(1-\sqrt{c_1})$ for $u \in S$. Let us put $c_2 = (1-\sqrt{c_1})$ $(c_2 > 0)$. We have

$$\frac{1}{h^{\alpha/2}} \int_{P \cap B_y} \frac{\delta_{B_y}^{\alpha/2}(u)}{|u-y|^d} du \geqslant \frac{1}{h^{\alpha/2}} \int_S \frac{c_2^{\alpha/2} h^{\alpha/2}}{|u-y|^d} du$$

$$= c_2^{\alpha/2} \int_{h/2}^{h} \int_0^{\pi/6} \int_0^{\pi} \ldots \int_0^{\pi} \int_0^{2\pi} \frac{1}{\varrho^d} \varrho^{d-1} \sin^{d-2} \varphi_1 \ldots \sin \varphi_{d-2} \, d\varphi_{d-1} \ldots d\varphi_1 \, d\varrho.$$

But

$$\int_{h/2}^{h} \frac{1}{\varrho} \, d\varrho = \ln h - \ln(h/2) = \ln 2 > 0$$

and (4.3) is proved.

Case 3. $h < \delta(y)/4$ ($y \in B_x$ or $y \notin B_x$).

Set $B_1 = B(y, \delta(y)/2)$. Of course, $B_1 \subset B_y$. By Theorem 2.5 (i) for $x \in B_x \subset D$ we get

$$G(x, y) \geqslant \int_{B_x^c} P_r(x-o_x, u-o_x) G(u, y) du \tag{4.4}$$

$$\geqslant \int_{B_x^c \cap B_1} P_r(x-o_x, u-o_x) G_{B_y}(u, y) du$$

$$= c_\alpha^d \int_{B_x^c \cap B_1} \frac{(r^2-|x-o_x|^2)^{\alpha/2}}{(|u-o_x|^2-r^2)^{\alpha/2}} \frac{1}{|x-u|^d} G_{B_y}(u, y) du.$$

Notice that $r^2-|x-o_x|^2 \geqslant r\delta(x)$. Let $u \in B_x^c \cap B_1$ and let us write $\delta_{B_y}(u) = \operatorname{dist}(u, \partial B_y)$. We have

$$\delta_{B_y}(y) = \delta(y) > |u-y|$$

and

$$\delta_{B_y}(u) \geqslant \operatorname{dist}(y, \partial B_y) - |u-y| > \delta(y)/2 > |u-y|.$$

By Theorem 3.4 we get

$$G_{B_y}(u, y) \geqslant A' \min\left(\frac{1}{|u-y|^{d-\alpha}}, \frac{\delta_{B_y}^{\alpha/2}(u)\,\delta_{B_y}^{\alpha/2}(y)}{|u-y|^d}\right) = \frac{A'}{|u-y|^{d-\alpha}}.$$

If $u \in B_x^c \cap B_1$, we also have

$$|x-u| \leqslant |x-y|+|y-u| < |x-y|+\delta(y)/2 < 2|x-y|,$$

$$|u-o_x| \leqslant |o_x-y|+|u-y| < r+h+\delta(y)/2 \leqslant r+\delta(y),$$

$$|u-o_x|^2-r^2 = (|u-o_x|+r)(|u-o_x|-r) \leqslant 3r\delta(y).$$

Thus the right-hand side of (4.4) is greater than or equal to

$$c_\alpha^d \int_{B_x^c \cap B_1} \frac{r^{\alpha/2}\delta^{\alpha/2}(x)}{\delta^{\alpha/2}(y)\,3^{\alpha/2} r^{\alpha/2}} \frac{1}{2^d|x-y|^d} \frac{A'}{|u-y|^{d-\alpha}} du$$

$$= A' c_\alpha^d 2^{-d} 3^{-\alpha/2} \frac{\delta^{\alpha/2}(x)}{|x-y|^d \delta^{\alpha/2}(y)} \int_{B_x^c \cap B_1} \frac{1}{|u-y|^{d-\alpha}} du.$$

It is enough to show that there exists a constant $c = c(d, \alpha)$ such that

(4.5) $$\int_{B_x^c \cap B_1} \frac{1}{|u-y|^{d-\alpha}} du > c\delta^{\alpha}(y).$$

To do this, introduce spherical coordinates $(\varrho, \varphi_1, \ldots, \varphi_{d-1})$ with origin y and principal axis yo_x. Let us consider

$$P = \{u = (\varrho, \varphi_1, \ldots, \varphi_{d-1}) \colon \sqrt{2}h < \varrho < \delta(y)/2,\ 3\pi/4 \leqslant \varphi_1 \leqslant \pi\}.$$

Clearly, $P \subset B_1$. We will show that $P \subset B_x^c$. If $u = (\varrho, \varphi_1, \ldots, \varphi_{d-1}) \in P$, we have

(4.6) $$\begin{aligned} |u-o_x|^2 &= |y-o_x|^2 + |u-y|^2 - 2|y-o_x|\,|u-y| \cos \varphi_1 \\ &\geqslant |y-o_x|^2 + |u-y|^2 - 2|y-o_x|\,|u-y| \cos(3\pi/4). \end{aligned}$$

If $y \in B_x$, then $|y-o_x| = r-h$, and if $y \notin B_x$, then $|y-o_x| = r+h$. We also have $h < \delta(y)/4 < r$. So $|y-o_x| \geqslant r-h > 0$. Thus, the right-hand side of (4.6) is greater than or equal to

$$(r-h)^2 + \varrho^2 + \sqrt{2}(r-h)\varrho.$$

But $\varrho > \sqrt{2}h$. Hence this is greater than

$$(r-h)^2 + 2h^2 + 2h(r-h) = r^2 + h^2 \geqslant r^2.$$

It is obvious that $\sqrt{2}h < \sqrt{2}\delta(y)/4$, since in the case 3 we have $h < \delta(y)/4$. Thus

$$\int_{B_x^c \cap B_1} \frac{1}{|u-y|^{d-\alpha}} du \geqslant \int_P \frac{1}{|u-y|^{d-\alpha}} du$$

$$\geqslant \int_{\sqrt{2}\delta(y)/4}^{\delta(y)/2} \int_{3\pi/4}^{\pi} \int_0^{\pi} \ldots \int_0^{\pi} \int_0^{2\pi} \frac{1}{\varrho^{d-\alpha}} \varrho^{d-1} \sin^{d-2}\varphi_1 \ldots \sin\varphi_{d-2}\, d\varphi_{d-1} \ldots d\varphi_1\, d\varrho.$$

But

$$\int_{\sqrt{2}\delta(y)/4}^{\delta(y)/2} \varrho^{\alpha-1} d\varrho = \alpha^{-1}(2^{-\alpha} - 2^{-3\alpha/2})\delta^{\alpha}(y),$$

which gives (4.5) and completes the proof.

We are now in a position to show the lower bound inequality of $G(x, y)$.

THEOREM 4.3. *There exists a constant $C = C(d, \alpha, D)$ such that for any $x, y \in D$ we have*

$$G(x, y) \geqslant \begin{cases} \dfrac{C}{|x-y|^{d-\alpha}} & \text{if } |x-y| \leqslant \max\left(\dfrac{\delta(x)}{2}, \dfrac{\delta(y)}{2}\right), \\[2ex] C\,\dfrac{\delta^{\alpha/2}(x)\,\delta^{\alpha/2}(y)}{|x-y|^d} & \text{if } |x-y| > \max\left(\dfrac{\delta(x)}{2}, \dfrac{\delta(y)}{2}\right). \end{cases}$$

Proof. If $|x-y| \leqslant \delta(x)/2$, take $B = B(x, \delta(x))$ and write $\delta_B(u) = \operatorname{dist}(u, B^c)$. Then $y \in B \subset D$ and

$$\delta_B(y) \geqslant \operatorname{dist}(x, B^c) - |x-y| \geqslant \delta(x) - \delta(x)/2 = \delta(x)/2 \geqslant |x-y|.$$

By Theorem 3.4 we obtain

$$G(x, y) \geqslant G_B(x, y) \geqslant A' \min\left(\frac{1}{|x-y|^{d-\alpha}}, \frac{\delta_B^{\alpha/2}(x)\,\delta_B^{\alpha/2}(y)}{|x-y|^d}\right) = \frac{A'}{|x-y|^{d-\alpha}}.$$

If $|x-y| \leqslant \delta(y)/2$, the proof is the same.

If $\max(\delta(x), \delta(y)) \leqslant 2|x-y|$ and $|x-y| \leqslant r_0 10^{-1}(1+r_0 s_0)^{-1}$, we apply Lemma 4.2.

When $|x-y| \geqslant r_0 10^{-1}(1+r_0 s_0)^{-1}$, then by Lemma 4.1 we have

$$\frac{G(x, y)}{\delta^{\alpha/2}(x)\,\delta^{\alpha/2}(y)} |x-y|^d \geqslant \frac{C_1 r_0^d}{10^d (1+r_0 s_0)^d}.$$

Now we are going to prove the upper bound estimate of the Green function for D. The classical proof of the upper bound estimate in [9] is based on the explicit formula for the harmonic function in the ring (say $r < |x| < 2r$), which is 0 on $\partial B(0, r)$ and 1 on $\partial B(0, 2r)$, and the fact that the support of the harmonic measure for an open set coincides with its boundary. We were unable to adapt these arguments to our case. Instead, we exploited direct estimates of the kernel $P_r(x, \cdot)$.

Lemma 4.4. *Let us choose $n \in N$ such that $(n+1)\alpha/(2n) < 1$. Let $k \in N$, $0 \leqslant k \leqslant n-1$, and $m = 0$ or $\alpha/2$. Assume that for each $l \in N$, $0 \leqslant l \leqslant k$ we have for all $x, y \in D$*

$$G(x, y) \leqslant A_l \frac{\delta^{l\alpha/(2n)}(x)\,\delta^m(y)}{|x-y|^{d-\alpha+m+l\alpha/(2n)}}, \tag{4.7}$$

with constants $A_l = A_l(n, m, l, d, \alpha, D)$ $(0 \leqslant l \leqslant k)$. Then inequality (4.7) holds for $l = k+1$ with another constant $A_{k+1} = A_{k+1}(n, m, k+1, d, \alpha, D)$.

Notice that for $\alpha < 1$ we can choose $n = 1$. Once we prove this key lemma we are able to prove the upper bound estimate of $G(x, y)$.

Theorem 4.5. *There exists a constant $C = C(d, \alpha, D)$ such that for any $x, y \in D$ we have*

$$G(x, y) \leqslant \min\left(\frac{A_{d,\alpha}}{|x-y|^{d-\alpha}}, C\frac{\delta^{\alpha/2}(x)\,\delta^{\alpha/2}(y)}{|x-y|^d}\right).$$

Proof of Theorem 4.5. We know that inequality (4.7) holds for $l = 0$ and $m = 0$. Using Lemma 4.4 n times, we see that the inequality holds for $l = n$ and $m = 0$. Since $G(x, y) = G(y, x)$, we obtain

$$G(x, y) \leqslant A \frac{\delta^{\alpha/2}(y)}{|x-y|^{d-\alpha/2}}, \qquad x, y \in D,$$

with a constant $A = A(d, \alpha, D)$. So, inequality (4.7) holds for $l = 0$ and $m = \alpha/2$. By using Lemma 4.4 again, the inequality holds for $l = n$ and $m = \alpha/2$ for a constant $C = C(d, \alpha, D)$. This proves Theorem 4.5.

Proof of Lemma 4.4. There are three kinds of situations.

Case 1. $\delta(x) \geqslant r_0$.

From (4.7) we have

$$G(x, y) \leqslant A_0 \frac{\delta^m(y)}{|x-y|^{d-\alpha+m}} \leqslant \frac{A_0 (\operatorname{diam}(D))^{(k+1)\alpha/(2n)}}{r_0^{(k+1)\alpha/(2n)}} \frac{\delta^{(k+1)\alpha/(2n)}(x)\,\delta^m(y)}{|x-y|^{d-\alpha+m+(k+1)\alpha/(2n)}}.$$

Case 2. $\delta(x) < r_0$, $|x-y| \leqslant 4\delta(x)$.

From (4.7) we have

$$G(x, y) \leqslant A_0 \frac{\delta^m(y)}{|x-y|^{d-\alpha+m}} \leqslant A_0\, 4^{(k+1)\alpha/(2n)} \frac{\delta^{(k+1)\alpha/(2n)}(x)\,\delta^m(y)}{|x-y|^{d-\alpha+m+(k+1)\alpha/(2n)}}.$$

Case 3. $\delta(x) < r_0$, $|x-y| > 4\delta(x)$.

Set $r = \min(r_0, |x-y|/4)$. We will use in the sequel the following easy inequality:

$$\frac{1}{r} \leqslant \frac{C_0}{|x-y|},$$

where $C_0 = \max(4, \operatorname{diam}(D)/r_0)$. Let x_0 be the point on ∂D such that $|x-x_0| = \delta(x)$. Set

$$B_1^{x_0} = B(x_1, r) \subset D \quad \text{and} \quad B_2^{x_0} = B(x_2, r) \subset \mathbf{R}^d \setminus \overline{D}.$$

Obviously, $y \notin \overline{B(x_1, r)}$, since $|x-y| \geqslant 4r$. Points x_2, x_0, x, x_1 lie on the same line and since $\delta(x) < \min(r_0, |x-y|/4) = r$, we have $x \in B(x_1, r)$, and x lies between x_0 and x_1. Therefore, by Theorem 2.5 (ii), we obtain

$$G(x, y) = \int_{B(x_1, r)^c} G(u, y) P_r(x-x_1, u-x_1)\, du. \tag{4.8}$$

Let us consider four sets:

$$P = B(x_0, r) \setminus (B(x_1, r) \cup B(x_2, r)), \qquad R = B(x_1, 2r) \setminus (B(x_1, r) \cup B(x_2, r) \cup P),$$

$$S = B(y, 2|x-y|) \setminus (B(x_1, r) \cup B(x_2, r) \cup P \cup R), \qquad T = B(y, 2|x-y|)^c.$$

It is immediate that $B(x_1, r)^c \subset B(x_2, r) \cup P \cup R \cup S \cup T$. We also have $G(\cdot, y) = 0$ on $B(x_2, r)$. We will estimate the integral in (4.8) separately on P, R, S, T. Estimate on P is the most difficult.

At first we will do this on R, S, T. Now we prove an easy lemma.

LEMMA 4.6. *If $u \notin (P \cup B(x_1, r) \cup B(x_2, r))$, then $|u-x| \geqslant r\sqrt{3}/2$.*

Proof. Let us consider the triangle ux_0x_1. The point x lies between x_0 and x_1. Since $|u-x_1| \geqslant r$, $|u-x_0| \geqslant r$ and $|x_1-x_0| = r$, it is easy to notice that the angle $\sphericalangle x_1 u x_0 \leqslant \pi/3$. Hence one of the angles $\sphericalangle x_1 x_0 u$ and $\sphericalangle x_0 x_1 u$ must be $\geqslant \pi/3$. We may and do assume that $\sphericalangle x_0 x_1 u \geqslant \pi/3$. Now we will consider the triangle $x x_1 u$. Denote angles $\varphi = \sphericalangle x x_1 u \geqslant \pi/3$ and $\psi = \sphericalangle x_1 x u$. If $\varphi \geqslant \pi/2$, then $|u-x| \geqslant |u-x_1| \geqslant r$. If $\varphi \leqslant \pi/2$, then $\sin\varphi \geqslant \sqrt{3}/2$. We have

$$\frac{|u-x|}{\sin\varphi} = \frac{|u-x_1|}{\sin\psi}.$$

Hence $|u-x| \geqslant \sin\varphi\, |u-x_1| \geqslant r\sqrt{3}/2$.

Notice that $r+|x-x_1| \leqslant 2r \leqslant 2(r+|u-x_1|)$ and $r-|x-x_1| = \delta(x)$. Hence

$$(4.9)\qquad \int\limits_{R\cup S\cup T} G(u, y) P_r(x-x_1, u-x_1)\, du$$

$$\leqslant A_0 c_\alpha^d \delta^m(y) \int\limits_{R\cup S\cup T} \frac{1}{|u-y|^{d-\alpha+m}} \frac{(r^2-|x-x_1|^2)^{\alpha/2}}{(|u-x_1|^2-r^2)^{\alpha/2}} \frac{1}{|u-x|^d}\, du$$

$$\leqslant C_1 \delta^{\alpha/2}(x) \delta^m(y) \int\limits_{R\cup S\cup T} \frac{1}{|u-y|^{d-\alpha+m}} \frac{1}{(|u-x_1|-r)^{\alpha/2}} \frac{1}{|u-x|^d}\, du,$$

where $C_1 = A_0 c_\alpha^d 2^{\alpha/2}$.

Now we will estimate the integral on the right-hand side of (4.9) separately on R, S, T.

Let $u\in R$. Then, by Lemma 4.6, $|u-x| \geqslant r\sqrt{3}/2$. Recall that $r \leqslant |x-y|/4$. We have

$$|u-y| \geqslant |x-y|-|x-x_1|-|x_1-u| \geqslant |x-y|-3r \geqslant |x-y|-3|x-y|/4 = |x-y|/4.$$

Hence

$$\int\limits_R \frac{1}{|u-y|^{d-\alpha+m}} \frac{1}{(|u-x_1|-r)^{\alpha/2}} \frac{1}{|u-x|^d}\, du \leqslant \frac{2^d 4^{d-\alpha+m}}{3^{d/2} r^d |x-y|^{d-\alpha+m}} \int\limits_R \frac{1}{(|u-x_1|-r)^{\alpha/2}}\, du.$$

We have

$$\int\limits_R \frac{1}{(|u-x_1|-r)^{\alpha/2}}\, du \leqslant \int\limits_{B(x_1,2r)\setminus B(x_1,r)} \frac{1}{(|u-x_1|-r)^{\alpha/2}}\, du = \omega_d \int\limits_r^{2r} \frac{1}{(\varrho-r)^{\alpha/2}} \varrho^{d-1}\, d\varrho.$$

After substituting $t=\varrho-r$, this is equal to

$$\omega_d \int\limits_0^r \frac{1}{t^{\alpha/2}} (t+r)^{d-1}\, dt \leqslant \frac{\omega_d 2^d}{2-\alpha} r^{d-\alpha/2} \leqslant \frac{\omega_d 2^d C_0^{\alpha/2} r^d}{(2-\alpha)|x-y|^{\alpha/2}}.$$

Hence

$$(4.10)\qquad \int\limits_R \frac{1}{|u-y|^{d-\alpha+m}} \frac{1}{(|u-x_1|-r)^{\alpha/2}} \frac{1}{|u-x|^d}\, du \leqslant \frac{C_2}{|x-y|^{d-\alpha/2+m}},$$

with the constant $C_2 = C_2(m, \alpha, d, D)$.

Now let $u\in S$. We continue estimating the integral on the right-hand side of (4.9). By Lemma 4.6, if $u\in S$, then $|u-x| \geqslant r\sqrt{3}/2$. We also have

$|u-x_1|-r \geqslant r$. Hence

$$\int_S \frac{1}{|u-y|^{d-\alpha+m}} \frac{1}{(|u-x_1|-r)^{\alpha/2}} \frac{1}{|u-x|^d} du \leqslant \frac{2^d C_0^{d+\alpha/2}}{3^{d/2}|x-y|^{d+\alpha/2}} \int_S \frac{1}{|u-y|^{d-\alpha+m}} du.$$

We have

$$\int_S \frac{1}{|u-y|^{d-\alpha+m}} du \leqslant \int_{B(y,2|x-y|)} \frac{1}{|u-y|^{d-\alpha+m}} du = \frac{\omega_d 2^{\alpha-m}}{\alpha-m} |x-y|^{\alpha-m}.$$

Hence

$$\int_S \frac{1}{|u-y|^{d-\alpha+m}} \frac{1}{(|u-x_1|-r)^{\alpha/2}} \frac{1}{|u-x|^d} du \leqslant \frac{C_3}{|x-y|^{d-\alpha/2+m}}, \tag{4.11}$$

with the constant $C_3 = C_3(m, \alpha, d, D)$.

Now let $u \in T$. We keep on estimating the integral on the right-hand side of (4.9). We have

$$|u-x| \geqslant |u-y|-|y-x| \geqslant |u-y|/2$$

and

$$\begin{aligned} |u-x_1|-r &\geqslant |u-y|-|y-x|-|x-x_1|-r \geqslant |u-y|-|y-x|-2r \\ &\geqslant |u-y|-|x-y|-|x-y|/2 \geqslant |u-y|/4. \end{aligned}$$

Thus

$$\begin{aligned} &\int_T \frac{1}{|u-y|^{d-\alpha+m}} \frac{1}{(|u-x_1|-r)^{\alpha/2}} \frac{1}{|u-x|^d} du \\ &\leqslant 2^d 4^{\alpha/2} \int_{B(y,2|x-y|)^c} \frac{1}{|u-y|^{2d-\alpha/2+m}} du = C_4 \frac{1}{|x-y|^{d-\alpha/2+m}}, \end{aligned} \tag{4.12}$$

with the constant $C_4 = C_4(m, \alpha, d, D)$.

Finally, by (4.9)–(4.12), we get

$$\begin{aligned} &\int_{R \cup S \cup T} G(u, y) P_r(x-x_1, u-x_1) du \\ &\leqslant C_1(C_2+C_3+C_4) \frac{\delta^{\alpha/2}(x)\delta^m(y)}{|x-y|^{d-\alpha/2+m}} \\ &\leqslant \frac{C_1(C_2+C_3+C_4)}{4^{(n-k-1)\alpha/(2n)}} \frac{\delta^{(k+1)\alpha/(2n)}(x)\delta^m(y)}{|x-y|^{d-\alpha+m+(k+1)\alpha/(2n)}}, \end{aligned} \tag{4.13}$$

since in the case 3 we have $4\delta(x) < |x-y|$.

Now we are going to estimate the integral in (4.8) on the set P. Let us recall that

$$P = B(x_0, r) \backslash (B(x_1, r) \cup B(x_2, r)).$$

We have $r^2 - |x-x_1|^2 \leqslant 2r\delta(x)$. If $u \in P$, then

$$|u-y| \geqslant |y-x|-|x-x_0|-|x_0-u| \geqslant |y-x|-2r \geqslant |x-y|-|x-y|/2 = |x-y|/2.$$

We also have

$$|u-x| \geqslant \operatorname{dist}(x, \partial B(x_1, r)) = \delta(x).$$

From (4.7) it follows that

(4.14) $$\int_P G(u, y) P_r(x-x_1, u-x_1)\, du$$

$$\leqslant A_k c_\alpha^d \int_P \frac{\delta^{k\alpha/(2n)}(u)\,\delta^m(y)}{|u-y|^{d-\alpha+m+k\alpha/(2n)}} \frac{(r^2-|x-x_1|^2)^{\alpha/2}}{(|u-x_1|^2-r^2)^{\alpha/2}} \frac{1}{|u-x|^d}\, du$$

$$\leqslant A_k c_\alpha^d 2^d 2^{\alpha/2} \frac{\delta^{\alpha/2}(x)\,\delta^m(y)}{|x-y|^{d-\alpha+m+k\alpha/(2n)}} \frac{1}{\delta^{\alpha/2-(k+1)\alpha/(2n)}(x)}$$

$$\times \int_P \frac{\delta^{k\alpha/(2n)}(u)}{|u-x|^{d-\alpha/2+(k+1)\alpha/(2n)}} \frac{r^{\alpha/2}}{(|u-x_1|^2-r^2)^{\alpha/2}}\, du$$

$$= C_5 \frac{\delta^{(k+1)\alpha/(2n)}(x)\,\delta^m(y)}{|x-y|^{d-\alpha+m+k\alpha/(2n)}} \int_P \frac{\delta^{k\alpha/(2n)}(u)}{|u-x|^{d-\alpha/2+(k+1)\alpha/(2n)}} \frac{r^{\alpha/2}}{(|u-x_1|^2-r^2)^{\alpha/2}}\, du,$$

where $C_5 = A_k c_\alpha^d 2^{d+\alpha/2}$.

We will estimate the integral on the right-hand side of (4.14). To do this, introduce spherical coordinates $(\varrho, \varphi_1, \ldots, \varphi_{d-1})$ with origin x_0 and principal axis $x_0 x_1$.

Consider the triangle $u x_0 x_1$. We have

$$|u-x_1|^2 = |u-x_0|^2 + |x_0-x_1|^2 - 2|u-x_0|\,|x_0-x_1| \cos\varphi_1.$$

Since $|x_0-x_1| = r$ and $|u-x_0| = \varrho$, we get

(4.15) $$|u-x_1|^2 = \varrho^2 + r^2 - 2\varrho r \cos\varphi_1.$$

For $0 < \varrho < r$ let $\beta(\varrho)$ be the angle satisfying $0 \leqslant \beta(\varrho) \leqslant \pi/2$ and

(4.16) $$r^2 = \varrho^2 + r^2 - 2\varrho r \cos\beta(\varrho).$$

Let $u = (\varrho, \varphi_1, \ldots, \varphi_{d-1}) \in B(x_0, r)$. The angle $\beta(\varrho)$ has the following property. If $\pi \geqslant \varphi_1 \geqslant \beta(\varrho)$, then $u \in B(x_0, r)\backslash B(x_1, r)$, and if $\beta(\varrho) > \varphi_1 \geqslant 0$, then $u \in B(x_1, r)$. Indeed, if $\pi \geqslant \varphi_1 \geqslant \beta(\varrho)$, then $\cos\varphi_1 \leqslant \cos\beta(\varrho)$. From (4.15) and (4.16) it follows that

$$|u-x_1|^2 = \varrho^2 + r^2 - 2\varrho r \cos\varphi_1 \geqslant \varrho^2 + r^2 - 2\varrho r \cos\beta(\varrho) = r^2.$$

By similar arguments, if $\pi - \beta(\varrho) \geqslant \varphi_1 \geqslant 0$, then $u \in B(x_0, r)\backslash B(x_2, r)$, and if $\pi \geqslant \varphi_1 > \pi - \beta(\varrho)$, then $u \in B(x_2, r)$. Hence $u = (\varrho, \varphi_1, \ldots, \varphi_{d-1}) \in P$ if and only if

(4.17) $$0 \leqslant \varrho < r \quad \text{and} \quad \beta(\varrho) \leqslant \varphi_1 \leqslant \pi - \beta(\varrho).$$

From (4.16) it is immediate that

(4.18) $$\cos\beta(\varrho) = \varrho/(2r).$$

Thus, if $u \in P$, we have $\cos\beta(\varrho) < 1/2$. Hence

$$\pi/2 \geqslant \beta(\varrho) \geqslant \pi/3 \quad \text{and} \quad \sin\beta(\varrho) \geqslant \sqrt{3}/2. \tag{4.19}$$

Now, we need the following easy fact:

$$\pi\sin\gamma \geqslant 2\gamma \quad \text{for } \gamma \in [0, \pi/2]. \tag{4.20}$$

Using this and (4.18) we obtain

$$\pi\varrho/(2r) = \pi\sin(\pi/2-\beta(\varrho)) \geqslant \pi - 2\beta(\varrho). \tag{4.21}$$

Now, we are going to estimate terms in the integrand on the right-hand side of (4.14). Let $u \in P$. We can simply replace $\delta(u)$ by ϱ. Indeed,

$$\delta(u) \leqslant \operatorname{dist}(u, B(x_2, r)) = |u-x_2|-r \leqslant |u-x_0|+|x_0-x_2|-r = \varrho. \tag{4.22}$$

By (4.15), (4.16) and (4.19) we obtain

$$\begin{aligned} |u-x_1|^2-r^2 &= \varrho^2-2\varrho r\cos\big((\varphi_1-\beta(\varrho))+\beta(\varrho)\big) \\ &= \varrho^2-2\varrho r\cos\beta(\varrho)\cos(\varphi_1-\beta(\varrho))+2\varrho r\sin\beta(\varrho)\sin(\varphi_1-\beta(\varrho)) \\ &\geqslant 2\varrho r\sin\beta(\varrho)\sin(\varphi_1-\beta(\varrho)) \geqslant \varrho r\sin(\varphi_1-\beta(\varrho)). \end{aligned} \tag{4.23}$$

Let us put $t = |x-x_0|$. We have

$$\begin{aligned} |u-x|^2 &= |u-x_0|^2+|x-x_0|^2-2|u-x_0|\,|x-x_0|\cos\varphi_1 \\ &= \varrho^2+t^2-2\varrho t\cos\varphi_1 \geqslant \varrho^2+t^2-2\varrho t\cos\beta(\varrho) \\ &= \varrho^2+t^2-\varrho^2 t/r \geqslant \varrho^2+t^2-\varrho t \geqslant 3\varrho^2/4. \end{aligned}$$

Thus

$$|u-x| \geqslant \varrho/2. \tag{4.24}$$

Now we estimate the integral on the right-hand side of (4.14). From (4.17) and (4.22)–(4.24) it follows that

$$\begin{aligned} &\int_P \frac{\delta^{k\alpha/(2n)}(u)}{|u-x|^{d-\alpha/2+(k+1)\alpha/(2n)}} \frac{r^{\alpha/2}}{(|u-x_1|^2-r^2)^{\alpha/2}}\,du \\ &\leqslant \int_0^r \int_{\beta(\varrho)}^{\pi-\beta(\varrho)} \int_0^{\pi} \ldots \int_0^{\pi} \int_0^{2\pi} \varrho^{k\alpha/(2n)} \frac{2^{d-\alpha/2+(k+1)\alpha/(2n)}}{\varrho^{d-\alpha/2+(k+1)\alpha/(2n)}} \\ &\quad \times \frac{r^{\alpha/2}}{\varrho^{\alpha/2} r^{\alpha/2} \sin^{\alpha/2}(\varphi_1-\beta(\varrho))} \varrho^{d-1} \sin^{d-2}\varphi_1 \ldots \sin\varphi_{d-2}\,d\varphi_{d-1} \ldots d\varphi_1\,d\varrho \\ &\leqslant C_6 \int_0^r \int_{\beta(\varrho)}^{\pi-\beta(\varrho)} \frac{1}{\varrho^{1+\alpha/(2n)} \sin^{\alpha/2}(\varphi_1-\beta(\varrho))}\,d\varphi_1\,d\varrho, \end{aligned} \tag{4.25}$$

with a constant $C_6 = C_6(d)$. Substitute $\varphi = \varphi_1-\beta(\varrho)$, and then use (4.20) for $\varphi \in [0, \pi-2\beta(\varrho)] \subset [0, \pi/2]$ (see (4.19)). The right-hand side of (4.25) is equal to

$$C_6 \int_0^r \int_0^{\pi-2\beta(\varrho)} \frac{1}{\varrho^{1+\alpha/(2n)} \sin^{\alpha/2}\varphi} d\varphi\, d\varrho \leqslant \frac{\pi^{\alpha/2} C_6}{2^{\alpha/2}} \int_0^r \frac{1}{\varrho^{1+\alpha/(2n)}} \int_0^{\pi-2\beta(\varrho)} \frac{1}{\varphi^{\alpha/2}} d\varphi\, d\varrho$$
$$= \frac{\pi^{\alpha/2} C_6}{2^{\alpha/2}(1-\alpha/2)} \int_0^r \frac{(\pi-2\beta(\varrho))^{1-\alpha/2}}{\varrho^{1+\alpha/(2n)}} d\varrho.$$

By (4.21), this is less than or equal to

$$\frac{\pi C_6}{2-\alpha} \int_0^r \frac{1}{\varrho^{1+\alpha/(2n)}} \frac{\varrho^{1-\alpha/2}}{r^{1-\alpha/2}} d\varrho = \frac{\pi C_6}{(2-\alpha)} \frac{1}{r^{1-\alpha/2}} \int_0^r \frac{1}{\varrho^{(n+1)\alpha/(2n)}} d\varrho.$$

Using our first assumption in Lemma 4.4 that $(n+1)\alpha/(2n) < 1$, we see that the expression above is equal to

$$\frac{\pi C_6}{(2-\alpha)(1-[(n+1)\alpha/(2n)])} \frac{1}{r^{1-\alpha/2}} r^{1-[(n+1)\alpha/(2n)]} = \frac{C_7}{r^{\alpha/(2n)}},$$

with a constant $C_7 = C_7(d, \alpha, n)$. Hence by (4.14) we have

$$\int_P G(u, y) P_r(x-x_1, u-x_1)\, du \leqslant C_5 C_7 C_0^{\alpha/(2n)} \frac{\delta^{(k+1)\alpha/(2n)}(x)\, \delta^m(y)}{|x-y|^{d-\alpha+m+(k+1)\alpha/(2n)}}.$$

This with (4.8) and (4.13) proves Lemma 4.4.

The following corollary is an easy extension of our main results:

COROLLARY 4.7. *Let a function F be defined by the formula*

$$F(x, y) = \min\left(\frac{1}{|x-y|^{d-\alpha}}, \frac{\delta^{\alpha/2}(x)}{|x-y|^{d-\alpha/2}}, \frac{\delta^{\alpha/2}(y)}{|x-y|^{d-\alpha/2}}, \frac{\delta^{\alpha/2}(x)}{\delta^{\alpha/2}(y)|x-y|^{d-\alpha}},\right.$$
$$\left.\frac{\delta^{\alpha/2}(y)}{\delta^{\alpha/2}(x)|x-y|^{d-\alpha}}, \frac{\delta^{\alpha/2}(x)\,\delta^{\alpha/2}(y)}{|x-y|^{d}}\right).$$

Then there exist constants $C_1 = C_1(d, \alpha, D)$ and $C_2 = C_2(d, \alpha, D)$ such that for all $x, y \in D$ we have

$$C_1 F(x, y) \leqslant G(x, y) \leqslant C_2 F(x, y).$$

The next theorem is known as "3G Theorem". It is an easy consequence of Theorems 4.3 and 4.5 and Corollary 4.7. The proof of this theorem is the same as in the classical case (see [5]).

THEOREM 4.8. *There exists a constant $C = C(d, \alpha, D)$ such that*

$$\frac{G(x, y)\, G(y, z)}{G(x, z)} \leqslant C \frac{u(x, y)\, u(y, z)}{u(x, z)} \quad \textit{for all } x, y, z \in D.$$

As a simple corollary to Theorem 4.3 and Corollary 4.7 we can obtain some estimates of $E^x(\tau_D)$. We will use the following formula (cf. [6]):

$$E^x(\tau_D) = \int G(x, y)\, dy. \tag{4.26}$$

PROPOSITION 4.9. *There exist constants* $c_1 = c_1(d, \alpha, D)$ *and* $c_2 = c_2(d, \alpha, D)$ *such that*

$$c_1\, \delta^{\alpha/2}(x) \leqslant E^x(\tau_D) \leqslant c_2\, \delta^{\alpha/2}(x), \quad x \in D.$$

Proof. By (4.26) and Corollary 4.7 we have

$$E^x(\tau_D) = \int G(x, y)\, dy \leqslant C_2\, \delta^{\alpha/2}(x) \int_D \frac{1}{|x-y|^{d-\alpha/2}}\, dy$$

$$\leqslant C_2\, \delta^{\alpha/2}(x) \int_{B(x,\operatorname{diam}(D))} \frac{1}{|x-y|^{d-\alpha/2}}\, dy = \frac{2C_2 (\operatorname{diam}(D))^{\alpha/2}}{\alpha}\, \delta^{\alpha/2}(x).$$

Now we prove the left-hand inequality. Set $D_a = \{y \in D\colon \delta(y) > a\}$. Choose a constant $a = a(D)$ such that D_{2a} has a positive Lebesgue measure $m(D_{2a})$. We will consider two cases: $\delta(x) < a$ and $\delta(x) \geqslant a$.

Case 1. $\delta(x) < a$.

In this case $B(x, \delta(x)/2)$ and D_{2a} are disjoint. If $y \in D \setminus B(x, \delta(x)/2)$, we have

$$2|x-y| \geqslant \delta(x) \quad \text{and} \quad 3|x-y| \geqslant \delta(x) + |x-y| \geqslant \delta(y).$$

Thus, by Theorem 4.3, there exists a constant $c_3 = c_3(d, \alpha, D)$ such that

$$G(x, y) \geqslant c_3\, \frac{\delta^{\alpha/2}(x)\, \delta^{\alpha/2}(y)}{|x-y|^d}$$

for $y \in D \setminus B(x, \delta(x)/2)$. Hence

$$E^x(\tau_D) = \int G(x, y)\, dy \geqslant c_3 \int_{D_{2a}} \frac{\delta^{\alpha/2}(x)\, \delta^{\alpha/2}(y)}{|x-y|^d}\, dy \geqslant \frac{c_3 (2a)^{\alpha/2}\, m(D_{2a})}{(\operatorname{diam}(D))^d}\, \delta^{\alpha/2}(x).$$

Case 2. $\delta(x) \geqslant a$.

By Theorem 4.3 we have $G(x, y) \geqslant C|x-y|^{\alpha-d}$ for $y \in B(x, \delta(x)/2)$. Hence

$$E^x(\tau_D) = \int G(x, y)\, dy \geqslant \int_{B(x,\delta(x)/2)} \frac{C}{|x-y|^{d-\alpha}}\, dy = \frac{C}{\alpha 2^\alpha}\, \delta^\alpha(x) \geqslant \frac{C a^{\alpha/2}}{\alpha 2^\alpha}\, \delta^{\alpha/2}(x).$$

It is natural to ask whether the estimates of the Green function obtained in Theorems 4.3 and 4.5 hold for more general sets than bounded open sets with a $C^{1,1}$ boundary. We are not going to give necessary and sufficient conditions under which these inequalities hold. However, we point out some counterexamples. If we take a ball without its center (say $D = B(0, 1) \setminus \{0\}$), the Green function for D equals the Green function for $B(0, 1)$ and the upper bound estimate does not hold near 0. So, in Theorem 4.5 we must assume some regularity conditions on the boundary of a set. What is more, if we take the difference of a ball and a cone (with a sufficiently narrow opening and with its vertex inside the ball), the upper bound estimate does not hold either. On the other hand, the lower bound estimate does not hold for a (bounded) sufficiently "narrow" cone. Therefore, Theorems 4.3 and 4.5 are not true if we replace bounded open sets with a $C^{1,1}$ boundary by Lipschitz domains.

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