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DILATION THEOREMS FOR POSITIVE OPERATOR-VALUED MEASURES*

BY

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Abstract. Let $Q(\Delta)$ be a positive operator-valued measure defined on a measurable space (X, Σ). This means that $Q(\Delta): L_1(M, \mathscr{M}, \mu) \to L_1(M, \mathscr{M}, \mu)$ with $Q(\Delta) f \geqslant 0$ for $f \geqslant 0$. Then $Q(\cdot)$ has a "dilation" of the form $\tilde{Q}(\Delta) = 4\mathbf{E}^{\mathscr{A}} \mathbf{1}_{e(\Delta)} \mathbf{E}^{\mathscr{B}} \mathbf{1}_{\Omega_0}$ in $(\Omega, \mathscr{F}, P)$. Namely, for some "identification" map $i: \Omega \to M$, the equality $(Q(\Delta) f) \circ i = \tilde{Q}(\Delta)(f \circ i)$ holds. The indicator operators $\mathbf{1}_{e(\Delta)}$ are taken for a set $e(\Delta)$ with some σ-lattice homomorphism $e: \Sigma \to \mathscr{F}$. Other dilation formulas of that type are collected.

1. INTRODUCTION

Let $(M, \mathscr{M}, \mu)$ be a probability space, and for $Z \in \mathscr{M}$ let $\mathbf{1}_Z$ denote the indicator operator (of multiplication by an indicator function) of the form

$$(\mathbf{1}_Z f)(m) = \mathbf{1}_Z(m) f(m).$$

A map $Z \to \mathbf{1}_Z$, $Z \in \mathscr{M}$, is the simplest example of a *positive operator-valued measure* in $L_p(M, \mathscr{M}, \mu)$. A less trivial example is given by the formula

$$Q(Z) = B_1 \ldots B_m \mathbf{1}_{e(Z)} A_1 \ldots A_m, \quad Z \in \mathscr{M}, \tag{1}$$

where $A_1, \ldots, A_m, B_1, \ldots, B_m$ are arbitrary positive operators in L_p and e is a σ-lattice endomorphism of $\mathscr{M}$. For example, A_i, B_i may be indicator operators, conditional expectations or operators $\tilde{T}$ generated by a measurable transformation $T: M \to M$. $Q(\cdot)$ may be quite complicated, especially for large m. It is worth noting that nonnegative operator-valued measures (semispectral measures) play an important role in the noncommutative statistics (see, for example, [2]). That is why it seems to be interesting that pretty simple formulas for any positive operator measure can be obtained by using the following "dilation" idea. Namely, Q can be represented as follows:

$$(Q(Z) f) \circ i = \tilde{Q}(Z)(f \circ i),$$

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where i is an "identification map," $i\colon \Omega \to M$ is defined on a larger probability space $(\Omega, \mathscr{F}, P)$ with $\Omega \supset M$, and $\tilde{Q}(\cdot)$ is of the form (1) with m not greater than 2 and with A_j, B_j of the following forms:

(i) the indicator operator $\mathbf{1}_{\Omega_0}$, $\Omega_0 \in \mathscr{F}$;

(ii) the conditional expectation $\mathbf{E}^{\mathscr{A}}$ with respect to a σ-field $\mathscr{A} \subset \mathscr{F}$;

(iii) the operator $\tilde{T}$ generated by a measurable transformation $T\colon \Omega \to \Omega$, i.e. $\tilde{T}$ is given by the formula

$$(\tilde{T}f)(\omega) = f(T\omega), \qquad \omega \in \Omega.$$

For example, we have the formulas like

$$\tilde{Q}(\cdot) = 4\mathbf{E}^{\mathscr{A}}\mathbf{1}_{e(\cdot)}\mathbf{E}^{\mathscr{B}}\mathbf{1}_{\Omega_0} \tag{2}$$

or

$$\tilde{Q}(\cdot) = 2\mathbf{E}^{\mathscr{A}}\mathbf{1}_{e(\cdot)}\tilde{T}. \tag{3}$$

Let us mention that constructing a dilation of a positive operator measure via a larger probability space we follow a general idea of Rota [3] (see also [5]).

In the whole paper, a map $e(\cdot)$ (which appeared in (2) and (3)) will always transform σ-field $\mathscr{M}$ into $\mathscr{F}$ as a σ-homomorphism of lattices.

Our main goal is to prove several results in the spirit of (2) or (3). To formulate them precisely, let us begin with some notation and definitions.

Let $(M, \mathscr{M}, \mu)$ be a finite regular measure space, and let X be a topological space with a σ-field Σ of its subsets containing all Borel sets.

Denote by W the space of all bounded linear operators acting in $L_1(M, \mathscr{M}, \mu)$.

Definition. A map $Q\colon \Sigma \to W$ is said to be a *regular positive operator measure* (shortly, *PO-measure*) if the following conditions are satisfied:

1° $Q(\Delta)f \geqslant 0$ for $0 \leqslant f \in L_1$;

2° $Q\left(\bigcup_{s=1}^{\infty} \Delta_s\right) f = \sum_{s=1}^{\infty} Q(\Delta_s) f$ for $f \in L_1$ and pairwise disjoint Δ_i's, the series being convergent in $L_1(M, \mathscr{M}, \mu)$;

3° Q is regular in the sense that for each $\varepsilon > 0$ and each $\Delta \in \Sigma$ there exist a compact set Z and an open set $V \subset X$ such that

$$\int_M Q(V-Z)\mathbf{1}_M \, d\mu < \varepsilon, \qquad Z \subset \Delta \subset V;$$

4° $Q(X)\mathbf{1}_M \leqslant \mathbf{1}_M$.

We say that a PO-measure *satisfies* (*) when

$$\int_M Q(X) f \, d\mu \leqslant \int_M f \, d\mu, \qquad 0 \leqslant f \in L_1. \tag{*}$$

In the whole paper we keep the notation

$$\mathscr{M} \times \Sigma = \{A \times B;\ A \in \mathscr{M},\ B \in \Sigma\}, \qquad \mathscr{M} \times Z = \{A \times Z;\ A \in \mathscr{M}\},$$

$$\mathscr{M}_1 \cup \mathscr{M}_2 = \{A \cup B;\ A \in \mathscr{M}_1,\ B \in \mathscr{M}_2\}$$

for any σ-fields $\mathscr{M}$, $\mathscr{M}_1$, $\mathscr{M}_2$, Σ, and $Z \subset \{1, 2, \ldots\}$.

2. DILATION THEOREMS FOR PO-MEASURES

In this section we prove several theorems keeping the notation adapted in the Introduction. In particular, the spaces $(M, \mathcal{M}, \mu)$ and (X, Σ) are fixed, and Q denotes a PO-measure $Q\colon \Sigma \to W$. We start with the following result:

THEOREM 1. *There exist a measurable space $(\Omega, \mathcal{F})$, two measurable maps i, j from Ω onto M, a σ-field $\mathcal{A} \subset \mathcal{F}$, and a σ-lattice homomorphism $e\colon \Sigma \to \mathcal{F}$ such that for every PO-measure $Q\colon \Sigma \to W$ there exists a probability measure P on $(\Omega, \mathcal{F})$ for which the following formula holds*:

$$(Q(\Delta)f)\circ j = \mathbf{E}_P^{\mathcal{A}} \mathbf{1}_{e(\Delta)}(f\circ i), \quad \Delta \in \Sigma. \tag{4}$$

Proof. We put

$$\Omega = (M\times M\times X\times\{1\})\cup(M\times\{2\}),$$
$$\mathcal{F} = \sigma\big((\mathcal{M}\times\mathcal{M}\times\Sigma\times\{1\})\cup(\mathcal{M}\times\{2\})\big),$$
$$i(m, n, x, 1) = m, \quad i(m, 2) = m, \quad j(m, n, x, 1) = n, \quad j(m, 2) = m,$$
$$\mathcal{A} = \{(M\times A\times X\times\{1\})\cup(A\times\{2\});\ A\in\mathcal{M}\},$$
$$e(\Delta) = M\times M\times\Delta\times\{1\}, \quad \Delta\in\Sigma,$$
$$P(A\times B\times\Delta\times\{1\}) = \int_M \big(Q(\Delta)1_A\big)1_B\,d\mu,$$
$$P(A\times\{2\}) = \int_A \big(1_M - Q(X)1_M\big)\,d\mu.$$

P defined as above can be extended in a unique way to a probability measure P on $\mathcal{F}$ (see the Appendix).

Now, let us observe that, for $g\in L_1(M, \mathcal{M}, \mu)$, the formula

$$\int_{(M\times B\times X\times\{1\})\cup(B\times\{2\})} (g\circ j)\,dP = \int_B g\,d\mu \tag{5}$$

holds.

Indeed, for $g = 1_C$, we have

$$\begin{aligned}\int_{(M\times B\times X\times\{1\})\cup(B\times\{2\})} (1_C\circ j)\,dP &= \int_{(M\times B\times X\times\{1\})\cup(B\times\{2\})} 1_{(M\times C\times X\times\{1\})\cup(C\times\{2\})}\,dP\\ &= P\big((M\times(B\cap C)\times X\times\{1\})\cup((B\cap C)\times\{2\})\big)\\ &= \int_M \big(Q(X)1_M\big)1_{B\cap C}\,d\mu + \int_M 1_{B\cap C}\big(1_M - Q(X)1_M\big)\,d\mu\\ &= \int_M 1_{B\cap C}\,d\mu = \int_B 1_C\,d\mu,\end{aligned}$$

and (5) follows.

Since the $\mathcal{A}$-measurability of the left-hand side in (4) is obvious, to prove formula (4) it is enough to show the equality of integrals

$$\int_S \big(Q(\Delta)1_A\big)\circ j\,dP = \int_S 1_{e(\Delta)}(1_A\circ i)\,dP$$

for

$$S = (M \times B \times X \times \{1\}) \cup (B \times \{2\}) \in \mathscr{A}.$$

By (5), we have

$$\int\limits_{(M \times B \times X \times \{1\}) \cup (B \times \{2\})} Q(\Delta) 1_A \circ j\, dP = \int\limits_B Q(\Delta) 1_A\, d\mu.$$

On the other hand,

$$\int\limits_{(M \times B \times X \times \{1\}) \cup (B \times \{2\})} 1_{e(\Delta)} (1_A \circ i)\, dP = \int\limits_{M \times B \times \Delta \times \{1\}} 1_{(A \times M \times X \times \{1\}) \cup (A \times \{2\})}\, dP$$

$$= P(A \times B \times \Delta \times \{1\}) = \int\limits_M \big(Q(\Delta) 1_A\big) 1_B\, d\mu,$$

so we get formula (4). ∎

THEOREM 2. *There exist a measurable space $(\Omega, \mathscr{F})$, a one-to-one measurable map $T\colon \Omega \to \Omega$, a measurable map $i\colon \Omega \to M$ (onto), a σ-field $\mathscr{A} \subset \mathscr{F}$, a σ-lattice homomorphism $e\colon \Sigma \to \mathscr{F}$ such that for every PO-measure Q satisfying condition* $(*)$ (see the Definition), *there exists a probability measure P on $(\Omega, \mathscr{F})$ preserved by T, for which the following formula holds*:

$$(6) \qquad \big(Q(\Delta) f\big) \circ i = 2\mathbf{E}_P^{\mathscr{A}}\, 1_{e(\Delta)}\, \tilde{T}(f \circ i), \qquad \Delta \in \Sigma.$$

Proof. Let us put

$$\Omega = (M \times M \times X \times \{1\}) \cup (M \times M \times X \times \{2\}) \cup (M \times \{3\}),$$

$$\mathscr{F} = \sigma\big((\mathscr{M} \times \mathscr{M} \times \Sigma \times \{1\}) \cup (\mathscr{M} \times \mathscr{M} \times \Sigma \times \{2\}) \cup (\mathscr{M} \times \{3\})\big),$$

$$T(m, n, x, 1) = (n, m, x, 2), \quad T(m, n, x, 2) = (n, m, x, 1), \quad T(m, 3) = (m, 3),$$

$$e(\Delta) = M \times M \times \Delta \times \{2\},$$

$i(\omega) = m$ for all $\omega = (m, \ldots) \in \Omega$, i.e. $i(\omega)$ is the first coordinate of ω,

$$\mathscr{A} = \{(A \times M \times X \times \{1\}) \cup (A \times M \times X \times \{2\}) \cup (A \times \{3\});\ A \in \mathscr{M}\},$$

$$P(A \times B \times \Delta \times \{1\}) = \int\limits_B \big(Q(\Delta) 1_A\big) d\mu, \quad P(A \times B \times \Delta \times \{2\}) = \int\limits_A \big(Q(\Delta) 1_B\big) d\mu,$$

$$P(A \times \{3\}) = \int\limits_A \left(2 - Q(X) 1_M - \frac{d\varrho}{d\mu}\right) d\mu \quad \text{for } \varrho(A) = \int\limits_M Q(X) 1_A\, d\mu$$

(observe here that the Radon–Nikodym derivative $d\varrho/d\mu \leqslant 1$ μ-a.e. since $\varrho(A) \leqslant \mu(A)$ by $(*)$).

P defined as above can be extended in a unique way to a probability measure P on $(\Omega, \mathscr{F})$. Since the $\mathscr{A}$-measurability of the left-hand side in (6) is obvious, it is enough to show the following equality:

$$(7) \qquad \int\limits_E \big(Q(\Delta) f\big) \circ i\, dP = 2 \int\limits_E 1_{e(\Delta)}\, \tilde{T}(f \circ i)\, dP \quad \text{for } E \in \mathscr{A}.$$

To this end, let us observe that, for $f \in L_1(M, \mathscr{M}, \mu)$ and $E = (B\times M\times X\times\{1\})\cup(B\times M\times X\times\{2\})\cup(B\times\{3\})$, we have

$$\int_E (f\circ i)\,dP = 2\int_B f\,d\mu. \tag{8}$$

Indeed, for $f = 1_C$, $C\in\mathscr{M}$, we have

$$\begin{aligned}\int_E (1_C\circ i)\,dP &= \int_{B\times M\times X\times\{1\}} 1_{C\times M\times M\times\{1\}}\,dP \\ &\quad + \int_{B\times M\times X\times\{2\}} 1_{C\times M\times M\times\{2\}}\,dP + \int_{B\times\{3\}} 1_{C\times\{3\}}\,dP \\ &= P(B\cap C\times M\times X\times\{1\}) + P(B\cap C\times M\times X\times\{2\}) + P(B\cap C\times\{3\}) \\ &= \int_M Q(X)\,1_{B\cap C}\,d\mu + \int_M (Q(X)\,1_M)\,1_{B\cap C}\,d\mu + \int_{B\cap C}\left(2 - Q(X)\,1_M - \frac{d\varrho}{d\mu}\right)d\mu \\ &= 2\int_M 1_{B\cap C}\,d\mu = 2\int_B 1_C\,d\mu,\end{aligned}$$

and (8) follows.

Now it is easy to prove (7). Indeed, for $A\in\mathscr{M}$ we have

$$\begin{aligned}\int_E \mathbf{1}_{e(\varDelta)}\,\tilde{T}(1_A\circ i)\,dP &= \int_{B\times M\times\varDelta\times\{2\}} 1_{(M\times A\times X\times\{1\})\cup(M\times A\times X\times\{2\})\cup(A\times\{3\})}\,dP \\ &= \int_{B\times M\times\varDelta\times\{2\}} 1_{(M\times A\times X\times\{2\})}\,dP = P(B\times A\times\varDelta\times\{2\}) = \int_B (Q(\varDelta)\,1_A)\,d\mu,\end{aligned}$$

and, by (8),

$$\int_E (Q(\varDelta)\,1_A)\circ i\,dP = 2\int_B Q(\varDelta)\,1_A\,d\mu,$$

which completes the proof. ∎

Remark. Theorem 2 may be rewritten by changing (6) into

$$\big(Q(\varDelta)f\big)\circ i = 2\mathbf{E}_P^{\mathscr{A}}\,\tilde{T}\mathbf{1}_{e(\varDelta)}(f\circ i). \tag{6'}$$

Indeed, in this case, in the proof it is enough to put $e(\varDelta) = M\times M\times\varDelta\times\{1\}$ (instead of $M\times M\times\varDelta\times\{2\}$).

THEOREM 3. *There exist a measurable space $(\Omega, \mathscr{F})$, a measurable map $i\colon \Omega\to M$ (onto), σ-fields $\mathscr{A}, \mathscr{B}\subset\mathscr{F}$, a σ-lattice homomorphism $e\colon \Sigma\to\mathscr{F}$, a set $\Omega_0\in\mathscr{F}$ such that, for every PO-measure $Q\colon\Sigma\to W$ satisfying $(*)$,*

(α) *there exists a probability measure P on $(\Omega, \mathscr{F})$ for which the following formula holds:*

$$\big(Q(\varDelta)\,f\big)\circ i = 4\mathbf{E}_P^{\mathscr{A}}\,\mathbf{1}_{e(\varDelta)}\,\mathbf{E}_P^{\mathscr{B}}\,\mathbf{1}_{\Omega_0}(f\circ i),\qquad \varDelta\in\Sigma; \tag{9}$$

(β) *there exists a probability measure $\bar{P}$ on $(\Omega, \mathscr{F})$ for which*

$$\big(Q(\varDelta)\,f\big)\circ i = 4\mathbf{E}_P^{\mathscr{A}}\,\mathbf{1}_{\Omega_0}\,\mathbf{E}_P^{\mathscr{B}}\,\mathbf{1}_{e(\varDelta)}(f\circ i),\qquad \varDelta\in\Sigma. \tag{9'}$$

Proof. We put

$$\Omega = (M\times M\times X\times\{1\})\cup(M\times M\times X\times\{2\})\cup(M\times\{3\}),$$

$$\mathcal{F} = \sigma\big((\mathcal{M}\times\mathcal{M}\times\Sigma\times\{1\})\cup(\mathcal{M}\times\mathcal{M}\times\Sigma\times\{2\})\cup(\mathcal{M}\times\{3\})\big),$$

$$i(\omega) = m \quad \text{for all } \omega = (m, \ldots)\in\Omega,$$

$$\mathcal{A} = \{(A\times M\times X\times\{1\})\cup(A\times M\times X\times\{2\})\cup(A\times\{3\});\ A\in\mathcal{M}\},$$

$$\mathcal{B} = \sigma\big[\{(M\times B\times\Delta\times\{1\})\cup(M\times B\times\Delta\times\{2\});\ B\in\mathcal{M},\ \Delta\in\Sigma\}\cup\{(M\times\{3\})\}\big],$$

$$e(\Delta) = M\times M\times\Delta\times\{2\}, \qquad \Omega_0 = M\times M\times X\times\{1\}.$$

To prove (α), let us define

$$\varrho(A) = \int_M Q(X)\,1_A\,d\mu.$$

We define a probability measure P on $(\Omega, \mathcal{F})$ by putting

$$\begin{aligned} P(A\times B\times\Delta\times\{1\}) &= \int_M \big(Q(\Delta)\,1_A\big)\,1_B\,d\mu, \\ P(A\times B\times\Delta\times\{2\}) &= \int_M 1_{A\cap B}\big(Q(\Delta)\,1_M\big)\,1_B\,d\mu, \qquad (10)\\ P(A\times\{3\}) &= \int_A \left(2-Q(X)\,1_M-\frac{d\varrho}{d\mu}\right)d\mu, \end{aligned}$$

and then extending it to a σ-additive measure on $\mathcal{F}$ (see the Appendix).

Now, let us remark that, for

$$E = (B\times M\times X\times\{1\})\cup(B\times M\times X\times\{2\})\cup(B\times\{3\}) \quad \text{with } B\in\mathcal{M},$$

the equality

$$\int_E (f\circ i)\,dP = 2\int_B f\,d\mu \tag{11}$$

holds for $f\in L_1(M, \mathcal{M}, \mu)$.

Indeed, for $f = 1_C$, $C\in\mathcal{M}$, we have

$$\begin{aligned} \int_E (1_C\circ i)\,dP &= \int_{(B\times M\times X\times\{1\})\cup(B\times M\times X\times\{2\})\cup(B\times\{3\})} 1_{(C\times M\times X\times\{1\})\cup(C\times M\times X\times\{2\})\cup(C\times\{3\})}\,dP \\ &= P\big((B\cap C)\times M\times X\times\{1\}\big) \\ &\quad + P\big((B\cap C)\times M\times X\times\{2\}\big) + P\big((B\cap C)\times\{3\}\big) \\ &= \int_M Q(X)\,1_{B\cap C}\,d\mu + \int_M 1_{B\cap C}\,Q(X)\,1_M\,d\mu + \int_{B\cap C}\left(2-Q(X)\,1_M-\frac{d\varrho}{d\mu}\right)d\mu \\ &= 2\int_B 1_C\,d\mu, \end{aligned}$$

so we get (11). Clearly, (11) implies

$$\int_E (Q(\Delta)f)\circ i\,dP = 2\int_B (Q(\Delta)f)\,d\mu, \qquad f\in L_1(M, \mathscr{M}, \mu).$$

Since the $\mathscr{A}$-measurability of the left-hand side in (9) is obvious, to get formula (9) it is enough to show that for $A\in\mathscr{M}$ and E as above we have

$$\int_E \mathbf{1}_{e(\Delta)}\mathbf{E}_P^{\mathscr{B}}\mathbf{1}_{\Omega_0}(1_A\circ i)\,dP = \tfrac{1}{2}\int_B Q(\Delta)\,1_A\,d\mu \tag{12}$$

(then, by a standard argument we get easily formula (12) for $f\in L_1(M, \mathscr{M}, \mu)$ instead of 1_A). The left-hand side of (12) can be written in the form

$$\int_{M\times B\times\Delta\times\{2\}} (\mathbf{E}_P^{\mathscr{B}} h)(\omega)\,dP(\omega),$$

where

$$\begin{aligned} h(m, \ldots) &= \mathbf{1}_{\Omega_0}(1_A\circ i)\,(m, \ldots) \\ &= \begin{cases} 1_A(m) & \text{for } (m, \ldots)\in M\times M\times X\times\{1\}, \\ 0 & \text{elsewhere,} \end{cases} \end{aligned} \tag{13}$$

where $\omega = (m, \ldots)$ is an arbitrary point in Ω.

Let us remark that if $g\in L_1\big((\Omega, P)\big)$ is $\mathscr{B}$-measurable, then

$$\int_{M\times B\times\Delta\times\{1\}} g\,dP = \int_{M\times B\times\Delta\times\{2\}} g\,dP. \tag{14}$$

Indeed, for $g = 1_{C\times\Delta'}$, we have

$$\begin{aligned} \int_{M\times B\times\Delta\times\{1\}} g\,dP &= \int_{M\times B\times\Delta\times\{1\}} 1_{M\times C\times\Delta'\times\{1\}}\,dP \\ &= P\big(M\times(B\cap C)\times(\Delta\times\Delta')\times\{1\}\big) = \int_M \big(Q(\Delta\times\Delta')\,1_M\big)\,1_{B\cap C}\,d\mu \end{aligned}$$

and

$$\begin{aligned} \int_{M\times B\times\Delta\times\{2\}} g\,dP &= \int_{M\times B\times\Delta\times\{2\}} 1_{M\times C\times\Delta'\times\{2\}}\,dP \\ &= P\big(M\times(B\cap C)\times(\Delta\times\Delta')\times\{2\}\big) = \int_M \big(Q(\Delta\times\Delta')\,1_M\big)\,1_{B\cap C}\,d\mu. \end{aligned}$$

Putting $g = \mathbf{E}_P^{\mathscr{B}} h$ in (14), we obtain

$$\int_{M\times B\times\Delta\times\{2\}} \mathbf{E}_P^{\mathscr{B}} h\,dP = \tfrac{1}{2}\Big[\int_{M\times B\times\Delta\times\{1\}} \mathbf{E}_P^{\mathscr{B}} h\,dP + \int_{M\times B\times\Delta\times\{2\}} \mathbf{E}_P^{\mathscr{B}} h\,dP\Big].$$

Setting

$$Z = (M\times B\times\Delta\times\{1\})\cup(M\times B\times\Delta\times\{2\})$$

we have

$$\int_{M\times B\times\Delta\times\{2\}} \mathbf{E}_P^{\mathscr{B}} h\,dP = \tfrac{1}{2}\int_Z \mathbf{E}_P^{\mathscr{B}} h\,dP = \tfrac{1}{2}\int_Z h\,dP$$

since $Z \in \mathscr{B}$. Evidently,

$$\int_Z h\,dP = \int_{M\times B\times\Delta\times\{1\}} h\,dP,$$

so it remains to show that

$$\int_{M\times B\times\Delta\times\{1\}} h\,dP = \int_B Q(\Delta)1_A\,d\mu$$

for h given by (13). This is easy to check because

$$\int_{M\times B\times\Delta\times\{1\}} h\,dP = \int_{M\times B\times\Delta\times\{1\}} 1_{A\times B\times X\times\{1\}}\,dP$$
$$= P(A\times B\times\Delta\times\{1\}) = \int_B Q(\Delta)1_A\,d\mu.$$

To prove (β) it is enough to define $\bar{P}$ by changing P into $\bar{P}$, $\{1\}$ into $\{2\}$, $\{2\}$ into $\{1\}$ in formulas (10). The rest of calculations is in fact a repetition of those for P. ∎

3. FINAL REMARKS

3.1. It is interesting to compare our construction of dilations via conditional expectation with the classical Naimark dilation theorem [1]. If $H = L_2(M, \mathscr{M}, \mu)$, then for any measure space $(N, \mathscr{N}, \nu)$, H can be treated as the subspace of $L_2(M\times N, \mathscr{M}\otimes\mathscr{N}, \mu\otimes\nu)$ corresponding to the orthogonal projection $\mathbb{E}^{\mathscr{A}}$ being conditional expectation with respect to $\mathscr{A} = \mathscr{M}\otimes(N, \emptyset)$. If $(N, \mathscr{N}, \nu)$ is rich enough, then for any semispectral measure $Q(\cdot)$ acting in H there exists a spectral measure $E(\cdot)$ acting in $\mathscr{H} = L_2(M\times N, \mathscr{M}\otimes\mathscr{N}, \mu\otimes\nu)$ such that $Q(\cdot) = \mathbb{E}^{\mathscr{A}}E(\cdot)|_H$. More precisely, $(Q(\cdot)f)\circ i = \mathbb{E}^{\mathscr{A}}E(f\circ i)$, $f\in L_2(M, \mathscr{M}, \mu)$, $i(m, n) = m$ for $m\in M$, $n\in N$.

There is a temptation to specify the spectral measure $E(\cdot)$ or to use another operator-valued measure which would be natural in the case of a function $\mathscr{H}$.

Our all efforts have been devoted to find an operator measure $E(\cdot)$ as close as possible to the indicator measure $\mathbb{1}_{e(\cdot)}$ with $e(\cdot)$ being a homomorphism of set lattices (cf. formulas (2) and (3)).

3.2. It is sometimes useful to give a theorem about a measure in terms of integrals. In particular, our representations of positive operator measure $Q(\cdot)$ correspond to some representation of the positive transformation

$$\varphi \to \tilde{Q}(\varphi) = \int_X \varphi(x)Q(dx).$$

More precisely, let (X, Σ, λ) and $(M, \mathscr{M}, \mu)$ be regular finite measure spaces. By a *positive transformation*

$$\tilde{Q}\colon L_\infty(X, \Sigma, \lambda) \to B(L_1(M, \mathscr{M}, \mu))$$

we mean a linear mapping satisfying

$$\tilde{Q}(\varphi)f \geqslant 0 \quad \text{for } \varphi \geqslant 0,\ f \geqslant 0, \quad \varphi \in L_\infty(X, \Sigma, \lambda),\ f \in L_1(M, \mathcal{M}, \mu),$$

$$\tilde{Q}(\varphi_n) \to \tilde{Q}(\varphi) \text{ strongly in } B\big(L_1(M, \mathcal{M}, \mu)\big) \quad \text{for } \varphi_n \nearrow \varphi.$$

To present the consequences of our previous results we give the following corollary to Theorem 3.

COROLLARY. *There exist* $(\Omega, \mathcal{F})$, *measurable maps* $i\colon \Omega \to M$, $j\colon \Omega \to X$, Ω_0, $\Omega_1 \in \mathcal{F}$, *σ-fields* $\mathcal{A} \subset \mathcal{F}$ *and* $\mathcal{B} \subset \mathcal{F}$ *such that for every* $\tilde{Q}$ *there exists a probability measure* P *on* $(\Omega, \mathcal{F})$ *such that*

$$\big(\tilde{Q}(\varphi)f\big) \circ i = 4\mathbf{E}_P^{\mathcal{A}}(\varphi \circ j)\mathbf{1}_{\Omega_1}\mathbf{E}_P^{\mathcal{B}}\mathbf{1}_{\Omega_0}(f \circ i).$$

APPENDIX

We shall keep the notation used in the previous sections. In particular, $(M, \mathcal{M}, \mu)$, (X, Σ), Q will denote the same objects as in the Definition. Proving the dilation theorem (in any version that has been formulated in the paper) we defined a set function P on some decomposable family of sets and then we got a probability measure on $\mathcal{F}$ by the unique extension of P. Our aim is to indicate the method of this extension. For the sake of simplicity we shall confine ourselves to the following case:

Let $\Omega = M \times X$, $\mathcal{F} = \mathcal{M} \otimes \Sigma = \sigma(\mathcal{G})$, *where* $\mathcal{G} = \{A \times \Delta\colon A \in \mathcal{M},\ \Delta \in \Sigma\}$. *For* $S = A \times \Delta \in \mathcal{G}$, *we set* $P(S) = \int_M Q(\Delta)1_A\,d\mu$. *Then* P *can be extended, in a unique way, to a measure on* $\mathcal{F}$.

To prove this, it is enough to show that P is additive and σ-subadditive on $\mathcal{G}$. We split the proof into several steps.

Step 1. *P is additive on $\mathcal{G}$.*

We need some notation. Let $S = A \times \Delta \in \mathcal{G}$. By a *partition* of S we mean a system of mutually disjoint rectangles $S_j = A_j \times \Delta_j$ $(j = 1, \ldots, k)$ such that $S = \bigcup_{j=1}^k S_j$. Let $A = A_1 \cup \ldots \cup A_a$ and $\Delta = \Delta_1 \cup \ldots \cup \Delta_d$, where A_α and Δ_δ are mutually disjoint. A family of rectangles $S_{\alpha,\delta} = A_\alpha \times \Delta_\delta$ $(\alpha = 1, \ldots, a;$ $\delta = 1, \ldots, d)$ is said to be a *simple partition* of S.

Now, let $\Pi = \{S_0, \ldots, S_k\}$ be an arbitrary system of rectangles

$$S_\varkappa = A_\varkappa \times \Delta_\varkappa \in \mathcal{G}, \quad \varkappa = 0, \ldots, k.$$

Let us put

$$\mathcal{A} = \{A_0^{\eta_0} \cap \ldots \cap A_k^{\eta_k}\colon \eta_\varkappa = \pm 1,\ \varkappa = 0, \ldots, k\},$$

$$\mathcal{D} = \{\Delta_0^{\eta_0} \cap \ldots \cap \Delta_k^{\eta_k}\colon \eta_\varkappa = \pm 1,\ \varkappa = 0, \ldots, k\},$$

where $B^\eta = B$ or B^c depending on $\eta = 1$ or -1.

Put $\Pi^* = \{A \times \Delta\colon A \in \mathscr{A}, \Delta \in \mathscr{D}\}$.

Following Sikorski [4], let us remark that

(1) $\Pi^*_\varkappa = \{S \in \Pi^*\colon S \subset S_\varkappa\}$ is a simple partition of $S_\varkappa$;

(2) $S_0 \subset S_1 \cup \ldots \cup S_k$ implies $\Pi^*_0 \subset \Pi^*_1 \cup \ldots \cup \Pi^*_k$;

(3) rectangles $S_\varkappa$ and $S_{\varkappa'}$ are disjoint if and only if the systems $\Pi^*_\varkappa$ and $\Pi^*_{\varkappa'}$ are disjoint;

(4) if $\Pi = \{S_0, \ldots, S_k\}$ is a partition of some rectangle S, then Π^* is a simple partition of S, and $\Pi^* = \Pi^*_0 \cup \ldots \cup \Pi^*_k$.

Now we are in a position to show that, for an arbitrary partition Π of a rectangle $S \in \mathscr{G}$, we have

$$P(S) = \sum_{S' \in \Pi} P(S'),$$

which means the additivity of P on $\mathscr{G}$. Indeed, let $S = A \times \Delta$, and assume that Π is a simple partition of S. Let $A = A_1 \cup \ldots \cup A_a$ and $\Delta = \Delta_1 \cup \ldots \cup \Delta_d$. Then

$$P(S) = \int_M Q(\Delta) 1_A \, d\mu = \sum_{\alpha=1}^{a} \sum_{\delta=1}^{d} \int_M Q(\Delta_\delta) 1_{A_\alpha} \, d\mu$$

$$= \sum_{\alpha=1}^{a} \sum_{\delta=1}^{d} P(A_\alpha \times \Delta_\delta) = \sum_{S' \in \Pi} P(S').$$

Now, let $\Pi = \{S_0, \ldots, S_k\}$ be an arbitrary partition of S. By (1)–(4) and the equality just proved, we have

$$P(S) = \sum_{S' \in \Pi^*} P(S') = \sum_{\varkappa=0}^{k} \sum_{S' \in \Pi^*_\varkappa} P(S') = \sum_{\varkappa=0}^{k} P(S_\varkappa) = \sum_{S' \in \Pi} P(S').$$

Step 2. *P is subadditive on $\mathscr{G}$.*

Indeed, let $S_0 \subset \bigcup_{\varkappa=1}^{k} S_\varkappa$, $S_\varkappa \in \mathscr{G}$, $\varkappa = 0, \ldots, k$. Put $\Pi = \{S_0, \ldots, S_k\}$. Then, by (1)–(4), we have

$$P(S_0) = \sum_{S' \in \Pi^*_0} P(S') \leqslant \sum_{\varkappa=1}^{k} \sum_{S' \in \Pi^*_\varkappa} P(S') = \sum_{\varkappa=1}^{k} P(S_\varkappa).$$

Step 3. For every rectangle $S = A \times \Delta \times \mathscr{G}$ and $\varepsilon > 0$, there exist compact sets $K \in \mathscr{M}$ and $Z \in \Sigma$ and open sets $U \in \mathscr{M}$, $V \in \Sigma$ such that

$$K \subset A \subset U, \quad Z \subset \Delta \subset V$$

and

$$P(U \times V) - \varepsilon < P(S) < P(K \times Z) + \varepsilon. \tag{15}$$

Indeed, by the regularity of μ and Q (assumption 3°), one can find compact sets $K \subset A$ and $Z \subset \Delta$ such that

$$\mu(A \backslash K) < \varepsilon/2 \quad \text{and} \quad \int_M Q(\Delta \backslash Z) 1 \, d\mu < \varepsilon/2.$$

Then we have

$$P(S) = \int_M Q(\Delta) 1_A \, d\mu = \int_M Q(Z) 1_K \, d\mu + \int_M Q(Z) 1_{A \setminus K} \, d\mu + \int_M Q(\Delta \setminus Z) 1_A \, d\mu$$
$$\leqslant P(K \times Z) + \int_M 1_{A \setminus K} \, d\mu + \int_M Q(\Delta \setminus Z) 1_M \, d\mu < P(K \times Z) + \varepsilon.$$

The proof of the left-hand side in (15) is similar.

Step 4. *P is σ-subadditive on $\mathscr{G}$.*

Indeed, let

$$S \subset \bigcup_{\varkappa=1}^{\infty} S_\varkappa, \quad S = A \times \Delta, \quad S_\varkappa = A_\varkappa \times \Delta_\varkappa.$$

For $\varepsilon > 0$, let K, Z be compact sets as in Step 3. There exist open sets $U_\varkappa \supset A_\varkappa$ and $V_\varkappa \supset \Delta_\varkappa$ such that

$$P(U_\varkappa \times V_\varkappa) < P(S_\varkappa) + \varepsilon/2^\varkappa.$$

Since $K \times Z \subset \bigcup_{\varkappa=1}^{\infty} (U_\varkappa \times V_\varkappa)$, we have also

$$K \times Z \subset \bigcup_{\varkappa=1}^{k} (U_\varkappa \times V_\varkappa) \quad \text{for some } k.$$

By Step 2, we have

$$P(K \times Z) \leqslant \sum_{\varkappa=1}^{k} P(U_\varkappa \times V_\varkappa) < \sum_{\varkappa=1}^{k} P(S_\varkappa) + \varepsilon.$$

Thus, $P(S) \leqslant \sum_{\varkappa=1}^{k} P(S_\varkappa) + 2\varepsilon$ and, consequently, $P(S) \leqslant \sum_{\varkappa=1}^{\infty} P(S_\varkappa)$. ∎

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