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ON ESTIMATION IN THE MULTIPLICATIVE INTENSITY MODEL VIA HISTOGRAM SIEVE

BY

ROMAN RÓŻAŃSKI* AND ADAM ZAGDAŃSKI* (WROCŁAW)

Abstract. In the paper we consider the problem of estimating stochastic intensity of a point process from multiplicative intensity model using the method of sieves of Grenander [6]. Basic properties of the histogram sieve estimator including consistency and asymptotic normality are proved. Our approach extends results obtained in Leśkow and Różański [13].

Key words and phrases: Point processes, multiplicative intensity model, sieve method of estimation, histogram maximum likelihood estimator.

1. Introduction. The paper develops and generalizes results obtained earlier by Leśkow and Różański [13] and Leśkow [10]. We deal here with the existence and asymptotic distribution problems for an estimator constructed by the method of sieves in the multiplicative intensity model.

Let us start with a short description of the model introduced by Aalen [1]. Let $(\Omega, \mathscr{F}, \mathscr{P})$ be a probability space on which a sequence of point processes $\{N_n(t), t \in [0, 1], n \in N\}$ is defined. We assume that the processes are adapted to their filtrations $\{\mathscr{F}_{n,t}, t \in [0, 1], n \in N\}$. We consider such models for which a stochastic intensity $\alpha_n(t)$ of the process $N_n(t)$, $t \in [0, 1]$, exists and can be defined in the following way:

$$\alpha_n(t) = \lim_{h \to 0} \left(h^{-1} \mathscr{P}\left(N_n(t+h) - N_n(t) \geqslant 1 \mid \mathscr{F}_{n,t-}\right)\right).$$

We assume that the point processes $N_n(t)$, $t \in [0, 1]$, belong to the class of multiplicative intensity models. Therefore, the intensity function $\alpha_n(t)$ has the form

$$\alpha_n(t) = Y_n(t)\, \alpha_0(t),$$

* Institute of Mathematics, Wrocław University of Technology.

where $Y_n(t)$ is a nonnegative process, left-continuous with right-hand limits (or, more generally, predictable), and observable, and α_0 is the unknown nonnegative function to be estimated.

In 1983, Ramlau-Hansen [16] presented the kernel estimator for α_0 by smoothing the Nelson–Aalen estimator. The consistency and asymptotic normality of the estimator has been proved. Later, Leśkow and Różański [13] applied Grenander's idea of sieve estimation constructing histogram maximum likelihood estimator via histogram sieve. The consistency and asymptotic normality of the estimator has also been proved. However, the assumptions used were hard to prove, especially the mixing condition and separation from zero for the expectation of Y_n. Similar conditions are assumed in later papers of Leśkow [11], [12]. The estimator constructed in Leśkow and Różański [13] is a type of aggregating data procedure which may be appropriate in some applications.

In the paper, the consistency and asymptotic normality of the maximum likelihood estimator based on histogram sieve has been proved under rather mild conditions similar to those in Ramlau-Hansen [16]. It is worth noting that the conditions contain the cases and assumptions considered by Leśkow and Różański [13] and Leśkow [10].

In Section 2 the sieve method is described and the maximum likelihood estimator is derived. Section 3 contains theorems on consistency and asymptotic normality of the estimator. In Section 4, proofs of the results presented in Section 3 are given. Section 5 is devoted to simulation results.

2. Sieve maximum likelihood estimation. Let $\mathscr{P}^n = \{\mathscr{P}^n_\alpha, \alpha \in A\}$ be the set of probability measures corresponding to the point processes $\{N_n(t), t \in [0, 1], n \in N\}$, where the processes N_n are defined as

$$N_n(t) = \int_0^t \alpha(s) Y_n(s) ds + M_n(t), \qquad t \in [0, 1],$$

and $M_n(t)$ is a martingale with respect to the filtration $\{\mathscr{F}_{n,t}\}$. For instance, the set A may be a subset of the $L^p_+ [0, 1]$. It is known (Andersen et al. [2], Karr [8], Liptser and Shiryayev [14] or Prakasa Rao [15]) that for each n the family $\mathscr{P}^n$ is dominated. As a dominating measure one can choose the probability measure $\mathscr{P}$ corresponding to the Poisson process with intensity 1. The density of $\mathscr{P}^n_\alpha$ with respect to $\mathscr{P}$ is of the following form:

$$\frac{d\mathscr{P}^n_\alpha}{d\mathscr{P}} = \exp\Big\{\int_0^1 \log\big(\alpha(s) Y_n(s)\big) dN_n(s) + \int_0^1 \big(1 - \alpha(s) Y_n(s)\big) ds\Big\}. \tag{2.1}$$

In the sequel, the logarithm of the right-hand side of (2.1) will be denoted by $L_n(\alpha)$. Following Grenander [6], the family $S(n)$ of subsets of the space A is called a *sieve* if $S(n)$ is increasing in n and $\bigcup_n S(n)$ is dense in A. In the paper, we assume that the family $S(n)$ is a histogram sieve, that is:

$$S(n) = \{\alpha \in A\colon\ \alpha(s) = \sum_{l=1}^{m(n)} x_l 1_{B_{l,m(n)}}(s)\}, \tag{2.2}$$

where $1_{B_{l,m(n)}}$ denotes the indicator of the set $B_{l,m(n)}$,

$$B_{l,m(n)} = \left(\frac{l-1}{m(n)}, \frac{l}{m(n)}\right] \text{ for } l = 2, \ldots, m(n), \qquad B_{1,m(n)} = \left[0, \frac{1}{m(n)}\right],$$

while $x_l \geqslant 0$, $\sum_l x_l^2 > 0$, $s \in [0, 1]$.

The sequence $\{m(n)\}$ denotes the speed of the growth of the sieve $S(n)$. Note that the sequence $\{B_{l,m(n)}\}$ defines the partition of $[0, 1]$ into subintervals of the length $1/m(n)$.

Under the assumption that both processes $Y_n(s)$ and $N_n(s)$, $s \in [0, 1]$, are observable the maximum likelihood estimator $\hat{\alpha}_n$ based on the sieve $S(n)$ is defined by the following equation:

$$L_n(\hat{\alpha}_n) = \max_{\alpha \in S(n)} L_n(\alpha). \tag{2.3}$$

The likelihood function $L_n(\alpha)$ on the histogram sieve $S(n)$ is merely a function $L_n(x_1, \ldots, x_{m(n)})$ of $m(n)$ variables. Therefore, we obtain the following

LEMMA 2.1. *The maximum likelihood estimator $\hat{\alpha}_n$ based on the histogram sieve $S(n)$ is of the following form*:

$$\hat{\alpha}_n(s) = \sum_{l=1}^{m(n)} \frac{N_n(B_{l,m(n)})}{\int_{B_{l,m(n)}} Y_n(u)\,du} 1_{C_{l,m(n)}} 1_{B_{l,m(n)}}(s), \tag{2.4}$$

where $C_{l,m(n)} = \{\int_{B_{l,m(n)}} Y_n(u)\,du > 0\}$, $l = 1, \ldots, m(n)$.

Let $s \in [0, 1]$ be fixed and choose $l(n, s) \in \{1, \ldots, m(n)\}$ such that $s \in B_{l(n,s),m(n)}$. Putting $B_{m(n)} = B_{l(n,s),m(n)}$ and $C_{m(n)} = C_{l(n,s),m(n)}$ the estimator $\hat{\alpha}_n(s)$ may be written in the following form:

$$\hat{\alpha}_n(s) = \frac{N_n(B_{m(n)})}{\int_{B_{m(n)}} Y_n(u)\,du} 1_{C_{m(n)}}. \tag{2.5}$$

3. Consistency and asymptotic normality of the histogram sieve estimator $\hat{\alpha}_n$. Let us assume that the following conditions hold:

(C.1) There exist $\delta > 0$ and a function $y\colon [0, 1] \to R_+$, positive and continuous at s such that

$$\sup_{t \in [s-\delta, s+\delta]} |Y_n(t)/n - y(t)| \xrightarrow{P} 0 \quad \text{as } n \to \infty,$$

where $\xrightarrow{P}$ denotes the convergence in probability.

(C.2) The function $\alpha_0(s)$ is continuous on $[0, 1]$.

(C.3) The speed of growth of the sieve $m_n = n^{1/2}$.

THEOREM 3.1. *If the conditions* (C.1)–(C.3) *hold, then the maximum likelihood estimator* $\hat{\alpha}_n(s)$ *defined in* (2.4) *and* (2.5) *is weakly consistent for each* $s \in [0, 1]$, *that means* $\hat{\alpha}_n(s)$ *converges to* $\alpha_0(s)$ *in probability for any* $s \in [0, 1]$ *as* $n \to \infty$.

Assume additionally that the following condition is satisfied:

(C.4) There exist $\delta > 0$, $\alpha > \frac{1}{2}$ and a positive constant $C(\alpha_0, s)$ such that

$$\forall_{t \in [s-\delta, s+\delta]} \ |\alpha_0(s) - \alpha_0(t)| \leqslant C(\alpha_0, s)|t-s|^{\alpha}.$$

Under the assumptions above it is possible to obtain the asymptotic distribution of $\hat{\alpha}_n$.

THEOREM 3.2. *Let* $\{s_1, \ldots, s_p\}$ *be an arbitrary finite collection of points from the interval* $[0, 1]$. *If the conditions* (C.1)–(C.4) *are fulfilled, then the sequence of random vectors*

$$n^{1/4}\big(\hat{\alpha}_n(s_1) - \alpha_0(s_1), \ldots, \hat{\alpha}_n(s_p) - \alpha_0(s_p)\big)$$

converges in distribution, as $n \to \infty$, *to the p-dimensional normal distribution with zero expectation and the covariance* $\sigma' I$, *where* I *is the unit diagonal matrix and the i-th component of the vector* $\sigma \in R^p$, $\sigma_i = \alpha_0(s_i)/y(s_i)$, $i = 1, \ldots, p$.

REMARKS AND EXAMPLES.

(i) All the results presented in previous sections can be immediately generalized to the case of multivariate point processes with multiplicative intensity model. In this context it is possible to describe the competing risks model through a marked multivariate point process as a multivariate point process belonging to the multiplicative intensity models (see Andersen et al. [2], pp. 77–78, and Example III.1.5).

(ii) Theorem 3.2 implies that for any fixed $s \in [0, 1]$ the limit distribution of $n^{1/4}\big(\hat{\alpha}_n(s) - \alpha_0(s)\big)$ is normal with zero expectation and variance $\alpha_0(s)/y(s)$.

(iii) *Estimation of the hazard rate function in a model without any censoring.* Let $\{X_1, \ldots, X_n\}$, $n \in N$, be a sequence of identically distributed, nonnegative random variables with distribution function F, density function f and hazard rate function $\alpha_0(t) = f(t)/(1-F(t))$. One can assume that the random variables X_i are i.i.d. as considered in Ramlau-Hansen [16] and Aalen [1]. Let $N_n(t) = \sum_{k=1}^{n} 1(X_k \leqslant t)$. Then the stochastic intensity of the process $N_n(t)$ is of the form $\alpha_n(t) = \alpha_0(t) \sum_{k=1}^{n} 1(X_k \geqslant t)$. In this case all the assumptions of our theorems are satisfied and one can construct the consistent and asymptotically normal histogram sieve estimator of α_0.

(iv) *Censoring.*

Assume that the random variables $X_1, X_2, \ldots, X_n$, $n \in N$, are as above and let $\{T_i\}$ be a sequence of censoring random variables, that means we either observe X_i or we only observe that X_i is larger than T_i. In this censorship model, the censoring times $\{T_i\}_{i=1}^n$ are assumed to be an i.i.d. sequence and they are also assumed to be independent of the $\{X_i\}$. Introducing the 0-1 valued random variables D_i called *censoring indicators* we observe the random variables $(\tilde{X}_i, D_i)$, where $\tilde{X}_i = X_i$ if $D_i = 1$ and $\tilde{X}_i < X_i$ if $D_i = 0$. Let

$$Y_n(t) = \sum_{i=1}^{n} 1(\tilde{X}_i \geqslant t) \quad \text{and} \quad N_n(t) = \sum_{i=1}^{n} 1(\tilde{X}_i \leqslant t,\ D_i = 1).$$

The intensity function $\alpha_n(t)$ of the process $N_n(t)$ is of the form

$$\alpha_n(t) = Y_n(t)\,\alpha_0(t)$$

and one can construct the histogram sieve estimator $\hat{\alpha}_n$ of α_0.

(v) *Hazard rate estimation for dependent data.*

In this case the random variables $\{X_1, \ldots, X_n\}$ are not assumed to be mutually independent as in (iii) or (iv). However, they have a common distribution function F, density function f and hazard rate function α_0. Furthermore, we assume that the sequence $\{X_i\}$ is ψ-mixing as considered in Leśkow and Różański [13]. Let us write $N_n(t) = \sum_{k=1}^{n} 1(X_k \leqslant t)$. One can show that the point process $N_n(t)$ has the stochastic intensity of the form

$$\alpha_n(t) = C\big(\psi(1)\big)\,\alpha_0(t) \sum_{k=1}^{n} 1(X_k \geqslant t).$$

Therefore, once again the histogram sieve estimator may be applied. We can also consider the possibility of censoring the dependent survival data $\{X_1, \ldots, X_n\}$ introducing independent censoring times as in (iv).

(vi) *Type* II *censoring* (Andersen et al. [2], Example III.2.2).

Let $X_1, X_2, \ldots, X_n$ be i.i.d. as in (iii). In this case, the censoring random variables $T_i = X_{r(n)}$, where $X_{r(n)}$ denotes the time of $r(n)$-th failure. Observe that in this example the censored random variables $\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n$ are dependent. Nevertheless, we are still in the frames of the multiplicative intensity model.

(vii) Observe that if the conditions (A.1)–(A.4) from Leśkow and Różański [13] hold, then the conditions (C.1)–(C.4) are in force.

(viii) *Hazard rate estimation for exchangeable random variables.*

Let $\{X_1, \ldots, X_n\}$ be a sequence of identically distributed and conditionally independent random variables given some σ-algebra $\mathscr{B}$. Under these assumptions $\{X_i\}$ forms a sequence of exchangeable random variables. Putting now $\tilde{N}_i(t) = 1(X_i \leqslant t)$ and $N_n(t) = \sum_{i=1}^{n} \tilde{N}_i(t)$ and using consideration concerned with combining conditionally independent components (Andersen et al. [2], Section II.4.3) we see that the process $N_n(t)$ has the multiplicative intensity form.

(ix) *Periodic observations.*

Let us construct the sequence $\tilde{N}_i(t)$, $t \in [0, 1]$, $i \in \{1, 2, \ldots, n\}$, $n \in N$, from a single realization of the point process $\{N(s), s \in [0, n]\}$ in the following way:

$$\tilde{N}_i(t) = N(t+i-1) - N(i-1), \qquad \tilde{Y}_i(t) = Y(t+i-1).$$

Assume that the process $N(t)$ is of the multiplicative intensity model, the unknown function α_0 is periodic with known period equal to 1, and the sequence $\{\tilde{N}_i(t)\}$ is ψ-mixing. Under weaker conditions than in Leśkow [10] a maximum likelihood estimator of the function α_0 may be constructed. As in (v), we infer that the process $N_n(t) = \sum_{i=1}^{n} \tilde{N}_i(t)$ belongs to the multiplicative intensity model and the conditions (C.1)–(C.4) are in force. Consequently, all the results concerned with the pointwise consistency and asymptotic normality of the histogram estimator $\hat{\alpha}_n$ based on a single realization $N(t)$ follow from Theorems 3.1 and 3.2. Details will be given in a forthcoming paper.

(x) *The assumption* (C.3) *can be changed.*

Namely, the sequence $m_n = n^{1/2}$ can be replaced by any sequence m_n tending to infinity slower than $n^{1/2}$. Under this assumption we get the consistency of the estimator $\hat{\alpha}_n(s)$. To obtain the limit theorem concerned with the asymptotic distribution of the estimator $\hat{\alpha}_n(s)$ we need to impose some additional assumption on the smoothness of the estimated function α_0 and use the normalizing sequence $(n/m_n)^{1/2}$. For instance, assuming that the function α_0 is differentiable at s and m_n is such that $n^{1/2}/m_n \to \infty$ and $n/m_n^3 \to 0$, we see that $\hat{\alpha}_n(s)$ is asymptotically normal

$$\mathcal{N}\left(\alpha_0(s), \frac{m_n^2}{n}\frac{\alpha_0(s)}{y(s)}\right).$$

It is also possible to consider the problem of choosing optimal m_n which minimizes the mean integrated squared error of the estimator. Some results connected with this problem will be presented in another paper.

4. Proofs. According to the assumptions of the multiplicative intensity model and the Doob–Meyer decomposition the process $N_n(t)$ can be decomposed into

$$N_n(t) = \int_0^t \alpha_0(u)\, Y_n(u)\, du + M_n(t), \qquad t \in [0, 1],\ n \in N,$$

where $M_n(t)$ is a martingale with respect to the filtration $\mathcal{F}_{n,t}$.

To prove the consistency and asymptotic normality of the estimator $\hat{\alpha}_n(s)$ let us write the difference $\hat{\alpha}_n(s) - \alpha_0(s)$, $s \in [0, 1]$, in the following form:

$$(4.1) \qquad \hat{\alpha}_n(s) - \alpha_0(s) = \frac{n^{-1/2} M_n(B_{m(n)}) + n^{-1/2} \int_{B_{m(n)}} \big(\alpha_0(u) - \alpha_0(s)\big)\, Y_n(u)\, du}{n^{-1/2} \int_{B_{m(n)}} Y_n(u)\, du},$$

where $M_n(B_{m(n)})$ is the increment of the martingale M_n on the interval $B_{m(n)}$. In the sequel the following lemma will be useful:

LEMMA 4.1. *The sequence* $n^{-1/2} M_n(B_{m(n)}) \xrightarrow{P} 0$ *as* $n \to \infty$.

Proof. Let us consider the sequence of martingales $n^{-1/2} M_n(t)$, $t \in [0, 1]$. The sequence is also a sequence of random elements of $\mathscr{D}([0, 1])$ (the space of right continuous functions having left-hand limits with Skorohod topology). By Rebolledo's theorem (Rebolledo [17]) the sequence $n^{-1/2} M_n(\cdot)$ is convergent in distribution in $\mathscr{D}([0, 1])$ to an element $\tilde{M}(\cdot)$ which is a continuous Gaussian martingale with independent increments for which

$$E\tilde{M}^2(t) = \int_0^t \alpha(u) y(u)\, du.$$

Further, using Theorem 5.5 from Billingsley [4] on weak convergence of a sequence of continuous mappings of random elements we obtain

$$n^{-1/2} M_n(B_{m(n)}) = n^{-1/2}\big(M_n(t_{m(n)+1}(s)) - M_n(t_{m(n)}(s))\big) \xrightarrow{D} 0 \quad \text{as } n \to \infty,$$

where $B_{m(n)} = \big(t_{m(n)}(s),\ t_{m(n)+1}(s)\big]$, $t_{m(n)}(s) \nearrow s$, $t_{m(n)+1}(s) \searrow s$.

Proof of Theorem 3.1. By the conditions (C.1)–(C.3) we have

$$n^{-1/2} \int_{B_{m(n)}} Y_n(u)\, du \xrightarrow{P} y(s),$$

and

$$n^{-1/2} \int_{B_{m(n)}} \big(\alpha_0(u) - \alpha_0(s)\big) Y_n(u)\, du \xrightarrow{P} 0 \quad \text{as } n \to \infty,$$

which together with Lemma 4.1 implies that (4.1) converges to zero in probability.

Proof of Theorem 3.2. To prove Theorem 3.2 we will first show the convergence of one-dimensional distributions. One can write

$$(4.2) \quad n^{1/4}\big(\hat{\alpha}_n(s) - \alpha_0(s)\big) = \frac{n^{1/4}\big(n^{-1/2} M_n(B_{m(n)})\big) + n^{1/4}\big(n^{-1/2} \int_{B_{m(n)}} (\alpha_0(u) - \alpha_0(s))\, Y_n(u)\, du\big)}{n^{-1/2} \int_{B_{m(n)}} Y_n(u)\, du}.$$

We have shown that the denominator of (4.2) converges in probability to $y(s)$. Let us note that by the condition (C.4) the second term in the numerator of (4.2) converges to zero in probability.

By previously mentioned Rebolledo's theorem (Rebolledo [17]) the sequence $n^{-1/2} M_n(\cdot)$ is convergent in distribution in $D([0, 1])$ to an element $\tilde{M}(\cdot)$ which is a continuous Gaussian martingale with independent increments. Let us denote by Q_n and Q the measures generated by $n^{-1/2} M_n(\cdot)$ and $\tilde{M}(\cdot)$ in $D([0, 1])$, respectively.

Let $\mathscr{A}$ denote the following class of subsets of $D([0, 1])$:

$$\mathscr{A} = \Big\{A \subset D([0, 1]):\ A = \bigcap_k \{x:\ x(t_k(s)) \leqslant x_k\} \cap \bigcap_l \{x:\ x(t_l(s)) \leqslant y_l\}\Big\},$$

where $\{t_k(s)\}$ and $\{t_l(s)\}$ are finite or infinite subsets of the sets $\{t_{m(n)}(s)\}$ and $\{t_{m(n)+1}(s)\}$ for arbitrary real numbers $\{x_k\}$ and $\{y_l\}$, respectively. From Theorem 3 in Topsøe [18] it follows that the $\mathscr{A}$ is the Q-uniformity class. Thus we have

$$\sup_{A\in\mathscr{A}} |Q_n(A)-Q(A)| \to 0 \quad \text{as } n\to\infty.$$

It is easy to see that the σ-algebra $\sigma(\mathscr{A})$ generated by the class $\mathscr{A}$ is also the Q-uniformity class.

Let F_n and G_n denote the distribution functions of $n^{-1/2} M_n(B_{m(n)})$ and $\tilde{M}(B_{m(n)})$, respectively. Since

$$\sigma\big(n^{-1/2} M_n\big(t_{m(n)}(s)\big);\ n^{-1/2} M_n\big(t_{m(n)+1}(s)\big),\ n\geqslant 1\big) \subset \sigma(\mathscr{A}),$$

where $\sigma(\mathscr{A})$ is the Q-uniformity class, we obtain

$$\sup_x |F_n(x)-G_n(x)| \to 0 \quad \text{as } n\to\infty. \tag{4.3}$$

Obviously,

$$n^{1/4}\big(\tilde{M}(B_{m(n)})\big) \xrightarrow{D} U \quad \text{as } n\to\infty,$$

where U is a random variable normally distributed with zero expectation and the variance equal to $\alpha_0(s)\,y(s)$. Thus, from (4.3) it follows that also

$$n^{1/4}\big(n^{-1/2} M_n(B_{m(n)})\big) \xrightarrow{D} U \quad \text{as } n\to\infty.$$

Now, we have proved that the numerator of (4.2) converges in distribution to the random variable U and the denominator of (4.2) converges in probability to $y(s)$.

This shows the asymptotical normality with zero expectation and the variance equal to $\alpha_0(s)/y(s)$. Applying Slutzky's lemma and the Cramer–Wold device we conclude the assertion of the theorem.

5. Simulation results. In order to illustrate the behaviour of the histogram sieve estimator we have made some computer experiments. Namely, in our numerical example $n = 500$ independent lifetimes $X_1, \ldots, X_n$ were generated from the Weibull, log-normal, gamma and Gompertz distibutions which are chosen because of their popularity in analyzing survival data (see, for example, Weibull [19], Feinleib [5], Horner [7], Klein and Moeschberger [9]). The corresponding hazard rate functions and parameters used in simulation are summarized in the table below (I denotes the incomplete gamma function).

Distribution	Hazard rate	Parameters
Weibull	$\gamma \varrho t^{\varrho - 1}$	$\gamma = 2,\ \varrho = 0.5$
Log-normal	$\dfrac{\exp\left[-\frac{1}{2}((\ln t - \mu)/\sigma)^2\right]}{\sqrt{2\pi}\sigma t\left(1 - \Phi\left((\ln t - \mu)/\sigma\right)\right)}$	$\mu = 0,\ \sigma = 1$
Gompertz	$\theta e^{\alpha t}$	$\theta = 0.2,\ \alpha = 2$
Gamma	$\dfrac{\lambda^{\beta} t^{\beta - 1} \exp(-\lambda t)}{\Gamma(\beta)(1 - I(\lambda t, \beta))}$	$\lambda = 1,\ \beta = 2$

Additionally, simulated lifetimes were censored by n independent censoring times $T_1, \ldots, T_n$ generated from an exponential distribution with mean 1. If we define $\tilde{X}_i = \min(X_i, T_i)$ and the indicator of censoring $D_i = 1(\tilde{X}_i = X_i)$, $i = 1, \ldots, n$, then the counting process $N_n(\cdot)$ is given by

$$N_n(t) = \sum_{i=1}^{n} 1(\tilde{X}_i \leqslant t,\ D_i = 1).$$

This process counts the total number of observed deaths in $[0, t]$ and has the intensity process α_n given by $\alpha_n(t) = \alpha_0(t)\, Y_n(t)$, where $Y_n(t) = \sum_{i=1}^{n} 1(\tilde{X}_i \geqslant t)$ is the number of individuals "at risk" just prior to t.

Figures 1–4 show the histogram sieve estimator of the hazard rate function constructed for all considered distributions on the interval $[0, 1]$ and for $m(n) = \sqrt{n} = 22$. We have also drawn 95% pointwise confidence intervals based on asymptotic normality and the variance estimator $\hat{\sigma}_n(s) = n\hat{\alpha}_n(s)/Y_n(s)$.

For each figure the true underlying hazard rate function is denoted by a solid line, histogram sieve estimate is denoted by a dotted line, and confidence intervals are marked as a dash-dot line.

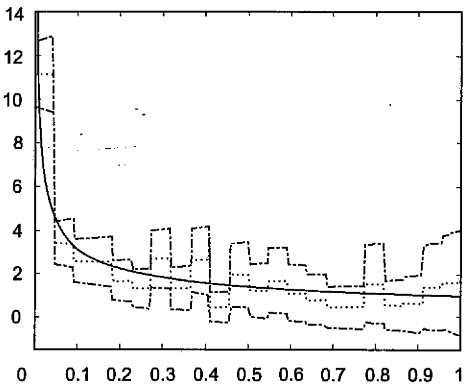

Fig. 1. Histogram sieve estimator for the Weibull distribution

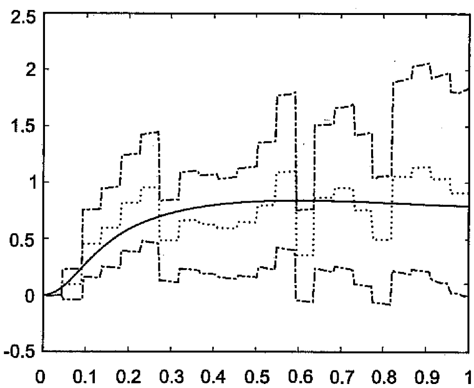

Fig. 2. Histogram sieve estimator for the log-normal distribution

The accuracy of constructed pointwise intervals was investigated in terms of empirical coverage. Table 1 contains results of empirical coverage for selected points and different choices of the sequence $m(n)$ obtained for the Weibull distribution. Apart from the sequence $m(n) = \sqrt{n}$ we used in simulation $m(n) = n^{4/9}$ and $m(n) = n^{2/5}$ which satisfy the conditions given in Remark (x).

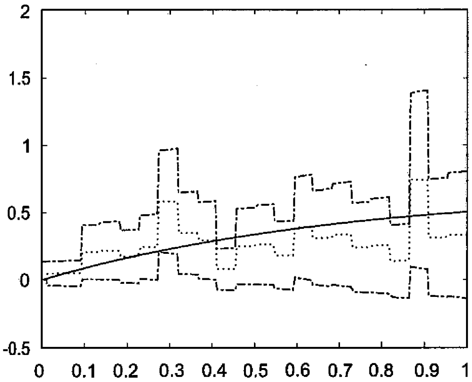

Fig. 3. Histogram sieve estimator for the gamma distribution

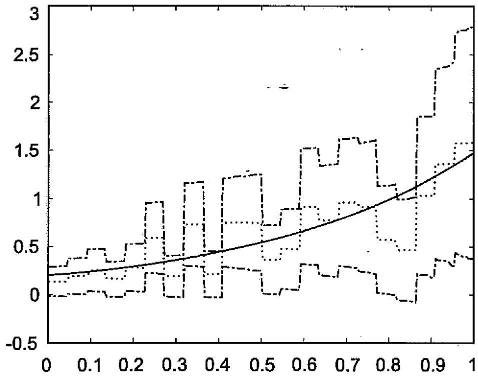

Fig. 4. Histogram sieve estimator for the Gompertz distribution

TABLE 1. Empirical coverage for the Weibull hazard rate function. Nominal coverage is equal to 95%

$m(n)$	0.01	0.12	0.23	0.33	0.44	0.55	0.65	0.76	0.87	0.97
22	49%	95%	96%	94%	93%	90%	88%	90%	79%	71%
16	91%	89%	95%	93%	91%	92%	91%	86%	90%	82%
12	52%	95%	96%	91%	93%	92%	93%	86%	91%	88%

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Institute of Mathematics
Wrocław University of Technology
50-370 Wrocław, Poland

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