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RECURRENCE THEOREMS FOR MARKOV RANDOM WALKS

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Abstract. Let $(M_n, S_n)_{n\geqslant 0}$ be a Markov random walk whose driving chain $(M_n)_{n\geqslant 0}$ with general state space $(\mathscr{S}, \mathfrak{S})$ is ergodic with unique stationary distribution ξ. Providing $n^{-1}S_n \to 0$ in probability under P_ξ, it is shown that the recurrence set of $(S_n-\gamma(M_0)+\gamma(M_n))_{n\geqslant 0}$ forms a closed subgroup of $\boldsymbol{R}$ depending on the lattice-type of $(M_n, S_n)_{n\geqslant 0}$. The so-called shift function γ is bounded and appears in that lattice-type condition. The recurrence set of $(S_n)_{n\geqslant 0}$ itself is also given but may look more complicated depending on γ. The results extend the classical recurrence theorem for random walks with i.i.d. increments and further sharpen results by Berbee, Dekking and others on the recurrence behavior of random walks with stationary increments.

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1. Introduction and main results. Given a random walk $(S_n)_{n\geqslant 0}$ with $S_0 = 0$ and i.i.d. real-valued non-degenerate increments $X_1, X_2, \ldots$ such that $n^{-1}S_n \xrightarrow{P} 0$, it is well-known that its associated set $\mathfrak{R}$ of recurrence points, defined as

$$\mathfrak{R} \stackrel{\text{def}}{=} \{x\in \boldsymbol{R}\colon S_n \in (x-\varepsilon, x+\varepsilon) \text{ infinitely often for all } \varepsilon > 0\}, \tag{1.1}$$

a.s. forms a closed subgroup of $\boldsymbol{R}$ (see e.g. [3]). More precisely, $\mathfrak{R}$ a.s. equals $\boldsymbol{R}$ or $d\boldsymbol{Z}$ for some $d > 0$, depending on whether X_1 has lattice-span $d = 0$ (nonarithmetic case) or $d\in(0, \infty)$ (d-arithmetic case), respectively. Our purpose is to show a corresponding result for driftless Markov random walks (MRW) which are introduced below. This comprises the class of driftless random walks with stationary increments as will be explained below.

Let $(\mathscr{S}, \mathfrak{S})$ be a measurable space with countably generated σ-field $\mathfrak{S}$, $\boldsymbol{P}\colon \mathscr{S}\times(\mathfrak{S}\otimes\mathfrak{B}) \to [0, 1]$ a transition kernel, $\mathfrak{B}$ the Borel σ-field on $\boldsymbol{R}$, and $(M_n, X_n)_{n\geqslant 0}$ an associated Markov chain, defined on any probability space $(\Omega, \mathscr{A}, P)$, with state space $\mathscr{S}\otimes\boldsymbol{R}$, i.e.

$$P(M_{n+1}\in A, X_{n+1}\in B \mid M_n, X_n) = \boldsymbol{P}(M_n, A\times B) \text{ a.s.}$$

for all $n \geqslant 0$ and $A \in \mathfrak{S}$, $B \in \mathfrak{B}$. Thus (M_{n+1}, X_{n+1}) depends on the past only through M_n, and $M = (M_n)_{n \geqslant 0}$ forms a Markov chain with state space $\mathscr{S}$ and transition kernel $\boldsymbol{P}(x, A) \overset{\text{def}}{=} \boldsymbol{P}(x, A \times \boldsymbol{R})$. Given M, the X_n, $n \geqslant 0$, are conditionally independent with

$$P(X_n \in B \mid M) = Q(M_{n-1}, M_n, B) \text{ a.s.}$$

for all $n \geqslant 1$, $B \in \mathfrak{B}$ and a kernel $Q: \mathscr{S}^2 \times \mathfrak{B} \to [0, 1]$. Throughout we assume that a canonical model is given with probability measures P_s, $s \in \mathscr{S}$, on $(\Omega, \mathscr{A})$ such that $P_s(M_0 = s, X_0 = 0) = 1$. For any distribution ν on $\mathscr{S}$ put $P_\nu(\cdot) = \int_{\mathscr{S}} P_s(\cdot)\,\nu(ds)$ in which case (M_0, X_0) has initial distribution $\nu \otimes \delta_0$ under P_ν, where δ_0 is Dirac measure at 0.

The MRW associated with $(M_n, X_n)_{n \geqslant 0}$ is defined by $(M_n, S_n)_{n \geqslant 0}$, where $S_0 = 0$ and $S_n = X_1 + \ldots + X_n$ for each $n \geqslant 0$. We assume that M has a unique stationary distribution ξ, whence the chain is ergodic under P_ξ in the usual sense that any a.s. invariant event $A \in \mathfrak{S}^\infty$, i.e. $\mathbb{1}_A = \mathbb{1}_A \circ \theta$ P_ξ^M-a.s., θ the shift operator on $\mathscr{S}^\infty$, has probability 0 or 1 under P_ξ^M. Further we assume that

$$S_n/n \xrightarrow{P_\xi} 0, \tag{1.2}$$

which holds in particular when $E_\xi X_1 = 0$.

Next let us define the lattice-type of $(M_n, S_n)_{n \geqslant 0}$, which is more complicated than in the i.i.d. case. Following [9], the latter as well as $\boldsymbol{P}$ are called *d-arithmetic* if $d > 0$ is the maximal number for which there exists a function $\gamma: \mathscr{S} \to [0, d)$, called a *shift function,* such that

$$P\big(X_1 \in \gamma(x) - \gamma(y) + d\mathbb{Z} \mid M_0 = x, M_1 = y\big) = 1 \quad \xi \otimes \boldsymbol{P}\text{-a.s.}, \tag{1.3}$$

where $\xi \otimes \boldsymbol{P}$ is given through $\xi \otimes \boldsymbol{P}(A \times B) = \int_A \boldsymbol{P}(x, B)\,\xi(dx)$ for $A, B \in \mathscr{S}$. If no such d exists, $(M_n, S_n)_{n \geqslant 0}$ and $\boldsymbol{P}$ are called *nonarithmetic*. Note that d may also be ∞, namely when

$$X_n = \gamma(M_{n-1}) - \gamma(M_n) \quad P_\xi\text{-a.s.} \tag{1.4}$$

for all $n \geqslant 1$ and some measurable γ. This is called *null-homology* in [6] and corresponds to the trivial case $X_n \equiv 0$ for random walks with i.i.d. increments.

Although trivially obtained, it is important to note that every random walk $(S_n)_{n \geqslant 0}$ with ergodic stationary increments $X_1, X_2, \ldots$ and property (1.2) may be investigated in the previous framework. We must simply define an appropriate driving chain of that random walk, the canonical candidate being

$$M_n \overset{\text{def}}{=} (X_{n+k})_{k \geqslant 1}, \qquad n \geqslant 0.$$

In view of our purpose to describe the recurrence set of $(S_n)_{n \geqslant 0}$ the possibly occurring shift function in the arithmetic case is somewhat annoying because it means that the S_n "live" on the same lattice only modulo a time-varying adjustment with respect to that shift function. On the other hand, one can overcome this problem by observing that, given a d-arithmetic MRW $(M_n, S_n)_{n\geq 0}$ with non-vanishing shift function γ, its transformation $(M_n, S_n+\gamma(M_n)-\gamma(M_0))_{n\geq 0}$ forms again a d-arithmetic MRW but with shift function 0. It also satisfies (1.2) if and only if the former does.

Our first result shows that the dichotomy $\mathfrak{R} = \emptyset$ (transience) or $=$ closed subgroup of $\boldsymbol{R}$ (recurrence) extends from the i.i.d. case to general MRW with ergodic driving chain and shift function 0 if d-arithmetic for some $d > 0$. Further it gives a corresponding dichotomy for the renewal measures U_s of $(S_n)_{n\geq 0}$ under the P_s, $s\in\mathcal{S}$, defined as

$$U_s(A) \stackrel{\text{def}}{=} E_s N(A) = \sum_{n\geq 0} P(S_n \in I),$$

where $N(A) \stackrel{\text{def}}{=} \sum_{n\geq 0} \mathbf{1}_{\{S_n\in A\}}$, $A\in\mathfrak{B}$. Finally, put $\boldsymbol{L}_0 = \boldsymbol{R}$, $\boldsymbol{L}_d = d\boldsymbol{Z}$ for $d\in(0, \infty)$, and $\boldsymbol{L}_\infty = \{0\}$.

THEOREM 1. *Let $(M_n, S_n)_{n\geq 0}$ be an MRW with lattice-span $d\in[0, \infty)$, shift function 0 in the case $d > 0$, and driving chain $(M_n)_{n\geq 0}$ having a unique stationary distribution ξ. Denote by $\mathfrak{R}$ the set of recurrence points of $(S_n)_{n\geq 0}$, as defined in* (1.1). *Then either $\mathfrak{R} = \boldsymbol{L}_d$ or $\mathfrak{R} = \emptyset$ P_ξ-a.s., and the following assertions are equivalent*:

(a) $(M_n, S_n)_{n\geq 0}$ *is transient* ($\mathfrak{R} = \emptyset$ P_ξ*-a.s.*).

(b) $P_s(N(I) < \infty) = 1$ *for ξ-almost all $s\in\mathcal{S}$ and all bounded intervals I.*

(c) *There is a ξ-positive set $\mathcal{S}_0$ and a bounded open interval I, $I\cap \boldsymbol{L}_d \neq \emptyset$, such that $U_s(I) < \infty$ for all $s\in\mathcal{S}_0$.*

(d) $U_s(I) < \infty$ *for ξ-almost all $s\in\mathcal{S}$ and all bounded intervals I.*

Adding the condition (1.2) to the assumptions in Theorem 1 yields the following recurrence theorem which is the canonical extension of the result for random walks with i.i.d. increments stated at the beginning of this section.

THEOREM 2. *Let $(M_n, S_n)_{n\geq 0}$ be as in Theorem* 1. *Then* (1.2) *implies $\mathfrak{R} = \boldsymbol{L}_d$ P_ξ-a.s. (and thus P_s-a.s. for ξ-almost all $s\in\mathcal{S}$).*

In order to describe $\mathfrak{R}$ in the d-arithmetic case with non-vanishing shift function γ, denote by $\mathscr{C}_\gamma$ the set of points of increase (support) of $P_\xi^{\gamma(M_0)}$, i.e.

$$\mathscr{C}_\gamma \stackrel{\text{def}}{=} \{x\in[0, d]\colon\ P_\xi(|\gamma(M_0)-x| < \varepsilon) > 0 \text{ for all } \varepsilon > 0\}.$$

Clearly, $\mathscr{C}_\gamma$ is closed and $P_\xi(\gamma(M_0)\in\mathscr{C}_\gamma) = 1$.

THEOREM 3. *Let $(M_n, S_n)_{n\geq 0}$ be as in Theorem* 1 *with $d\in(0, \infty)$, but with non-vanishing shift function γ. Then*

$$\mathfrak{R} = \overline{\bigcup_{x\in\mathscr{C}_\gamma} (\gamma(M_0)-x+d\boldsymbol{Z})}\ P_\xi\text{-a.s.}$$

In particular, $P_\xi(d\boldsymbol{Z} \subset \mathfrak{R}) = 1$.

Remarks. (a) It is interesting to note and obvious now that even in the d-arithmetic case we can have $\mathfrak{R} = \mathbf{R}$ P_ξ-a.s., namely for $\mathscr{C}_\gamma = [0, d)$.

(b) As a trivial consequence of Theorem 2 or 3 we obtain

$$\liminf_{n\to\infty} S_n = -\infty \quad \text{and} \quad \limsup_{n\to\infty} S_n = \infty \quad P_\xi\text{-a.s.} \tag{1.5}$$

(c) Providing $E_\xi|X_1| < \infty$ in Theorem 1, one can show by slightly modifying an argument by Lalley [6] (see his Proposition 6) that $(M_n, S_n)_{n\geq 0}$ is null-homologous (see (1.4)) if and only if $\sup_{n\geq 1} E_\xi|S_n| < \infty$.

(d) All previous results remain valid if the driving chain $(M_n)_{n\geq 0}$ merely has a σ-finite stationary measure ξ which is unique up to multiplicative constants. The basic technique for proving the extensions is to consider the MRW $(M_{\sigma_n(\varepsilon)}, S_{\sigma_n(\varepsilon)})_{n\geq 0}$, where $\sigma_n(\varepsilon)$ denotes the n-th visit of the driving chain to a set $\mathscr{S}_\varepsilon \subset \mathscr{S}$ such that $\xi(\mathscr{S}_\varepsilon) < \infty$ and $\mathscr{S}_\varepsilon \uparrow \mathscr{S}$ as $\varepsilon \downarrow 0$. We omit further details.

(e) Ornstein [7] and independently Stone [10] showed that a random walk $(S_n)_{n\geq 0}$ with i.i.d. increments $X_1, X_2, \ldots$ is recurrent if and only if

$$\int_{(-\varepsilon,\varepsilon)} \operatorname{Re}\left(\frac{1}{1-\varphi(t)}\right) dt = \infty,$$

where φ denotes the characteristic function of X_1. A slightly weaker equivalent condition was given earlier by Chung and Fuchs [4]. We conjecture that an MRW $(M_n, S_n)_{n\geq 0}$ whose driving chain has a unique stationary distribution ξ is recurrent if and only if

$$\int_{\mathscr{S}} \int_{(-\varepsilon,\varepsilon)} \operatorname{Re}\left(\frac{1}{1-\varphi(s,t)}\right) dt\, \xi(ds) = \infty,$$

where $\varphi(s, t) \stackrel{\text{def}}{=} E_s \exp(itX_1)$.

2. Proofs. Throughout this section we denote by $(M_n, S_n)_{n\geq 0}$ an MRW satisfying the assumptions of Theorem 1. Since $(M_n, X_{n+1})_{n\geq 0}$ is stationary under P_ξ, we may extend it to a doubly infinite sequence $(M_n, X_{n+1})_{n\in\mathbf{Z}}$. Notice that we have thus altered the definition of X_0 which is generally no longer equal to 0 under P_ξ as stipulated in the Introduction. Put $S_{-n} = -\sum_{k=0}^{n-1} X_{-k}$ for $n < 0$, i.e. $S_{n+1} = S_n + X_{n+1}$ for all $n \in \mathbf{Z}$. The time reversal

$$(M_n^*, X_{n+1}^*)_{n\in\mathbf{Z}} \stackrel{\text{def}}{=} (M_{-n}, X_{-n+1})_{n\in\mathbf{Z}}$$

is again stationary and Markovian with kernel

$$P^*(x, dy \times dz) = Q(y, x, dz)\, p(y, x)\, \xi(dy),$$

where $p(x, y)$ denotes the $\mathfrak{S}^2$-measurable (since $\mathfrak{S}$ is separable) ξ-density of $P(x, dy)$. Consequently, $(M_n^*, S_n^*)_{n\in\mathbf{Z}}$, where $S_n^* \stackrel{\text{def}}{=} -S_{-n}$, is also an MRW whose driving chain $M^* = (M_n^*)_{n\in\mathbf{Z}}$ has the transition kernel $P^*(x, dy) = p(y, x)\, \xi(dy)$

and the same unique stationary distribution ξ. We call $(M_n^*, S_n^*)_{n\in Z}$ the *dual* of $(M_n, S_n)_{n\in Z}$ and note that they have the same lattice-span and shift function and that either both or none of them satisfy the condition (1.2). This follows immediately from

$$(2.1) \qquad P_\xi(S_n^*\in\cdot) = P_\xi(-S_{-n}\in\cdot) = P_\xi(S_n\in\cdot)$$

for all $n\in \boldsymbol{Z}$. Finally, assume that N^* and U_s^* have the obvious meaning.

Our presentation follows to a far extent the one in Breiman's book [3] for the i.i.d. case. However, the loss of independence makes a number of crucial arguments more difficult. Moreover, we make use of the following result from Berbee's [2] thesis (see his Corollary 2.3.4):

PROPOSITION 1. *Either $P_\xi(N(I)<\infty)=1$ for all bounded intervals I or $P_\xi(N(I)\in\{0,\infty\})=1$ for all intervals I.*

PROPOSITION 2. *If d denotes the lattice-span of $(M_n, S_n)_{n\geqslant 0}$, then either $\mathfrak{R}=L_d$ or $\mathfrak{R}=\emptyset$.*

Proof. $\mathfrak{R}$ is clearly closed. Put $B_\xi(x) = (x-\varepsilon, x+\varepsilon)$ and call $x\in \boldsymbol{R}$ a *possible* state if

$$\sum_{n\geqslant 1} 2^{-n} P_\xi(S_n\in B_\varepsilon(x)) = \sum_{n\geqslant 1} 2^{-n} P_\xi(S_{-n}\in B_\varepsilon(-x)) > 0$$

for all $\varepsilon>0$. Let $\mathscr{P}$ be the collection of all possible states. We claim that $x\in\mathfrak{R}$ and $y\in\mathscr{P}$ implies $x-y\in\mathfrak{R}$. For the proof fix any $\varepsilon>0$ and observe that a ξ-positive set $\mathscr{S}_0$ exists such that $\sum_{n\geqslant 1} 2^{-n} P_s(S_{-n}\in B_\varepsilon(-y))>0$ for each $s\in\mathscr{S}_0$. It follows that

$$\begin{aligned}
0 &= P_\xi\big(N(B_\varepsilon(x))<\infty\big) = P_\xi\big(S_n\in B_\varepsilon(x) \text{ finitely often}\big)\\
&\geqslant \sum_{k\geqslant 1} 2^{-k} P_\xi\big(S_k\in B_\varepsilon(y),\ S_{k+n}-S_k\in B_{2\varepsilon}(x-y) \text{ finitely often}\big)\\
&= \sum_{k\geqslant 1} 2^{-k} \int_{\mathscr{S}} P_\xi\big(S_k\in B_\varepsilon(y),\ S_{k+n}-S_k\in B_{2\varepsilon}(x-y) \text{ finitely often} \mid M_k=s\big)\,\xi(ds)\\
&\geqslant \int_{\mathscr{S}_0} \sum_{k\geqslant 1} 2^{-k} P_s\big(S_{-k}\in B_\varepsilon(-y)\big) P_s\big(S_n\in B_{2\varepsilon}(x-y) \text{ finitely often}\big)\,\xi(ds)\\
&= \int_{\mathscr{S}_0} \sum_{k\geqslant 1} 2^{-k} P_s\big(S_{-k}\in B_\varepsilon(-y)\big) P_s\big(N(B_{2\varepsilon}(x-y)<\infty)\big)\,\xi(ds),
\end{aligned}$$

and hence $P_s\big(N(B_{2\varepsilon}(x-y))<\infty\big)=0$ for ξ-almost all $s\in\mathscr{S}_0$, i.e.

$$P_\xi\big(N(B_{2\varepsilon}(x-y))=\infty\big)>0.$$

Now use Proposition 1 to infer $P_s\big(N(B_{2\varepsilon}(x-y))=\infty\big)=1$, i.e. $x-y\in\mathfrak{R}$, since $\varepsilon>0$ was arbitrarily chosen.

If $\mathfrak{R}$ is not empty, then $x \in \mathfrak{R}$ and $\mathfrak{R} \subset \mathscr{P}$ imply $x - x = 0 \in \mathfrak{R}$, which further gives $-x = 0 - x \in \mathfrak{R}$. $\mathfrak{R}$ thus forms a closed subgroup of $\boldsymbol{R}$, i.e. $\mathfrak{R}$ equals $\boldsymbol{R}$, $d\boldsymbol{Z}$ for some $d > 0$, or $\{0\}$. The latter can obviously hold only if $P_\xi(X_1 = 0) = 1$. If $(M_n, S_n)_{n \geq 0}$ has lattice-span $d \in (0, \infty)$, we obtain $\mathfrak{R} = \boldsymbol{L}_d$ because $\mathfrak{R} = \boldsymbol{L}_{kd}$ for some $k \geq 2$ would imply $P_\xi(N(\{x\}) < \infty) = 1$ for all $x \in \boldsymbol{L}_d - \boldsymbol{L}_{kd}$, and thus, by another appeal to Proposition 1, $P_\xi(N(\{x\}) = 0) = 1$. This, however, would further lead to $P_\xi(S_n \in \boldsymbol{L}_{kd}$ for all $n \geq 0) = 1$, and thereby to a lattice-span greater than or equal to kd. By a similar argument, we obtain $\mathfrak{R} = \boldsymbol{R}$ in the nonarithmetic case.

PROPOSITION 3. *Let d be the lattice-span of $(M_n, S_n)_{n \geq 0}$. If there is a bounded interval I such that $U_s^*(I) = \infty$ ξ-a.s., then $\mathfrak{R} = \boldsymbol{L}_d$.*

Note as an immediate consequence that $\mathfrak{R}^*$, the set of recurrence points of the dual walk $(S_n^*)_{n \geq 0}$, always coincides with $\mathfrak{R}$. Indeed, $\mathfrak{R}^* = \boldsymbol{L}_d$ gives $U_s^*(I) = \infty$ ξ-a.s. for some finite interval I, which in turn implies $\mathfrak{R} = \boldsymbol{L}_d$, by Proposition 3, and thus $U_s(I) = \infty$ ξ-a.s. for some finite interval I. For the converse it suffices to note that $(M_n, S_n)_{n \geq 0}$ is the dual of $(M_n^*, S_n^*)_{n \geq 0}$.

Proof. Suppose there is a finite interval I such that $U_s^*(I) = \infty$ for ξ--almost all $s \in \mathscr{S}$. It is no loss of generality to assume $I = (0, 1]$ in the following argument. Obviously, $\mathscr{S}$ (minus a ξ-null set) can be split into two subsets $\mathscr{S}_{1,1}$ and $\mathscr{S}_{1,2}$, defined by

$$\mathscr{S}_{1,1} \stackrel{\text{def}}{=} \{s \in \mathscr{S}: \sum_{n \geq 0} P_s(S_n^* \in (0, 1/2]) = \infty\},$$

$$\mathscr{S}_{1,2} = \{s \in \mathscr{S} - \mathscr{S}_{1,1}: \sum_{n \geq 0} P_s(S_n^* \in (1/2, 1]) = \infty\}.$$

Going on this way we obtain, for each $m \geq 1$, a partition $\mathscr{S}_{m,k}$, $1 \leq k \leq 2^m$, of $\mathscr{S}$ (minus a ξ-null set) such that

$$\sum_{n \geq 0} P_s(S_n^* \in I_{m,k}) = \sum_{n \geq 0} P_s(S_{-n} \in -I_{m,k}) = \infty, \qquad I_{m,k} \stackrel{\text{def}}{=} ((k-1)2^{-m}, k2^{-m}]),$$

for all $s \in \mathscr{S}_{m,k}$. Now fix m, k and define the disjoint sets

$$A_0 \stackrel{\text{def}}{=} \{S_n \notin -I_{m,k} \text{ for all } n \geq 1\},$$

$$A_j \stackrel{\text{def}}{=} \{S_j \in -I_{m,k},\ S_{j+n} \notin -I_{m,k} \text{ for all } n \geq 1\}, \quad j \geq 1,$$

and observe that

$$\{N(-I_{m,k}) < \infty\} = \sum_{k \geq 0} A_k.$$

For $j \geqslant 1$,

$$P_\xi(A_j) \geqslant P_\xi(S_j \in -I_{m,k},\ |S_{j+n}-S_j| \geqslant 2^{-m} \text{ for all } n \geqslant 1)$$

$$\geqslant \int_{\mathscr{S}_{m,k}} P_s(S_j^* \in I_{m,k}) P_s(|S_n| \geqslant 2^{-m} \text{ for all } n \geqslant 1)\, \xi(ds),$$

whence, by summing over j,

$$P_\xi(N(-I_{m,k}) < \infty) \geqslant \int_{\mathscr{S}_{m,k}} P_s(|S_n| \geqslant 2^{-m} \text{ for all } n \geqslant 1) \sum_{j \geqslant 1} P_s(S_j^* \in I_{m,k})\, \xi(ds).$$

Since the sum under the integral is infinite, we have

$$P_s(|S_n| \geqslant 2^{-m} \text{ for all } n \geqslant 1) = 0$$

for ξ-almost all $s \in \mathscr{S}_{m,k}$. But m, k were chosen arbitrarily so that we obtain

$$P_s(|S_n| \geqslant \varepsilon \text{ for all } n \geqslant 1) = 0 \tag{2.2}$$

for ξ-almost all $s \in \mathscr{S}$ and all $\varepsilon > 0$.

Now consider the A_j with $I_{m,k}$ replaced by $B_\varepsilon(0) = -B_\varepsilon(0)$ with an arbitrary $\varepsilon > 0$. Then

$$A_j = \lim_{\delta \uparrow \varepsilon} \{S_j \in B_\delta(0),\ S_{j+n} \notin B_\varepsilon(0) \text{ for all } n \geqslant 1\}$$

implies

$$P_\xi(A_j) \leqslant P_\xi(S_j \in B_\delta(0),\ |S_{n+j}-S_j| \geqslant \varepsilon-\delta \text{ for all } n \geqslant 1)$$

$$= \int_{\mathscr{S}} P_s(S_j^* \in B_\delta(0)) P_s(|S_n| \geqslant \varepsilon-\delta \text{ for all } n \geqslant 1)\, \xi(ds) = 0,$$

that is $P_\xi(A_j) = 0$ for all $j \geqslant 1$. Use (2.2) directly to get

$$P_\xi(A_0) = P_\xi(|S_n| \geqslant \varepsilon \text{ for all } n \geqslant 1) = 0.$$

Thus we have shown that $P_\xi(N(B_\varepsilon(0)) = \infty) = 1$ for every $\varepsilon > 0$, that is $0 \in \mathfrak{R}$. Proposition 2 finally gives $\mathfrak{R} = L_d$.

Proof of Theorem 1. The first assertion of Theorem 1 follows from Proposition 2, whence it only remains to show the equivalence of (a) through (d). The implications (d) $\Rightarrow$ (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are obvious, the non-trivial step thus being only (c) $\Rightarrow$ (d), which we now prove by contradiction.

Suppose there is a ξ-positive set $\mathscr{S}_0$ and a bounded interval I such that $U_s(I) = \infty$ for all $s \in \mathscr{S}_0$. We must show that $\mathfrak{R} \neq \emptyset$. Choose an arbitrary ξ-positive subset C of $\mathscr{S}_0$, put $\xi(\cdot | C) = \xi(\cdot \cap C)/\xi(C)$, and consider the MRW

$$(M_n^C, S_n^C)_{n \geqslant 0} \stackrel{\text{def}}{=} (M_{\sigma_n}, S_{\sigma_n})_{n \geqslant 0}, \tag{2.3}$$

where $\sigma_n = \sigma_n(C)$ denote the successive visit times of M to C. The imbedded chain $M^C = (M_n^C)_{n\geqslant 0}$ has the unique stationary distribution $\xi(\cdot\,|\,C)$ and $E_{\xi(\cdot|C)}\sigma_1 = 1/\xi(C) < \infty$, where, by Birkhoff's ergodic theorem,

$$\xi(C) = \lim_{n\to\infty} \frac{1}{\sigma_n} \sum_{k=1}^{\sigma_n} 1_{\{M_k\in C\}} = \lim_{n\to\infty} \frac{n}{\sigma_n} = \frac{1}{E_{\xi(\cdot|C)}\sigma_1} \quad P_{\xi(\cdot|C)}\text{-a.s.}$$

Let $V_s(\cdot\,|\,C) = \sum_{n\geqslant 0} P_s((M_n^C, S_n^C)\in\cdot)$ be the Markov renewal measure of $(M_n^C, S_n^C)_{n\geqslant 0}$ under P_s and $U_s(\cdot\,|\,C) = \sum_{n\geqslant 0} P_s(S_n^C\in\cdot)$ its second marginal. It follows from the above assumption and by the strong Markov property that

$$\infty = U_{\xi(\cdot|C)}(I) = \sum_{n\geqslant 0} E_{\xi(\cdot|C)}\Big(\sum_{k=\sigma_n}^{\sigma_{n+1}-1} 1_{\{S_k\in I\}}\Big) = \int_{C\times I} E_s\Big(\sum_{k=0}^{\sigma_1-1} 1_{\{S_k\in I-t\}}\Big)\, V_{\xi(\cdot|C)}(ds\times dt)$$

$$\leqslant E_{\xi(\cdot|C)}\sigma_1 V_{\xi(\cdot|C)}(C\times I) = \frac{U_{\xi(\cdot|C)}(I\,|\,C)}{\xi(C)},$$

and therefore (use $U_s(\cdot\,|\,\mathscr{S}_0) \geqslant U_s(\cdot\,|\,C)$ for all $s\in\mathscr{S}$ and $C\subset\mathscr{S}_0$)

$$\infty = U_{\xi(\cdot|C)}(I\,|\,C) = \xi(C)^{-1}\int_C U_s(I\,|\,C)\,\xi(ds) \leqslant \xi(C)^{-1}\int_C U_s(I\,|\,\mathscr{S}_0)\,\xi(ds)$$

for all ξ-positive $C\subset\mathscr{S}_0$. From this we conclude that $U_s(I\,|\,\mathscr{S}_0) = \infty$ for ξ-almost all $s\in\mathscr{S}_0$, which together with Proposition 3 shows that the recurrence set of $(S_n^{\mathscr{S}_0})_{n\geqslant 0}$, and thus also of $(S_n)_{n\geqslant 0}$ itself, is non-empty.

Theorem 2 follows immediately when combining Theorem 1 with the following proposition, a proof of which may be found in [5] (see also the references therein). It is based on a clever subadditivity argument and an application of Kingman's subadditive ergodic theorem.

PROPOSITION 4. *If* (1.2) *holds true, then* $P_\xi(N(B_\varepsilon(0)) = \infty) = 1$ *for all* $\varepsilon > 0$, *i.e.* $0\in\mathfrak{R}$.

We must finally prove Theorem 3.

Proof of Theorem 3. Without loss of generality suppose $d = 1$ and define for $k\geqslant 2$

$$\mathscr{C}_\gamma(k) = \{y\in Q\colon\ |y-x| < 1/2k \text{ for some } x\in\mathscr{C}_\gamma\},$$

and furthermore

$$\mathfrak{R}_k = \overline{\bigcup_{y\in\mathscr{C}_\gamma(k)} (\gamma(M_0) - B_{1/k}(y) + Z)}.$$

Fix any $k\geqslant 2$ and $y\in\mathscr{C}_\gamma(k)$ and put $C = \gamma^{-1}(B_{1/k}(y))$. Notice that

$$\xi(C) = P_\xi^{\gamma(M_0)}(B_{1/k}(y)) \geqslant P_\xi^{\gamma(M_0)}(B_{1/2k}(x)) > 0 \quad \text{for some } x\in\mathscr{C}_\gamma.$$

Hence M visits the set C infinitely often P_ξ-a.s. and we can consider the MRW $(M_n^C, S_n^C)_{n\geq 0}$ defined in (2.3), where $M_0^C = M_0$ and $S_0^C = S_0 = 0$. Under a probability measure P' equivalent to P_ξ (Palm duality, see e.g. [11]), the sequence $(M_n^C)_{n\geq 1}$ is stationary with $P'(M_n^C\in\cdot) = \xi(\cdot\,|\,C)$. It follows by Lemma 1 below that $(M_n^C, S_n^C)_{n\geq 0}$ is again 1-arithmetic with shift function γ. Consequently, by Theorem 2,

$$P_\xi\big(S_n^C-\gamma(M_0)+\gamma(M_0^C) = m \text{ i.o.}\big) = 1 \quad \text{for all } m\in \boldsymbol{Z},$$

which in turn yields

$$P_\xi\big(S_n^C\in B_{2/k}\big(\gamma(M_0)-x+m\big) \text{ i.o.}\big) = 1 \quad \text{for all } m\in \boldsymbol{Z},$$

because $\gamma(M_n^C)\in B_{1/k}(y)\subset B_{2/k}(x)$ for each $n\geq 1$. But y and k were arbitrarily chosen and each $(S_n^C)_{n\geq 0}$ is a subsequence of $(S_n)_{n\geq 0}$. Hence $\mathfrak{R}$ contains

$$\bigcap_{k\geq 2}\ \bigcap_{y\in\mathscr{C}_\gamma(k)} \mathfrak{R}_k = \overline{\bigcup_{x\in\mathscr{C}_\gamma} \big(\gamma(M_0)-x+d\boldsymbol{Z}\big)}\ P_\xi\text{-a.s.}$$

On the other hand, the recurrence set of $\big(S_n-\gamma(M_0)+\gamma(M_n)\big)_{n\geq 0}$ being $d\boldsymbol{Z}$, the recurrence set $\mathfrak{R}$ of $(S_n)_{n\geq 0}$ itself cannot be larger than

$$\overline{\big\{\gamma(M_0)-x+m\colon m\in\boldsymbol{Z} \text{ and } x \text{ a recurrence point for } \big(\gamma(M_n)\big)_{n\geq 0}\big\}}\ P_\xi\text{-a.s.}$$

and it follows from the ergodicity of $(M_n)_{n\geq 0}$ that the latter set coincides with the one in the previous display line. Finally, $P_\xi(d\boldsymbol{Z}\subset\mathfrak{R}) = 1$ is a trivial consequence of $P_\xi\big(\gamma(M_0)\in\mathscr{C}_\gamma\big) = 1$.

It was crucial for the proof of Theorem 3 that, for any ξ-positive set C, $(M_n^C, S_n^C)_{n\geq 0}$ is of the same lattice-type as $(M_n, S_n)_{n\geq 0}$ itself. This is not as obvious as one might think at first glance and thus shown as a separate lemma.

LEMMA 1. *Let $(M_n, S_n)_{n\geq 0}$ be an MRW as in Theorem 1 with lattice-span $d\in[0, \infty)$ and shift function γ in the case $d > 0$. Then, for each ξ-positive $C\in\mathfrak{S}$, the imbedded MRW $(M_n^C, S_n^C)_{n\geq 0}$ has the same lattice-span and the same shift function.*

Proof. Suppose that, for some ξ-positive C, $(M_n^C, S_n^C)_{n\geq 0}$ is d'-arithmetic, $d' > d$, with shift function γ'. Without loss of generality let $d' = 1$ and $\gamma'\equiv 0$. Hence

$$E_{\xi(\cdot|C)}\exp(2\pi i S_1^C) = E_{\xi(\cdot|C)}\exp(2\pi i S_{\sigma_1(C)}) = 1.$$

Let η be a geometric (1/2)-variable on the positive integers, which is independent of $(M_n, S_n)_{n\geq 0}$ and $\varrho_n = \inf\{k\geq \eta+n\colon M_k\in C\}$. We claim that, for suitable $\vartheta\colon \mathscr{S}^2\to[0, 1)$,

$$E\big(\exp(2\pi i S_\eta)\mid M_0, M_\eta\big) = \exp\big(2\pi i\vartheta(M_0, M_\eta)\big)\ P_\xi\text{-a.s.} \tag{2.4}$$

Indeed, by conditional independence of S_n, $S_{n+\eta}-S_n$ and $S_{\varrho_n}-S_{n+\eta}$ given M_0, M_n, $M_{n+\eta}$ and M_{ϱ_n}, we obtain, for each $n\geq 0$,

$$\begin{aligned}1 &= E_{\xi(\cdot|C)}\exp(2\pi i S_{\varrho_n})\\ &= E_{\xi(\cdot|C)}\Big(E\big(\exp(2\pi i S_n)\mid M_0, M_n\big)E\big(\exp\big(2\pi i(S_{n+\eta}-S_n)\big)\mid M_n, M_{n+\eta}\big)\\ &\quad\times E\big(\exp\big(2\pi i(S_{\varrho_n}-S_{n+\eta})\big)\mid M_{n+\eta}, M_{\varrho_n}\big)\Big).\end{aligned}$$

Thus

$$\left|E\left(\exp(2\pi i(S_{n+\eta}-S_n)) \mid M_n = x,\ M_{n+\eta} = y\right)\right| = 1 \quad P_{\xi(\cdot|C)}^{(M_n, M_{n+\eta})}\text{-a.s.}$$

Then the equality (2.4) follows from

$$E\left(\exp(2\pi i(S_{n+\eta}-S_n)) \mid M_n = x,\ M_{n+\eta} = y\right)$$
$$= E\left(\exp(2\pi i S_\eta) \mid M_0 = x,\ M_\eta = y\right) \quad P_{\xi(\cdot|C)}^{(M_n, M_{n+\eta})}\text{-a.s.}$$

and the fact that $P_\xi^{(M_0, M_\eta)}$ and $\sum_{n\geqslant 0} 2^{-n} P_{\xi(\cdot|C)}^{(M_n, M_{n+\eta})}$ are equivalent measures (ergodicity). As a consequence,

(2.5) $$E\left(\exp(2\pi i S_n) \mid M_0,\ M_n\right) = \exp\left(2\pi i \vartheta(M_0, M_n)\right) \quad P_\xi\text{-a.s.}$$

for all $n \geqslant 1$, because

$$1 = E_\xi \exp\left(2\pi i(S_\eta - \vartheta(M_0, M_\eta))\right) = \sum_{n\geqslant 1} \int_{\{\eta = n\}} \exp\left(2\pi i(S_n - \vartheta(M_0, M_n))\right) dP_\xi$$
$$= \sum_{n\geqslant 1} \frac{1}{2^n} E_\xi \exp\left(2\pi i(S_n - \vartheta(M_0, M_n))\right).$$

Further the equality (2.5) can easily be extended to arbitrary stopping times $\tau \geqslant 1$ for $M = (M_n)_{n\geqslant 0}$, i.e.

$$E\left(\exp(2\pi i S_\tau) \mid M_0,\ M_\tau\right) = \exp\left(2\pi i \vartheta(M_0, M_\tau)\right) \quad P_\xi\text{-a.s.}$$

Indeed, $P_\xi\left(S_n - \vartheta(M_0, M_n) \in \mathbf{Z} \text{ for all } n \geqslant 1\right) = 1$, by (2.5), implies

$$P_\xi\left(S_\tau - \vartheta(M_0, M_\tau) \in \mathbf{Z}\right) = 1.$$

Notice next that, for each $n \geqslant 1$ and each stopping time $\tau > n$, by conditional independence of $S_\tau - S_n$ and S_n given M_0, M_n and M_τ, we obtain

$$1 = E_{\xi(\cdot|C)} \exp\left(2\pi i(S_\tau - \vartheta(M_0, M_\tau))\right)$$
$$= E_{\xi(\cdot|C)} \exp\left(-2\pi i \vartheta(M_0, M_\tau)\right)$$
$$\times\left(E\left(\exp(2\pi i S_n) \mid M_0, M_n\right) E\left(\exp(2\pi i(S_\tau - S_n)) \mid M_n, M_\tau\right)\right)$$
$$= E_{\xi(\cdot|C)} \exp\left(2\pi i(\vartheta(M_0, M_n) + \vartheta(M_n, M_\tau) - \vartheta(M_0, M_\tau))\right),$$

and therefore

(2.6) $$\vartheta(M_0, M_n) + \vartheta(M_n, M_\tau) \equiv_{\mathbf{Z}} \vartheta(M_0, M_\tau) \quad P_{\xi(\cdot|C)}\text{-a.s.}$$

as well as (put $\tau = n+1$)

(2.7) $$\vartheta(M_0, M_{n+1}) - \vartheta(M_0, M_n) \equiv_{\mathbf{Z}} \vartheta(M_n, M_{n+1}) \quad P_{\xi(\cdot|C)}\text{-a.s.}$$

for all $n \geqslant 1$, where $\equiv_{\mathbf{Z}}$ means equivalence modulo integers.

Now let $(\sigma_m^n(C))_{m\geqslant 1}$ denote the sequence of successive visit epochs after n of the chain M to the set C, in particular $\sigma_m^0(C) = \sigma_m(C)$ for all $m \geqslant 1$.

By assumption, $\vartheta(M_0, M_{\sigma_m^n}(C)) = 0$ for all n, m. Consequently, from (2.6) we infer that

$$\vartheta(M_0, M_n) + \vartheta(M_n, M_{\sigma_m^n(C)}) \in Z \ P_{\xi(\cdot|C)}\text{-a.s.}$$

for all m, n. But $\vartheta(\cdot, \cdot) \in [0, 1)$ implies that $P_{\xi(\cdot|C)}$-a.s.

$$\vartheta(M_n, M_{\sigma_m^n(C)}) = \begin{cases} \vartheta(M_0, M_n) & \text{if } \vartheta(M_0, M_n) = 0, \\ 1 - \vartheta(M_0, M_n) & \text{if } \vartheta(M_0, M_n) \in (0, 1) \end{cases} \tag{2.8}$$

for all m, n. In any case we conclude that

$$P_{\xi(\cdot|C)}\Big(\lim_{N\to\infty} \frac{1}{N} \sum_{m=1}^{N} \vartheta(M_n, M_{\sigma_m^n(C)}) \text{ exists}\Big) = 1$$

for each $n \geq 0$. Put $\xi_n(\cdot \mid C) \overset{\text{def}}{=} P_{\xi(\cdot|C)}^{M_n}$ and use time-homogeneity to see further that

$$P_{\xi(\cdot|C)}\Big(\lim_{N\to\infty} \frac{1}{N} \sum_{m=1}^{N} \vartheta(M_n, M_{\sigma_m^n(C)}) \text{ exists}\Big) = P_{\xi_n(\cdot|C)}\Big(\lim_{N\to\infty} \frac{1}{N} \sum_{m=1}^{N} \vartheta(M_0, M_{(m)}^{C}) \text{ exists}\Big)$$

for all $n \geq 0$. Since ξ and $\sum_{n\geq 0} 2^{-(n+1)} \xi_n(\cdot \mid C)$ are equivalent distributions, it follows that

$$\theta(M_0) \overset{\text{def}}{=} \lim_{N\to\infty} \frac{1}{N} \sum_{m=1}^{N} \vartheta(M_0, M_{(m)}^{C}) \in [0, 1]$$

exists P_ξ-a.s. Now go back to (2.8), sum over $m = 1, \ldots, N$ to get

$$0 = \vartheta(M_0, M_n) = \frac{1}{N} \sum_{m=1}^{N} \vartheta(M_n, M_{\sigma_m^n(C)}) \to \theta(M_n) \ P_{\xi(\cdot|C)}\text{-a.s.}$$

or

$$(0, 1) \ni \vartheta(M_0, M_n) = 1 - \frac{1}{N} \sum_{m=1}^{N} \vartheta(M_n, M_{\sigma_m^n(C)}) \to 1 - \theta(M_n) \ P_{\xi(\cdot|C)}\text{-a.s.},$$

which can be summarized as

$$\vartheta(M_0, M_n) = \hat{\theta}(M_n) \ P_{\xi(\cdot|C)}\text{-a.s.} \quad \text{with } \hat{\theta}(x) \overset{\text{def}}{=} (1 - \theta(x)) \mathbf{1}_{\{\theta(x) > 0\}}.$$

Finally, this gives in (2.7)

$$\vartheta(M_n, M_{n+1}) \equiv_Z \hat{\theta}(M_{n+1}) - \hat{\theta}(M_n) \ P_{\xi(\cdot|C)}\text{-a.s.}$$

for all $n \geq 1$ or, equivalently,

$$\vartheta(M_0, M_1) \equiv_Z \hat{\theta}(M_1) - \hat{\theta}(M_0) \ P_\xi\text{-a.s.},$$

an obvious contradiction to the assumption that $(M_n, S_n)_{n\geq 0}$ is d-arithmetic for some $d < 1$.

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