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NOTE ON ASYMPTOTIC NORMALITY OF KERNEL DENSITY ESTIMATOR FOR LINEAR PROCESS UNDER SHORT-RANGE DEPENDENCE

BY

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Abstract. We consider the problem of density estimation for a one-sided linear process $X_t = \sum_{r=0}^{\infty} a_r Z_{t-r}$ with i.i.d. square integrable innovations $(Z_i)_{i=-\infty}^{\infty}$. We prove that under weak conditions on $(a_i)_{i=0}^{\infty}$, which imply short-range dependence of the linear process, finite-dimensional distributions of kernel density estimate are asymptotically multivariate normal. In particular, the result holds for $|a_n| = \mathcal{O}(n^{-a})$ with $a > 2$, which is much weaker than previously known sufficient conditions for asymptotic normality. No conditions on bandwidths b_n are assumed besides $b_n \to 0$ and $nb_n \to \infty$. The proof uses Chanda's [1], [2] conditioning technique as well as Bernstein's "large block-small block" argument.

1. Introduction. Let $X_1, X_2, \ldots, X_n$ be n consecutive observations of a linear process

$$X_t = \sum_{r=0}^{\infty} a_r Z_{t-r}, \quad t = 1, 2, \ldots, n, \tag{1.1}$$

where $(Z_i)_{i=-\infty}^{\infty}$ is an innovation process consisting of i.i.d. random variables with mean zero and finite variance. Assume that X_1 has a probability density f, which we wish to estimate. As an estimator of f we will consider the standard kernel type estimator (see e.g. Chanda [1]) given by

$$f_n(x) = \sum_{j=1}^{n} K\big((x - X_j)/b_n\big)/(nb_n) \tag{1.2}$$

for $x \in \boldsymbol{R}$, where b_n is a sequence of positive numbers such that $b_n \to 0$ and $nb_n \to \infty$ as $n \to \infty$, and K is a bounded density function.

Chanda [1], [2] showed that one-dimensional distributions of f_n are asymptotically normal under a general condition on $(a_i)_{i=0}^{\infty}$ and provided $E|Z_1|^{\varepsilon} < \infty$

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for some $\varepsilon > 0$, $b_n \to 0$ but $nb_n \to \infty$, and assumptions (A.1) and (A.4) (listed in our Section 2) hold. The condition on $(a_i)_{i=0}^{\infty}$ is $\sum_{r=j}^{\infty} r\,|a_r|^{\alpha} = \mathcal{O}(j^{-\theta})$ for some $\theta \geqslant 1$, where $\alpha = \delta/2(1+\delta)$ and $\delta = \varepsilon$ if $0 \leqslant \varepsilon < 2$ and $\delta = 2$ if $\varepsilon \geqslant 2$. In particular, if the innovations have a finite second moment ($\varepsilon = 2$), it yields $\sum_{r=j}^{\infty} r\,|a_r|^{1/3} = \mathcal{O}(j^{-\theta})$, which is much stronger than conditions (A.5) and (A.6) assumed in Section 2. In the case of innovations having finite second moment, Hallin and Tran [4] proved asymptotic multivariate normality of finite-dimensional distributions of scaled and centered f_n provided that assumptions (A.0) and (A.3) hold, the characteristic function of Z_1 belongs to $L^1(\mathbf{R})$, the coefficients of the linear process X_t tend to zero in such a way that $|a_r| = \mathcal{O}(r^{-(4+\sigma)})$ for some $\sigma > 0$ as $r \to \infty$ and the bandwidth b_n tends to zero sufficiently slowly so that

$$nb_n^{(13+2\sigma)/(3+2\sigma)} (\log\log n)^{-1} \to \infty.$$

The aim of the present note is to show that under assumptions on the distribution of Z_1 and K, which are comparable to those of Hallin and Tran [4], finite-dimensional distributions of f_n are asymptotically normal for a much wider class of linear processes. Moreover, in contrast to Hallin and Tran [4], the sole conditions imposed on the bandwidths are $b_n \to 0$ and $nb_n \to \infty$, which are identical to the usual condition imposed in the independent case.

2. Assumptions and the statement of the main result.

(A.0) The kernel K satisfies the Lipschitz condition.

(A.1) For every real a, $\int_{-\infty}^{\infty} |K(y+a) - K(y)|\,dy \leqslant M|a|$, where M denotes a generic constant.

(A.2) The support of K is compact.

(A.3) $Q(x) := \sup\{K(y)\colon\ |y| \geqslant |x|\}$ is integrable.

(A.4) $\int_{-\infty}^{\infty} |u\phi(u)|\,du < \infty$, where ϕ denotes the characteristic function of Z_1.

(A.5) $\sum_{r=j}^{\infty} a_r^2 = \mathcal{O}(j^{-(3+\sigma)})$ for some $\sigma > 0$.

(A.6) $\sum_{r=j}^{\infty} |a_r| = \mathcal{O}(j^{-(2+\sigma)})$ for some $\sigma > 0$.

We now state the main result of the paper.

THEOREM 2.1. *Suppose assumptions* (A.1), (A.4) *and* (A.5) *hold true and* $x_1, \ldots, x_s$ *are* s *distinct points of* $\mathbf{R}$. *Then*

$$T_n(x_1, \ldots, x_s) = (nb_n)^{1/2}\big(f_n(x_1) - Ef_n(x_1), \ldots, f_n(x_s) - Ef_n(x_s)\big) \xrightarrow{\mathcal{D}} N(\mathbf{0}, \Sigma), \tag{2.1}$$

where Σ *is a diagonal matrix with diagonal elements* $\sigma_{i,i} = f(x_i)\int_{-\infty}^{\infty} K^2(u)\,du$ *for* $i = 1, \ldots, s$.

3. Some auxiliary results. Let $\|\cdot\|_\infty$ denote the supremum norm in the space under consideration.

LEMMA 3.1. *If assumption* (A.4) *holds true, then*

(3.1) *the probability density f is bounded and continuous*;

(3.2) *the density $h_{l,t}$ of $X_{t,l} := \sum_{r=0}^{l-1} a_r Z_{t-r}$ satisfies the Lipschitz condition for all $l > 0$ and* $\sup_{l\in N} \|h_{l,t}\|_\infty \leqslant M$;

(3.3) *the density $g_{l,t}$ of $R_{t,l} := X_t - X_{t,l} = \sum_{r=l}^{\infty} a_r Z_{t-r}$ exists for all $l > 0$*;

(3.4) *the joint probability density $f_j(x, y)$ of (X_1, X_{j+1}) exists for all $j \in N$, and*

(3.5) $\sup_{j\in N} \|f_j\|_\infty \leqslant M$;

(3.6) *the joint probability density $f_{i,j}(x, y, z)$ of $(X_1, X_{i+1}, X_{i+j+1})$ exists for all $i, j \in N$, and*

(3.7) $\sup_{i,j\in N} \|f_{i,j}\|_\infty \leqslant M$.

Proof. The proof of (3.1) is straightforward, and therefore will be omitted. Relation (3.2) follows from (2.6) in Chanda [1]. For a fixed $l \in N$ we have

$$\int_{-\infty}^{\infty} \prod_{r=l}^{\infty} |\phi(a_r u)|\, du \leqslant \int_{-\infty}^{\infty} |\phi(a_l u)|\, du.$$

Substituting $z = a_l u$, we get

$$\int_{-\infty}^{\infty} |\phi(a_l u)|\, du \leqslant |a_l|^{-1} \int_{-\infty}^{\infty} |\phi(z)|\, dz < \infty,$$

and thus the characteristic function of $R_{t,l}$ belongs to $L^1(\boldsymbol{R})$, and (3.3) follows.

To prove (3.4) and (3.6) it suffices to show that for any $i = 1, 2, \ldots$ and any $j = 1, 2, \ldots$ the characteristic function $\hat{f}_j$ of (X_1, X_{j+1}) belongs to $L^1(\boldsymbol{R}^2)$ and the characteristic function $\hat{f}_{i,j}$ of $(X_1, X_{i+1}, X_{i+j+1})$ belongs to $L^1(\boldsymbol{R}^3)$. We use the method employed in Giraitis et al. [3]. Let us note first that for any $j \in N$

$$|\hat{f}_j(u, v)| \leqslant |\phi(ua_0 + va_j)\, \phi(va_0)| \tag{3.8}$$

and for any $i, j \in N$

$$|\hat{f}_{i,j}(u, v, z)| \leqslant |\phi(ua_0 + va_i + za_{i+j})\, \phi(va_0 + za_j)\, \phi(za_0)|, \tag{3.9}$$

since

$$\hat{f}_j(u, v) = \prod_{s=-\infty}^{j} \phi(ua_{-s} + va_{j-s}), \qquad \hat{f}_{i,j}(u, v, z) = \prod_{s=-\infty}^{i+j} \phi(ua_{-s} + va_{j-s} + za_{i+j-s}),$$

where $a_j = 0$ for $j < 0$ by definition. Substituting $z_1 = ua_0 + va_i + za_{i+j}$, $z_2 = va_0 + za_j$, $z_3 = za_0$, we get

$$\int_{R^3} |\phi(ua_0+va_i+za_{i+j})\,\phi(va_0+za_j)\,\phi(za_0)|\,dudvdz = |a_0|^{-3}\Big(\int_{-\infty}^{\infty} |\phi(z)|\,dz\Big)^3. \tag{3.10}$$

Now it follows from (3.9) and (3.10) that $\hat{f}_{i,j}(u, v, z) \in L^1(R^3)$, and using the Fourier inversion formula we obtain

$$\|f_{i,j}\|_\infty \leqslant \frac{1}{2\pi}|a_0|^{-3}.$$

Thus (3.7) is satisfied. Similarly we can show (3.5).

Let

$$U_j := n^{-1/2} \sum_{t=(j-1)(p+q)+1}^{j(p+q)-q} Y_t, \tag{3.11}$$

$$Y_t := cY_t^{(x)} + dY_t^{(y)}, \qquad Y_t^{(\cdot)} = b_n^{-1/2}\big(K((\cdot - X_t)/b_n) - EK((\cdot - X_t)/b_n)\big). \tag{3.12}$$

LEMMA 3.2. *Let $p = p(n)$, $q = q(n)$ and $k = k(n)$ be sequences of positive integers such that*

$$k = \left[\frac{n}{p+q}\right],\ p, q, k \to \infty, \quad \text{and} \quad \frac{q}{p} \to 0 \ \text{as} \ n \to \infty.$$

Assume that one of the following conditions (i) *or* (ii) *is satisfied*:

(i) $(n/b_n)^{1/2} p^{-1} q^{-(1/2+\sigma/2)} = o(1)$ *and assumptions* (A.1), (A.4) *and* (A.5) *hold true*;

(ii) $(n/b_n^3)^{1/2} p^{-1} q^{-(1+\sigma)} = o(1)$ *and assumptions* (A.0), (A.4) *and* (A.6) *hold true.*

Then

$$E\exp\big(iu\sum_{j=1}^{k} U_j\big) - \prod_{j=1}^{k} E\exp(iuU_j) = o(1) \quad \textit{for every real } u.$$

Proof. Let $\varphi^{(j)}$ denote the characteristic function of $(U_1, \ldots, U_j)$ and let φ_j be the characteristic function of U_j. Then (see (2.15) in Chanda [1])

$$\Big|\varphi^{(k)}(u, \ldots, u) - \prod_{j=1}^{k}\varphi_j(u)\Big| \leqslant \sum_{j=2}^{k} |\varphi^{(j)}(u, \ldots, u) - \varphi_j(u)\,\varphi^{(j-1)}(u, \ldots, u)| = \sum_{j=2}^{k} |EN_jP_j| = \sum_{j=2}^{k} |EN_jP_j^*|, \tag{3.13}$$

where

$$N_j = \exp\Big(iu \sum_{r=1}^{j-1} U_r\Big) - \varphi^{(j-1)}(u, \ldots, u), \qquad P_j = \exp(iuU_j),$$

$$P_j^* := P_j - E(P_j \mid \eta_j), \qquad \eta_j := \sigma(Z_{(j-1)(p+q)-q+1}, \ldots, Z_{j(p+q)-q})$$

for $j = 2, \ldots, k$. We have

$$|EN_j P_j^*| \leqslant 2E|P_j^*| \leqslant 2E|P_j - \xi_j| + 2E\big|\xi_j - E(P_j \mid \eta_j)\big|,$$

where

$$\xi_j = \exp\left\{\frac{iu}{n^{1/2}} \sum_{l=1}^{p} \big(c\tilde{Y}^{(x)}_{(j-1)(p+q)+l,l} + d\tilde{Y}^{(y)}_{(j-1)(p+q)+l,l}\big)\right\}$$

and

$$\tilde{Y}^{(\cdot)}_{t,l} = b_n^{-1/2}\big(K((\cdot - X_{t,l+q})/b_n) - EK((\cdot - X_t)/b_n)\big).$$

Since ξ_j is an η_j-measurable random variable for $j = 1, \ldots, k$, we obtain

$$E\big|\xi_j - E(P_j \mid \eta_j)\big| = E\big|E(\xi_j - P_j \mid \eta_j)\big| \leqslant E|\xi_j - P_j|.$$

Moreover, since $|\exp(ia) - 1| \leqslant M|a|$ for every real a, we have

$$|EN_j P_j^*| \leqslant 4E|P_j - \xi_j| \leqslant \frac{M}{(nb_n)^{1/2}} \sum_{l=1}^{p} \big(I_x(l) + I_y(l)\big), \tag{3.14}$$

where

$$I_z(l) = E\left|K\left(\frac{z - X_{(j-1)(p+q)+l}}{b_n}\right) - K\left(\frac{z - X_{(j-1)(p+q)+l,l+q}}{b_n}\right)\right| \quad \text{for } z \in \boldsymbol{R}.$$

By (3.2) and (3.3) we have

$$I_x(l) = \int_{\boldsymbol{R}^2} \left|K\left(\frac{x-u-v}{b_n}\right) - K\left(\frac{x-u}{b_n}\right)\right| h(u)\, g(v)\, du\, dv,$$

where $h := h_{l+q,(j-1)(p+q)+l}$ and $g := g_{l+q,(j-1)(p+q)+l}$.

Put

$$\tilde{I}_x(v) = \int_{-\infty}^{\infty} \left|K\left(\frac{x-u-v}{b_n}\right) - K\left(\frac{x-u}{b_n}\right)\right| h(u)\, du.$$

Ad (i). Substituting $z = (x-u-v)/b_n$ and applying (3.2) and (A.1) we have

$$\tilde{I}_x(v) = b_n \int_{-\infty}^{\infty} \left|K(z) - K\left(z + \frac{v}{b_n}\right)\right| h(x - v - b_n z)\, dz \leqslant M|v|,$$

and thus $I_x(l)$ is not greater than

$$M\int_{R} |v|\, g(v)\, dv = ME\Big|\sum_{r=l+q}^{\infty} a_r Z_{(j-1)(p+q)+l-r}\Big|$$
$$\leqslant M\Big(\sum_{r=l+q}^{\infty} a_r^2\Big)^{1/2} \leqslant M(l+q)^{-(3/2+\sigma/2)}.$$

This implies that the right-hand side of inequality (3.13) is not greater than

$$M\frac{k}{(nb_n)^{1/2}} q^{-(1/2+\sigma/2)} \to 0 \quad \text{as } n\to\infty.$$

Ad (ii). By (A.0) we have $\tilde{I}_x(v) \leqslant M|v|/b_n$ and

$$I_x(l) \leqslant Mb_n^{-1} E\Big|\sum_{r=l+q}^{\infty} a_r Z_{(j-1)(p+q)+l-r}\Big| \leqslant Mb_n^{-1} \sum_{r=l+q}^{\infty} |a_r| \leqslant Mb_n^{-1}(l+q)^{-(2+\sigma)}.$$

This implies that the right-hand side of the inequality in (3.13) is not greater than

$$\frac{Mk}{(nb_n)^{1/2}} b_n^{-1} q^{-(1+\sigma)} \to 0 \quad \text{as } n\to\infty.$$

Write $\Omega_u = \sigma(\ldots, Z_{u-1}, Z_u)$, $u = 1, 2, \ldots$, and

$$J_{u,t}^x(s) = J_u^x(s) = E\Big\{K\Big(\frac{x-s-X_{t,t-u}}{b_n}\Big) - K\Big(\frac{x-X_{t,t-u}}{b_n}\Big) \,|\, \Omega_u\Big\}$$
$$= E\Big(K\Big(\frac{x-s-X_{t,t-u}}{b_n}\Big) - K\Big(\frac{x-X_{t,t-u}}{b_n}\Big)\Big), \quad u\in N, \tag{3.15}$$

$$Q_m = P(|X_k - X_{k,m}| \geqslant c_m), \quad c_m = b_n^{-\beta}\Big(\sum_{r=m}^{\infty} a_r^2\Big)^{\alpha}, \tag{3.16}$$

where

$$\alpha\in\Big(\frac{1}{3+\sigma}, \frac{2+\sigma}{2(3+\sigma)}\Big), \quad \beta > 0.$$

Observe that

$$J_{t-m}^x(s) = \int\Big(K\Big(\frac{x-s-w}{b_n}\Big) - K\Big(\frac{x-w}{b_n}\Big)\Big) h_{m,t}(w)\, dw. \tag{3.17}$$

LEMMA 3.3. *Let the conditions* (A.4) *and* (A.5) *hold. Then*

$$\sup_{m,t\in N} E\Big|K\Big(\frac{x-X_{t,m}}{b_n}\Big) - EK\Big(\frac{x-X_t}{b_n}\Big)\Big| \leqslant Mb_n \quad \textit{for every real } x; \tag{3.18}$$

$$|J_u^x(s)| \leqslant Mb_n \textit{ and } |J_u^x(s)| \leqslant Mb_n|s| \quad \textit{for every real } s,\ x \textit{ and } u\in N; \tag{3.19}$$

$$\sum_{j=1}^{\infty} |E(Y_1 Y_{j+1})| \leqslant Mb_n^{2/3}. \tag{3.20}$$

Proof. Using the triangle inequality it is enough to bound $E|K((x-X_{t,m})/b_n)|$ and $|EK((x-X_t)/b_n)|$. Since

$$EK\left(\frac{x-X_{t,m}}{b_n}\right) = b_n \int_{-\infty}^{\infty} K(z)\, h_{m,t}(x-b_n z)\, dz$$

and

$$EK\left(\frac{x-X_t}{b_n}\right) = b_n \int_{-\infty}^{\infty} K(z)\, f(x-b_n z)\, dz$$

and (3.1), (3.2) hold, we see that (3.18) is satisfied.

By argument as in the proof of (3.18), $|J_u^x(s)| \leqslant Mb_n$. On the other hand, writing

$$J_u^x(s) = b_n \int_{-\infty}^{\infty} K(z)\, h_{t-u,t}(x-s-b_n z)\, dz - b_n \int_{-\infty}^{\infty} K(z)\, h_{t-u,t}(x-b_n z)\, dz$$

and using (3.2) we get

$$|J_u^x(s)| \leqslant Mb_n |s| \int_{-\infty}^{\infty} K(z)\, dz \leqslant Mb_n |s|.$$

Thus (3.19) is satisfied.

It is clear that (3.20) is equivalent to $\sum_{j=1}^{\infty} |E(Y_1^{(x)} Y_{j+1}^{(y)})| \leqslant Mb_n^{2/3}$ for every real x, y. Of course,

$$\begin{aligned} |E(Y_1^{(x)} Y_{j+1}^{(y)})| &= b_n^{-1/2} \left|E\left(Y_1^{(x)} J_1^y (R_{j+1,j})\right)\right| \\ &\leqslant b_n^{-1/2} \left|E\left(Y_1^{(x)} J_1^y (R_{j+1,j}) I\{|R_{j+1,j}| \leqslant c_j\}\right)\right| \\ &\quad + b_n^{-1/2} \left|E\left(Y_1^{(x)} J_1^y (R_{j+1,j}) I\{|R_{j+1,j}| > c_j\}\right)\right| \leqslant Mb_n c_j + MQ_j \end{aligned}$$

by (3.16) and (3.19). Using the Chebyshev inequality we have

$$Q_m \leqslant \sum_{r=m}^{\infty} a_r^2/c_m^2 \quad \text{for every } m \in N. \tag{3.21}$$

The choice of $\beta := \frac{1}{3}$ yields (3.19).

LEMMA 3.4 (Liapunov condition). *If $\tilde{U}_1, \ldots, \tilde{U}_k$ are independent copies of* (3.11) *and*

$$\frac{p}{n^{1/2}} b_n^{1/2} \to 0 \quad \textit{as } n \to \infty,$$

then

$$\sum_{j=1}^{k} \left(\operatorname{Var}\left(\sum_{i=1}^{k} \tilde{U}_i\right)\right)^{-2} E|\tilde{U}_j|^4 \to 0 \quad \textit{as } n \to \infty. \tag{3.22}$$

Proof. Since $\operatorname{Var}(\sum_{i=1}^{k} \tilde{U}_i) = kEU_1^2$ and (3.20) implies $EU_1^2 \geqslant \sigma^2 p/2n$ for sufficiently large n (see Chanda [1], (2.22)), the left-hand side of (3.22) is equal to

$$\mathcal{O}\left(\frac{n}{p}E|\tilde{U}_1|^4\right) = \mathcal{O}\left(\frac{n}{p}E|U_1|^4\right) = \mathcal{O}\left(\frac{1}{np}E(\sum_{t=1}^{p} Y_t)^4\right).$$

Following Chanda [2] we write

$$E(\sum_{t=1}^{p} Y_t)^4 \leqslant M(pEY_1^4 + p\mathrm{I} + p^2\mathrm{II}_1 + p^2\mathrm{II}_2 + p^3\mathrm{III}), \tag{3.23}$$

where

$$\mathrm{I} = \sum_{i=1}^{p} \left(|E(Y_1^2 Y_{i+1}^2)| + |E(Y_1 Y_{i+1}^3)| + |E(Y_1^3 Y_{i+1})|\right),$$

$$\mathrm{II}_1 = \sum_{j=1}^{p} \left(|E(Y_1^2 Y_{i+1} Y_{i+j+1})| + |E(Y_1 Y_{i+1}^2 Y_{i+j+1})|\right),$$

$$\mathrm{II}_2 = \sum_{j=1}^{p} |E(Y_1 Y_{i+1} Y_{i+j+1}^2)|,$$

$$\mathrm{III} = \sum_{w=1}^{p} |E(Y_1 Y_{i+1} Y_{i+j+1} Y_{i+j+w+1})|.$$

From (3.1) and (3.5) we have $EY_1^4 = \mathcal{O}(b_n^{-1})$ and $\mathrm{I} = \mathcal{O}(p)$. Consequently,

$$\frac{pEY_1^4}{np} = \mathcal{O}((nb_n)^{-1}) = o(1) \quad \text{and} \quad \frac{p\mathrm{I}}{np} = \mathcal{O}\left(\frac{p}{n}\right) = o(1).$$

Moreover, since $Y_1^2 Y_{i+1}$ is an Ω_{i+1}-measurable random variable, we obtain

$$\begin{aligned} &|E(Y_1^2 Y_{i+1} Y_{i+j+1})| \\ &\leqslant b_n^{-1/2}|E(Y_1^2 Y_{i+1} J_{i+1}^x(R_{i+j+1,j}))| + b_n^{-1/2}|E(Y_1^2 Y_{i+1} J_{i+1}^y(R_{i+j+1,j}))| \\ &\quad + b_n^{-1/2}|E(Y_1^2 Y_{i+1})|\,|EJ_{i+1}^x(R_{i+j+1,j})| + b_n^{-1/2}|E(Y_1^2 Y_{i+1})|\,|EJ_{i+1}^y(R_{i+j+1,j})|. \end{aligned}$$

From (3.19) and (3.21) we have

$$\begin{aligned} &|E(Y_1^2 Y_{i+1} J_{i+1}^x(R_{i+j+1,j}))| \\ &\qquad \leqslant |E(Y_1^2 Y_{i+j+1} J_{i+1}^x(R_{i+j+1,j})\mathrm{I}\{|R_{i+j+1,j}| > c_j\})| \\ &\qquad\quad + |E(Y_1^2 Y_{i+1} J_{i+1}^x(R_{i+j+1,j})\mathrm{I}\{|R_{i+j+1,j}| \leqslant c_j\})| \\ &\qquad \leqslant Mb_n^{-1/2}Q_j + Mc_j b_n E|Y_1^2 Y_{i+1}| \end{aligned}$$

and

$$|EJ_{i+1}^x(R_{i+j+1,j})| \leqslant Mb_n c_j + Mb_n Q_j \quad \text{for every real } x.$$

By (3.5), $b_n^{-1/2}E|Y_1^2 Y_{i+1}| \leqslant M$ and we have

$$|E(Y_1^2 Y_{i+1} Y_{i+j+1})| \leqslant Mb_n c_j + Mb_n^{-1} Q_j.$$

Similarly, we can show that

$$|E(Y_1 Y_{i+1}^2 Y_{i+j+1})| \leqslant Mb_n c_j + Mb_n^{-1} Q_j.$$

Choosing $\beta := \frac{2}{3}$, we get $\text{II}_1 = \mathcal{O}(b_n^{1/3})$.

By (3.7), $|E(Y_1 Y_{i+1} Y_{i+j+1}^2)| \leqslant Mb_n$ and $\text{II}_2 = \mathcal{O}(pb_n)$. Then

$$\frac{p^2\,\text{II}_1}{np} = \mathcal{O}\left(\frac{p}{n} b_n^{1/3}\right) = o(1), \qquad \frac{p^2\,\text{II}_2}{np} = \mathcal{O}\left(\frac{p^2 b_n}{n}\right) = o(1).$$

Finally,

$$\begin{aligned}|E(Y_1 Y_{i+1} Y_{i+j+1} Y_{i+j+w+1})| &\leqslant b_n^{-1/2}\big(|E(Y_1 Y_{i+1} Y_{i+j+1} J_{i+j+1}^x (R_{i+j+w+1,w}))| \\ &+ |E(Y_1 Y_{i+1} Y_{i+j+1} J_{i+j+1}^y (R_{i+j+w+1,w}))| \\ &+ |E(Y_1 Y_{i+1} Y_{i+j+1})|\,|EJ_{i+j+1}^x (R_{i+j+w+1,w})| \\ &+ |E(Y_1 Y_{i+1} Y_{i+j+1})|\,|EJ_{i+j+1}^y (R_{i+j+w+1,w})|\big) \leqslant Mb_n^2 c_w + Mb_n^{-1} Q_w.\end{aligned}$$

Choosing $\beta := 1$ we have

$$\text{III} = \mathcal{O}(b_n), \qquad \frac{p^3\,\text{III}}{np} = \mathcal{O}\left(\frac{p^2 b_n}{n}\right) = o(1)$$

using the assumption of the lemma.

Define $V_j = n^{-1/2} \sum_{j(p+q)-q+1}^{j(p+q)} Y_t$ and $W = n^{-1/2} \sum_{k(p+q)+1}^{n} Y_t$.

LEMMA 3.5. *Let assumptions* (A.4) *and* (A.5) *hold true. Then*

$$k(n)\,\text{Var}\,U_1 \to c^2 \sigma_x^2 + d^2 \sigma_y^2 \quad \text{as } n \to \infty, \tag{3.24}$$

$$\sum_{j=1}^{k} V_j + W \xrightarrow{\mathscr{P}} 0. \tag{3.25}$$

Proof. Clearly,

$$k\,\text{Var}\,U_1 = \frac{k}{n} E\Big(\sum_{t=1}^{p} Y_t\Big)^2 = \frac{k}{n} \sum_{i,j=1}^{p} E(Y_i Y_j) + \frac{k}{n} pEY_1^2,$$

and, by (3.21),

$$\left|\frac{k}{n} \sum_{i,j=1}^{p} E(Y_i Y_j)\right| \leqslant \frac{Mk}{n} p \sum_{j=1}^{p} |E(Y_1 Y_{j+1})| \leqslant Mb_n^{2/3}$$

since $kp/n \to 1$. Moreover,

$$EY_1^2 = c^2 E(Y_1^{(x)})^2 + d^2 E(Y_1^{(y)})^2 + 2cdE(Y_1^{(x)} Y_1^{(y)})$$

and $E(Y_1^{(x)})^2 \to \sigma_x^2$ for every real x. In order to complete the proof of (3.24) it is sufficient to show that $E(Y_1^{(x)} Y_1^{(y)}) \to 0$ as $n \to \infty$. Obviously,

$$\begin{aligned}&|EY_1^{(x)} Y_1^{(y)}| \\ &\leqslant b_n^{-1} EK\left(\frac{x-X_1}{b_n}\right) K\left(\frac{y-X_1}{b_n}\right) + b_n^{-1} EK\left(\frac{x-X_1}{b_n}\right) EK\left(\frac{y-X_1}{b_n}\right) \leqslant Mb_n + I_n,\end{aligned}$$

where

$$I_n := b_n^{-1} \int_{-\infty}^{\infty} K\left(\frac{x-s}{b_n}\right) K\left(\frac{y-s}{b_n}\right) f(s)\, ds.$$

I_n is written as

$$\int_{-\infty}^{\infty} K(z) K\left(z+\frac{y-x}{b_n}\right) f(x-b_n z)\, dz = \int_{|z|\leqslant z_n} \dots dz + \int_{|z|>z_n} \dots dz,$$

where z_n is an arbitrary real sequence such that $z_n \to \infty$ but $z_n b_n \to 0$. Then

$$\int_{|z|\leqslant z_n} \dots dz \leqslant M \int_{|z|\leqslant z_n} K\left(z+\frac{y-x}{b_n}\right) dz$$

in view of (3.1) and the boundedness of K. Substituting $u = z+(y-x)/b_n$ we have

$$\int_{|z|\leqslant z_n} K\left(z+\frac{y-x}{b_n}\right) dz \leqslant \int_{|u|\geqslant Mb_n^{-1}} K(u)\, du = o(1),$$

$$\int_{|z|>z_n} \dots dz = \mathcal{O}\Big(\int_{|z|>z_n} K(z)\, dz\Big) = o(1) \quad \text{and} \quad I_n \to 0.$$

Next, by (3.20),

$$E\Big(\sum_{j=1}^{k} V_j + W\Big)^2 \leqslant M \frac{kq+n-k(p+q)}{n} \Big(EY_1^2 + \sum_{j=1}^{\infty} |E(Y_1 Y_{j+1})|\Big) = \mathcal{O}\left(\frac{q}{p}\right) = o(1)$$

and (3.25) is satisfied.

Proof of Theorem 2.1. It is clearly sufficient to consider the case $s = 2$. According to the Cramer–Wold device it suffices to prove that whenever $(c, d) \in R^2 \setminus \{0\}$

$$(c, d)^T \circ T_n(x, y) \xrightarrow{\mathscr{D}} N(0, c^2 \sigma_x^2 + d^2 \sigma_y^2), \tag{3.26}$$

where T denotes the vector transposition and $\circ$ the scalar product in R^2. Consider a partition of the set $\{1, \dots, n\}$ into consecutive "large" blocks of size p and "small" blocks of size q. If we take

$$p \sim n^{1/2} b_n^{-\alpha}, \quad \alpha := \frac{1}{2} - \frac{\sigma}{4}, \quad q \sim n^{1/2} b_n^{-\delta}, \quad \delta := \frac{1}{2} - \frac{\sigma}{2(\sigma+1)},$$

then we see that conditions (i) and (ii) of Lemma 3.2 hold. Thus it follows from Lemma 3.2 that in order to arrive at the asymptotic distribution of $\sum_{j=1}^{k} U_j$ we can assume that the U_j, $j = 1, \dots, k$, are i.i.d. random variables. It follows now from Lemmas 3.4 and 3.5 that

$$(c, d)^T \circ T_n(x, y) = \sum_{j=1}^{k} (U_j + V_j) + W \xrightarrow{\mathscr{D}} N(0, c^2 \sigma_x^2 + d^2 \sigma_y^2).$$

Remark. Theorem 2.1 holds true when conditions (A.1) and (A.5) are replaced by (A.0) and (A.6), respectively. This follows from the reasoning similar to that in the proof of the main result. Observe that in this case Lemmas 3.2–3.5 hold true since

$$\sum_{r=j}^{\infty} a_r^2 \leqslant \Big(\sum_{r=j}^{\infty} |a_r|\Big)^2 = \mathcal{O}(j^{-(4+2\sigma)}).$$

We omit the details.

Since (A.0) and (A.2) imply (A.1), we have

COROLLARY 3.6. *Assume that* (A.0), (A.2), (A.4), *and* (A.5) *hold true. Then the assertion of Theorem* 2.1 *remains valid.*

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