PROBABILITY
AND
MATHEMATICAL STATISTICS

Vol. 21, Fasc. 2 (2001), pp. 321–328

ON THE APPROXIMATION OF A RANDOM VARIABLE BY A CONDITIONING OF A GIVEN SEQUENCE

BY

KRZYSZTOF KANIOWSKI* (ŁÓDŹ)

Abstract. Let $(\Omega, \mathfrak{F}, P)$ be a non-atomic probability space. If (X_n) is a sequence of r.v.'s satisfying $X_n \to 0$ a.s. (respectively, in probability) as $n \to \infty$ and $EX_n^+ \to \infty$, $EX_n^- \to \infty$ as $n \to \infty$, then for any r.v. Y there exists a sequence $(\mathfrak{A}_n)$ of σ-fields such that $E(X_n | \mathfrak{A}_n) \to Y$ a.s. (respectively, in probability) as $n \to \infty$.

1991 Mathematics Subject Classification: 60A10.

Key words and phrases: Conditional expectation, almost sure convergence, stochastic convergence.

The main result of the paper is the following

THEOREM 1. *Let $(\Omega, \mathfrak{F}, P)$ be a non-atomic probability space. If (X_n) is a sequence of random variables satisfying the conditions*

$$\lim_{n\to\infty} X_n = 0 \ \textit{a.s.}, \tag{1}$$

$$\lim_{n\to\infty} EX_n^+ = \lim_{n\to\infty} EX_n^- = \infty, \tag{2}$$

then for any random variable Y there exists a sequence $(\mathfrak{A}_n)$ of σ-fields such that

$$\lim_{n\to\infty} E(X_n \mid \mathfrak{A}_n) = Y \ \textit{a.s.}$$

An analogous theorem for stochastic convergence is also proved.

In [2] and [3] Paszkiewicz describes, in particular, the construction of a σ-field $\mathfrak{A}$ such that $E(X_n \mid \mathfrak{A})$ does not converge to 0 for a sequence of random vectors X_n tending to 0 in some sense. We give a more precise description of $E(X \mid \mathfrak{A})$ when X is close to 0 in measure topology and EX^+, EX^- are large enough.

* Faculty of Mathematics, University of Łódź.

The idea of proof of Theorem 1 is based on the following observation. Let X be a random variable and Y be a simple random variable. If EX^+ and EX^- are large enough, then we can construct a σ-field $\mathfrak{A}$ such that $E(X\,|\,\mathfrak{A})$ equals Y on the set where X is small enough. For some sequence (Y_n) of simple random variables converging to Y we construct a sequence $(\mathfrak{A}_n)$ of σ-fields such that $E(X_n\,|\,\mathfrak{A}_n)$ equals Y_n on some large enough set.

All necessary properties of conditional expectation can be found in [1].

Proof of Theorem 1. From (1) we get immediately

$$\lim_{i\to\infty} P(\sup_{n\geqslant i}|X_n| > \varepsilon) = 0 \quad \text{for every } \varepsilon > 0. \tag{3}$$

Let (ε_j) be a sequence of real numbers such that $0 < \varepsilon_j < 1$ and $\varepsilon_j \searrow 0$ as $j \to \infty$. The equality (3) implies now the existence of a strictly monotone sequence (n_j) of integers such that

$$P(\sup_{n\geqslant n_j}|X_n| > \varepsilon_j) \leqslant 2^{-j} \quad \text{for } j \geqslant 1.$$

We put

$$A_{n_j} = \{\sup_{n\geqslant n_j}|X_n| > \varepsilon_j\} \quad \text{for } j \geqslant 1. \tag{4}$$

Let $p(n) = \max\{n_j\colon n_j \leqslant n\}$. From (2) it follows that

$$\int_{A_{p(n)}} X_n^+ \to \infty \quad \text{and} \quad \int_{A_{p(n)}} X_n^- \to \infty \quad \text{as } n \to \infty.$$

We put

$$M_n = \frac{1}{4}\int_{A_{p(n)}} X_n^+ \quad \text{and} \quad N_n = \frac{1}{4}\int_{A_{p(n)}} X_n^- \quad \text{for } n \geqslant 1$$

(we can assume that $M_n \geqslant 1$ and $N_n \geqslant 1$ for $n \geqslant 1$).

Let (Y_n) be a sequence of simple random variables of the form

$$Y_n = \sum_{i=1}^{k(n)} \alpha_i(n)\,\mathbf{1}_{G_i(n)} + \sum_{i=1}^{l(n)} \beta_i(n)\,\mathbf{1}_{H_i(n)}$$

and such that

$$\lim_{n\to\infty} Y_n = Y \text{ a.s.,}$$

$$\begin{aligned} &\alpha_i(n) > 0 \quad \text{for } n \geqslant 1,\ i = 1, 2, \ldots, k(n),\\ &\beta_i(n) < 0 \quad \text{for } n \geqslant 1,\ i = 1, 2, \ldots, l(n), \end{aligned} \tag{5}$$

$G_1(n), \ldots, G_{k(n)}(n), H_1(n), \ldots, H_{l(n)}(n)$ are mutually disjoint for $n \geqslant 1$.

We can also assume that for $n_j \leqslant n < n_{j+1}$ we have

$$(6) \qquad -N_n \leqslant Y_n(\omega) \leqslant -\varepsilon_j \quad \text{or} \quad \varepsilon_j \leqslant Y_n(\omega) \leqslant M_n \quad \text{for } \omega \in \Omega.$$

Since $(\Omega, \mathfrak{F}, P)$ is a non-atomic one, we can consider a random variable Z uniformly distributed on $[0, 1]$.

Before we construct the σ-fields $\mathfrak{A}_n$ we shall prove that for every $n \geqslant 1$ there exist real numbers $t_1, \ldots, t_{k(n)}$ and $s_1, \ldots, s_{l(n)}$ satisfying

$$(7) \qquad 0 \leqslant t_1 \leqslant \ldots \leqslant t_{k(n)} \leqslant 1,$$

$$(8) \qquad 0 \leqslant s_1 \leqslant \ldots \leqslant s_{l(n)} \leqslant 1,$$

$$(9) \qquad \alpha_i(n) P\big(C_i(n)\big) = \int\limits_{C_i(n)} X_n \quad \text{for } i = 1, \ldots, k(n),$$

$$(10) \qquad \beta_i(n) P\big(D_i(n)\big) = \int\limits_{D_i(n)} X_n \quad \text{for } i = 1, \ldots, l(n),$$

where

$$(11) \qquad C_i(n) = \big(G_i(n) \setminus A_{p(n)}\big) \cup \big(Z^{-1}([t_{i-1}, t_i)) \cap A_{p(n)} \cap \{X_n^+ > 0\}\big),$$

$$(12) \qquad D_i(n) = \big(H_i(n) \setminus A_{p(n)}\big) \cup \big(Z^{-1}([s_{i-1}, s_i)) \cap A_{p(n)} \cap \{X_n^- > 0\}\big).$$

From (4) and (6) we deduce that

$$(13) \qquad \varepsilon_j \leqslant \alpha_i(n) \leqslant M_n \quad \text{for } n_j \leqslant n < n_{j+1} \text{ and } i = 1, \ldots, k(n)$$

and

$$(14) \qquad |X_n| \leqslant \varepsilon_j \quad \text{for } n_j \leqslant n < n_{j+1} \text{ and } \omega \notin A_{n_j}.$$

For $t_1 \in [0, 1]$ we put

$$T_1(t_1) = \alpha_1(n) P\Big(\big(G_1(n) \setminus A_{p(n)}\big) \cup \big(Z^{-1}([0, t_1)) \cap A_{p(n)} \cap \{X_n^+ > 0\}\big)\Big)$$
$$- \int\limits_{(G_1(n) \setminus A_{p(n)}) \cup (Z^{-1}([0, t_1)) \cap A_{p(n)} \cap \{X_n^+ > 0\})} X_n.$$

It can easily be seen that T_1 is continuous.

From (13) and (14) we get

$$T_1(0) = \alpha_1(n) P\big(G_1(n) \setminus A_{p(n)}\big) - \int\limits_{G_1(n) \setminus A_{p(n)}} X_n \geqslant 0.$$

On the other hand, from (13), (14) and the definition of M_n we have

$$T_1(1) = \alpha_1(n) P\Big(\big(G_1(n) \setminus A_{p(n)}\big) \cup \big(A_{p(n)} \cap \{X_n^+ > 0\}\big)\Big)$$
$$- \int\limits_{G_1(n) \setminus A_{p(n)}} X_n - \int\limits_{A_{p(n)} \cap \{X_n^+ > 0\}} X_n$$
$$\leqslant M_n + 1 - 4M_n \leqslant 0.$$

Thus there exists $t_1 \in [0, 1]$ satisfying (9) for $i = 1$.

Let us assume that we have found $t_1, \ldots, t_{r-1}$ satisfying (7) and (9). Then from (9) we get

$$(15)\quad M_n\Big[P\Big(\big(G_i(n)\backslash A_{p(n)}\big)\cup\big(Z^{-1}([t_i, t_{i-1}))\cap A_{p(n)}\cap\{X_n^+>0\}\big)\Big)\Big]$$
$$\geqslant \int\limits_{Z^{-1}([t_{i-1},t_i))\cap A_{p(n)}\cap\{X_n^+>0\}} X_n - \varepsilon_1 P\big(G_i(n)\backslash A_{p(n)}\big), \qquad i = 1, \ldots, r-1.$$

By summing (15) over i we obtain

$$\begin{aligned} M_n &\geqslant \int\limits_{Z^{-1}([0,t_{r-1}))\cap A_{p(n)}\cap\{X_n^+>0\}} X_n - \varepsilon_1 \\ &= \int\limits_{A_{p(n)}\cap\{X_n^+>0\}} X_n - \int\limits_{Z^{-1}([t_{r-1},1])\cap A_{p(n)}\cap\{X_n^+>0\}} X_n - \varepsilon_1 \\ &\geqslant 3M_n - \int\limits_{Z^{-1}([t_{r-1},1])\cap A_{p(n)}\cap\{X_n^+>0\}} X_n. \end{aligned}$$

Thus

$$(16)\qquad \int\limits_{Z^{-1}([t_{r-1},1])\cap A_{p(n)}\cap\{X_n^+>0\}} X_n \geqslant 2M_n.$$

We put for $t_r \in [t_{r-1}, 1]$

$$T_r(t_r) = \alpha_r(n)P\big(C_r(n)\big) - \int\limits_{G_r(n)\backslash A_{p(n)}} X_n - \int\limits_{Z^{-1}([t_{r-1},t_r))\cap A_{p(n)}\cap\{X_n^+>0\}} X_n.$$

We have

$$T_r(t_{r-1}) \geqslant 0$$

and from (16) we obtain

$$T_r(1) \leqslant M_n + 1 - 2M_n \leqslant 0.$$

In consequence, there exists $t_r \in [t_{r-1}, 1]$ satisfying (9) for $i = r$.

By the above arguments we conclude that there exist $t_1, \ldots, t_{k(n)}$ such that (7) and (9) hold. In the same manner we can find $s_1, \ldots, s_{l(n)}$ satisfying (8) and (10).

Let us define now the sequence $(\mathfrak{A}_n)$. For $n \geqslant 1$ we put

$$\mathfrak{A}_n = \sigma\big(C_1(n), \ldots, C_{k(n)}(n), D_1(n), \ldots, D_{l(n)}(n)\big).$$

From (11) and (12) it can easily be seen that the sets $C_1(n), \ldots, C_{k(n)}(n)$ and $D_1(n), \ldots, D_{l(n)}(n)$ are mutually disjoint. From (9) and (10) it follows immediately that

$$E(X_n \mid \mathfrak{A}_n)(\omega) = Y_n(\omega) \qquad \text{for } \omega \notin A_{p(n)}.$$

As $\sum_{j=1}^{\infty} P(A_{nj}) < \infty$, we conclude by the Borel–Cantelli lemma that

$$\lim_{n\to\infty} E(X_n | \mathfrak{A}_n) = \lim_{n\to\infty} Y_n = Y \text{ a.s.}$$

This completes the proof. ∎

One can easily see that if Y is nonnegative, then the assumption that $EX_n^- \to \infty$ as $n \to \infty$ is not used in the proof. Thus the following version of the previous theorem holds:

THEOREM 2. *Let $(\Omega, \mathfrak{F}, P)$ be a non-atomic probability space. If (X_n) is a sequence of random variables satisfying the conditions*

$$\lim_{n\to\infty} X_n = 0 \text{ a.s.} \quad \text{and} \quad \lim_{n\to\infty} EX_n^+ = \infty,$$

then for any nonnegative random variable Y there exists a sequence $(\mathfrak{A}_n)$ of σ-fields such that

$$\lim_{n\to\infty} E(X_n | \mathfrak{A}_n) = Y \text{ a.s.}$$

The next example shows that we cannot omit the assumption that the probability space is non-atomic.

COUNTEREXAMPLE 1. Let $\Omega = [0, 1]$, $\mathfrak{F} = \sigma\big((2^{-n}, 2^{-n+1}], n \geqslant 1\big)$ and P be the Lebesgue measure on $[0, 1]$. Let us consider random variables X_n given by

$$X_n = 4^n \mathbb{1}_{(2^{-n}, 2^{-n+1}]} \quad \text{for } n \geqslant 1.$$

Then

$$\lim_{n\to\infty} X_n = 0 \text{ a.s.} \quad \text{and} \quad \lim_{n\to\infty} EX_n = \infty.$$

It can be also easily checked that for any σ-field $\mathfrak{A} \subset \mathfrak{F}$ we have

$$E(X_n | \mathfrak{A}) = 0 \quad \text{or} \quad 2^n \leqslant E(X_n | \mathfrak{A}) \leqslant 4^n \text{ a.s.}$$

Thus there exists no sequence $(\mathfrak{A}_n)$ of σ-fields such that

$$\lim_{n\to\infty} E(X_n | \mathfrak{A}_n) = 1 \text{ a.s.}$$

One can ask whether the assumption that the sequence (X_n) converges to zero almost surely can be replaced by the stochastic convergence. The answer is negative.

COUNTEREXAMPLE 2. Let $\Omega = [0, 1]$, $\mathfrak{F} = \text{Borel}([0, 1])$, and P be the Lebesgue measure on $[0, 1]$. Let (X_n) be a sequence of random variables given by

$$X_{2^n+k} = 4^n \mathbb{1}_{[k2^{-n}, (k+1)2^{-n}]} \quad \text{for } n \geqslant 0,\ k = 0, 1, \ldots, 2^n - 1.$$

It can easily be seen that the sequence (X_n) converges to zero in probability but does not converge with probability one. Let us assume now that there exists a sequence $(\mathfrak{A}_n)$ of σ-fields such that

$$\lim_{n\to\infty} E(X_n \mid \mathfrak{A}_n) = 0 \text{ a.s.}$$

Then we can find $m_0 \geqslant 1$ such that

$$P\big(\sup_{m\geqslant m_0} E(X_m \mid \mathfrak{A}_m) > \tfrac{1}{2}\big) \leqslant \tfrac{1}{2}.$$

Let us put

$$A = \{\omega\colon \sup_{m\geqslant m_0} E(X_m \mid \mathfrak{A}_m)(\omega) > \tfrac{1}{2}\}.$$

Then for $\omega \in A^c$ and $m \geqslant m_0$ we have

$$E(X_m \mid \mathfrak{A}_m)(\omega) \leqslant \tfrac{1}{2}.$$

Since $P(A^c) \geqslant \frac{1}{2}$, we can choose $n \geqslant 1$ and $k = 0, 1, \ldots, 2^n - 1$ such that $2^n + k \geqslant m_0$ and

$$P\big(A^c \cap [k2^{-n}, (k+1)2^{-n}]\big) \geqslant 2^{-n-1}. \tag{17}$$

From (17) we get

$$\tfrac{1}{2} \geqslant \int\limits_{\{E(X_{2^n+k}|\mathfrak{A}_{2^n+k})\leqslant 1/2\}} E(X_{2^n+k} \mid \mathfrak{A}_{2^n+k}) = \int\limits_{\{E(X_{2^n+k}|\mathfrak{A}_{2^n+k})\leqslant 1/2\}} X_{2^n+k}$$

$$\geqslant \int\limits_{A^c\cap[k2^{-n},(k+1)2^{-n}]} X_{2^n+k} \geqslant 4^n 2^{-n-1} \geqslant 1,$$

which gives a contradiction and proves that there exists no sequence $(\mathfrak{A}_n)$ such that

$$\lim_{n\to\infty} E(X_n \mid \mathfrak{A}_n) = 0 \text{ a.s.}$$

We have shown that under the assumptions of Theorem 1 the almost sure convergence cannot be replaced by the stochastic one. Nevertheless, weaker assumptions give a weaker conclusion. We shall prove the following

THEOREM 3. *Let $(\Omega, \mathfrak{F}, P)$ be a non-atomic probability space. If (X_n) is a sequence of random variables satisfying the conditions*

$$\lim_{n\to\infty} X_n = 0 \textit{ in probability}, \tag{18}$$

$$\lim_{n\to\infty} EX_n^+ = \lim_{n\to\infty} EX_n^- = \infty, \tag{19}$$

then for any random variable Y there exists a sequence $(\mathfrak{A}_n)$ of σ-fields such that

$$\lim_{n\to\infty} E(X_n\,|\,\mathfrak{A}_n) = Y \text{ in probability.}$$

Proof. From (18) we get immediately

$$\lim_{n\to\infty} P(|X_n| > \varepsilon) = 0 \quad \text{for every } \varepsilon > 0. \tag{20}$$

Let (ε_j) be a sequence of real numbers such that $0 < \varepsilon_j < 1$ and $\varepsilon_j \searrow 0$ as $j \to \infty$. The equality (20) implies now the existence of an increasing sequence (n_j) of integers such that

$$P(|X_n| > \varepsilon_j) \leqslant 2^{-j} \quad \text{for } n \geqslant n_j.$$

We put

$$A_n = \{\omega\colon\ |X_n(\omega)| > \varepsilon_j\} \quad \text{for } n_j \leqslant n < n_{j+1}.$$

It is obvious that $P(A_n) \to 0$ as $n \to \infty$. From (19) we deduce that

$$\int_{A_n} X_n^+ \to \infty \quad \text{and} \quad \int_{A_n} X_n^- \to \infty \quad \text{as } n \to \infty.$$

Now in the same manner as in the proof of Theorem 1 we can find a sequence (Y_n) of simple random variables and a sequence $(\mathfrak{A}_n)$ of σ-fields such that

$$\lim_{n\to\infty} Y_n = Y \text{ a.s.} \quad \text{and} \quad E(X_n\,|\,\mathfrak{A}_n)(\omega) = Y_n(\omega) \text{ for } \omega \notin A_n.$$

Fix $\varepsilon > 0$. For any $\eta > 0$ we can find $m \geqslant 1$ such that

$$P(|Y_n - Y| > \varepsilon) < \eta/2 \quad \text{for } n \geqslant m$$

and

$$P\big(E(X_n\,|\,\mathfrak{A}_n) \neq Y_n\big) \leqslant P(A_n) < \eta/2 \quad \text{for } n \geqslant m.$$

Therefore, we obtain

$$P\big(|E(X_n\,|\,\mathfrak{A}_n) - Y| > \varepsilon\big) \leqslant \eta/2 + \eta/2 = \eta \quad \text{for } n \geqslant m,$$

which means that $E(X_n\,|\,\mathfrak{A}_n) \to Y$ in probability as $n \to \infty$. This completes the proof of the theorem ■

In a similar way we obtain

THEOREM 4. *Let $(\Omega, \mathfrak{F}, P)$ be a non-atomic probability space. If (X_n) is a sequence of random variables satisfying the conditions*

$$\lim_{n\to\infty} X_n = 0 \textit{ in probability} \quad \textit{and} \quad \lim_{n\to\infty} EX_n^+ = \infty,$$

then for any nonnegative random variable Y there exists a sequence $(\mathfrak{A}_n)$ of σ-fields such that

$$\lim_{n\to\infty} E(X_n\,|\,\mathfrak{A}_n) = Y \textit{ in probability.}$$

REFERENCES

[1] P. Billingsley, *Probability and Measure*, Wiley, New York 1979.

[2] A. Paszkiewicz, *On the almost sure convergence of all conditionings of positive random variables*, to appear.

[3] A. Paszkiewicz, *The characterization of a.s. convergence of all conditionings for sequences of random vectors*, to appear.

Faculty of Mathematics
University of Łódź
Banacha 22
90-238 Łódź, Poland

Received on 28.12.2000