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PREDICTABLE EXTENSIONS OF GIVEN FILTRATIONS

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Abstract. Filtrations with the property that every stopping time is predictable are of some importance in stochastic analysis, especially in connection with the Girsanov transformation (cf. e.g. Chung and Williams [1]). Presumably for that reason, S. Kwapień stated the problem whether any given filtration can be extended (in a sense defined below) to a filtration for which every stopping time is predictable. In this paper, this problem of Kwapień is solved positively: Any filtration has a predictable extension.

The extension we construct has even the stronger property: any square integrable martingale is a stochastic integral process relative to a certain Brownian motion.

1. Statement of the problem. Let $(\Omega, \mathscr{F}, P)$ be a given probability space. If $\mathfrak{F} = (\mathscr{F}_t)_{t \geqslant 0}$ is a given filtration indexed by $\boldsymbol{R}_+$, we set as usual

$$\mathscr{F}_\infty := \bigvee_{t \geqslant 0} \mathscr{F}_t.$$

Let $\mathscr{N} := \mathscr{N}(\mathfrak{F})$ denote the family of all $\boldsymbol{P}$-null sets of the $\boldsymbol{P}$-completion of $\mathscr{F}_\infty$. Then $\mathfrak{F}$ is called a *standard filtration* if $\mathfrak{F}$ is right continuous and if $\mathscr{N} \subset \mathscr{F}_t$ for all $t \in \boldsymbol{R}_+$. We will also consider filtrations $\mathfrak{F} = (\mathscr{F}_t)_{t \in I}$ indexed by a subset $I \subset \boldsymbol{R}_+$. Such a filtration can always be naturally extended to a right continuous filtration $\mathfrak{F}' = (\mathscr{F}'_t)_{t \geqslant 0}$ indexed by $\boldsymbol{R}_+$: If $t = \inf\{s \in I \mid s > t\}$, we set

$$\mathscr{F}'_t = \bigcap_{s > t, s \in I} \mathscr{F}_s,$$

and if $\inf\{s \in I \mid s > t\} > t$ (with $\inf \emptyset = \infty$), we set

$$\mathscr{F}'_t = \bigvee_{s \leqslant t, s \in I} \mathscr{F}_s$$

in case of $\{s \in I \mid s \leqslant t\} \neq \emptyset$, and

$$\mathscr{F}'_t = \bigcap_{s > t, s \in I} \mathscr{F}_s$$

in case of $\{s \in I \mid s \leqslant t\} = \emptyset$. Sometimes, we will tacitly identify a filtration $(\mathscr{F}_t)_{t \in I}$ with its natural extension $(\mathscr{F}'_t)_{t \geqslant 0}$. For example, if $\mathfrak{F} = (\mathscr{F}_t)_{t \in \{a,b\}}$ $(0 \leqslant a < b < \infty)$, then $(\mathscr{F}'_t)_{t \geqslant 0}$ is just the filtration given by

$$\mathscr{F}'_t = \begin{cases} \mathscr{F}_a & \text{for } 0 \leqslant t < b, \\ \mathscr{F}_b & \text{for } b \leqslant t. \end{cases}$$

Now suppose that $\mathfrak{F} = (\mathscr{F}_t)_{t \geqslant 0}$ is a standard filtration and denote by $\mathscr{P}$ the predictable σ-field on $\boldsymbol{R}_+ \times \Omega$, i.e. the σ-field generated by the $(\mathscr{F}_t)$-adapted real-valued continuous processes. A stopping time $\tau\colon \Omega \to \overline{\boldsymbol{R}}_+$ is then called *predictable* if

$$[\tau] := \{(t, \omega) \mid \tau(\omega) = t\} \in \mathscr{P}$$

(cf. e.g. Metivier [3] for equivalent characterizations).

The following result is well known and not very difficult to prove (cf. e.g. Chung and Williams [1], p. 30).

PROPOSITION 1.1. *Every $(\mathscr{F}_t)$-stopping time is predictable if and only if every $(\mathscr{F}_t)$-martingale has a continuous version.*

We will call a filtration $\mathfrak{F} = (\mathscr{F}_t)_{t \geqslant 0}$ *predictable* if every $\mathfrak{F}$-stopping time is predictable.

Let $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P})$ be a second probability space. Then $\tilde{\Omega}$ will be called an *extension* of Ω if there exists a map $\pi\colon \tilde{\Omega} \to \Omega$ such that $\pi^{-1}(\mathscr{F}) \subset \tilde{\mathscr{F}}$ and $\pi(\tilde{P}) = P$. We will call π the *projection associated with* $\tilde{\Omega}$. If $\mathfrak{F} = (\mathscr{F}_t)_{t \geqslant 0}$ is a right continuous filtration on Ω, then a filtration $\tilde{\mathfrak{F}} = (\tilde{\mathscr{F}}_t)_{t \geqslant 0}$ on the extension $\tilde{\Omega}$ is called an *extension* of $\mathfrak{F}$ if $\pi^{-1}(\mathscr{F}_t) \subset \tilde{\mathscr{F}}_t$ for all $t \geqslant 0$ and $\tilde{\mathscr{F}}_0 \subset \pi^{-1}(\mathscr{F}_0) \vee \tilde{\mathscr{N}}$, $\tilde{\mathscr{N}} = \mathscr{N}(\tilde{\mathfrak{F}})$. For $(\mathscr{F}_t)_{t \in I}$ $(I \subset \boldsymbol{R}_+)$, a filtration $\tilde{\mathfrak{F}} = (\tilde{\mathscr{F}}_t)_{t \geqslant 0}$ on $\tilde{\Omega}$ is called an *extension* of $(\mathscr{F}_t)_{t \in I}$ if $\tilde{\mathfrak{F}}$ is an extension of the associated right continuous filtration $\mathfrak{F}' = (\mathscr{F}'_t)_{t \geqslant 0}$ of $(\mathscr{F}_t)_{t \in I}$. Finally, if an extension $\tilde{\mathfrak{F}} = (\tilde{\mathscr{F}}_t)_{t \geqslant 0}$ of a filtration $(\mathscr{F}_t)_{t \in I}$ is a standard filtration and also predictable, then $\tilde{\mathfrak{F}}$ is called shortly a *predictable extension* of $(\mathscr{F}_t)_{t \in I}$.

The aim of this paper is to prove the general result that every filtration has a predictable extension.

Let us first show that this general problem can easily be reduced to a partial problem, which looks a little bit more simple. Let us call a filtration $(\mathscr{F}_t)_{t \in D}$ on Ω a *discrete filtration* if $D = \{t_n \mid n \in \boldsymbol{N}\}$ for a decreasing sequence $(t_n)_{n \geqslant 1}$ in $\boldsymbol{R}_+$. Then we have the following simple result:

PROPOSITION 1.2. *If every discrete filtration has a predictable extension, then every filtration has a predictable extension.*

Proof. Let $\mathfrak{F} = (\mathscr{F}_t)_{t \geqslant 0}$ be a given right continuous filtration on Ω. We take a strictly decreasing sequence $(t_n)_{n \geqslant 1}$ in $\boldsymbol{R}_+$ with $\lim_{n \to \infty} t_n = 0$ and set $D = \{t_n \mid n \in \boldsymbol{N}\}$. Then we define $\mathscr{G}_{t_1} := \mathscr{F}_\infty$ and $\mathscr{G}_{t_n} := \mathscr{F}_{t_{n-1}}$ for $n \geqslant 2$. By assumption, the filtration $\mathfrak{G} = (\mathcal{G}_t)_{t\in D}$ has a predictable extension $\tilde{\mathfrak{G}} = (\tilde{\mathcal{G}}_t)_{t\geq 0}$ on an extension $\tilde{\Omega}$ of Ω. If $\mathfrak{G}' = (\mathcal{G}'_t)_{t\geq 0}$ denotes the associated right continuous extension of $\mathfrak{G}$ on Ω, then obviously $\mathcal{F}_t \subset \mathcal{G}'_t$, and hence

$$\pi^{-1}(\mathcal{F}_t) \subset \tilde{\mathcal{G}}_t \quad \text{for all } t \geq 0,$$

and also

$$\tilde{\mathcal{G}}_0 \subset \pi^{-1}(\mathcal{G}'_0) \vee \tilde{\mathcal{N}} = \pi^{-1}(\mathcal{F}_0) \vee \tilde{\mathcal{N}}$$

by the right continuity of $\mathfrak{F}$. Hence $\tilde{\mathfrak{G}}$ is also a predictable extension of $\mathfrak{F}$. ∎

2. The solution of the problem for a special case. In this section we solve the problem for the very simple filtrations being of the form $\mathfrak{F} = (\mathcal{F}_t)_{t\in\{a,b\}}$ $(0 \leq a < b < \infty)$. An essential ingredient of the proof is to make use of a Brownian motion living on a different probability space. In the next result we collect some simple properties of a Brownian motion which we need later.

LEMMA 2.1. *Suppose that $B = (B_t)_{t\geq 0}$ is a Brownian motion on a probability space (S, Σ, Q) and let $(\Sigma_t)_{t\geq 0}$ denote the standard filtration generated by B. Consider the Brownian motion $(B_t)_{a\leq t<b}$ restricted to the interval $[a, b[$ and define for $a \leq t < b$*

$$N_t = \int_a^t (b-u)^{-1/2}\, dB_u,$$

$$\bar{B}_s = N_{b-(b-a)e^{-s}}, \quad \textit{and} \quad \bar{\Sigma}_s = \Sigma_{b-(b-a)e^{-s}} \;\; (\textit{for } 0 \leq s < \infty).$$

Then $\bar{B} = (\bar{B}_s)_{s\geq 0}$ is a $(\bar{\Sigma}_s)$-Brownian motion.

Suppose that $\mathcal{G}$ is a sub-σ-algebra of Σ such that $(B_t)_{a\leq t\leq b}$ is a Brownian motion for the filtration $(\mathcal{G}_t)_{a\leq t\leq b}$ defined by $\mathcal{G}_t = \mathcal{G} \vee \Sigma_t$ for $t \in [a, b]$. Then for every square integrable $(\mathcal{G}_t)$-martingale $(M_t)_{a\leq t\leq b}$ with $M_a = 0$ a.s. there exists a $(\mathcal{G}_t)$-progressively measurable function $f_M\colon [a, b]\times S \to \mathbf{R}$ such that

$$M_t = \int_a^t f_M(s)\, dB_s \;\; \textit{a.s.} \quad \textit{for all } t \in [a, b].$$

Proof. By definition, the process $(N_t)_{a\leq t<b}$ is a martingale with quadratic variation $[N]$ given by

$$[N](t) = \int_a^t (b-u)^{-1}\, du = -\log\frac{b-t}{b-a} \quad (a \leq t < b).$$

It follows that $[\bar{B}](s) = s$ for every $s \geq 0$, and hence $\bar{B}$ is a $(\bar{\Sigma}_s)$-Brownian motion.

If $(B_t)_{a\leq t\leq b}$ is a $(\mathcal{G}_t)$-Brownian motion, then the assertion on the representation of $(\mathcal{G}_t)$-martingales as stochastic integrals is probably well known (cf. Karatzas and Shreve [2], Theorem 3.4.15, for the basic theorem), but for lack of an exact reference we give the proof.

We set $B'_t = B_t - B_a$ for $a \leqslant t \leqslant b$ and denote by $(\Sigma'_t)_{a \leqslant t \leqslant b}$ the standard filtration of the canonical filtration generated by $(B'_t)_{a \leqslant t \leqslant b}$. Then $\mathscr{G}_t = \mathscr{G}_a \vee \Sigma'_t$ for $a \leqslant t \leqslant b$ and $\mathscr{G}_a$ and Σ'_t are independent.

Now suppose first that Y is a bounded $\mathscr{G}_a$-measurable random variable on S and that Z is a bounded Σ'_b-measurable random variable on S such that $E\{Z \mid \Sigma'_a\} = 0$ a.s. Then

$$E\{YZ \mid \mathscr{G}_t\} = YE\{Z \mid \mathscr{G}_t\} = YE\{Z \mid \Sigma'_t\} \quad \text{for every } t \in [a, b].$$

If $(M_t(Z))_{a \leqslant t \leqslant b}$ denotes a cadlag-version of the martingale $(E\{Z \mid \Sigma'_t\})_{a \leqslant t \leqslant b}$, then it follows from Theorem 3.4.15 in Karatzas and Shreve [2] that there exists a (Σ'_t)-progressively measurable function g_Z such that

$$M_t(Z) = \int_a^t g_Z(s)\, dB'_s = \int_a^t g_Z(s)\, dB_s.$$

Hence we have

$$E\{YZ \mid \mathscr{G}_t\} = \int_a^t Y g_Z(s)\, dB_s \text{ a.s.} \quad \text{for every } t \in [a, b].$$

Now let $\mathscr{E}$ denote the vector space of all $\mathscr{G}_b$-measurable random variables on S of the form $X = \sum_{i=1}^n Y_i Z_i$, where the Y_i are bounded $\mathscr{G}_a$-measurable and the Z_i are bounded Σ'_b-measurable with $E\{Z_i \mid \Sigma'_a\} = 0$ a.s. By linearity it follows from the above argument that

$$E\{X \mid \mathscr{G}_t\} = \int_a^t f_X(s)\, dB_s \text{ a.s.} \quad \text{for every } t \in [a, b],$$

where f_X is the progressively measurable function $f_X = \sum_{i=1}^n Y_i g_{Z_i}$.

Finally, let $M = (M_t)_{a \leqslant t \leqslant b}$ be a given square integrable $(\mathscr{G}_t)$-martingale with $M_a = 0$ a.s. Then there exists, by a monotone class argument, a sequence (X_n) in $\mathscr{E}$ such that $\lim X_n = M_b$ in $L^2(S, \mathscr{G}_b, Q)$. Especially, (X_n) is a Cauchy sequence and

$$E(E\{X_m \mid \mathscr{G}_t\} - E\{X_n \mid \mathscr{G}_t\})^2 = E\int_a^t \big(f_{X_m}(s) - f_{X_n}(s)\big)^2 ds \leqslant E(X_m - X_n)^2$$

implies that there exists a progressively measurable function f_M such that

$$M_t = \int_a^t f_M(s)\, dB_s \quad \text{for all } t \in [a, b].$$

Thus the lemma is proved. ∎

Remark. The second part of Lemma 2.1 gives especially non-trivial examples of predictable filtrations.

THEOREM 2.2. *Let $\mathfrak{F} = (\mathscr{F}_t)_{t\in\{0,1\}}$ be the filtration on $(\Omega, \mathscr{F}, P)$ given by $\mathscr{F}_0 = \{\emptyset, \Omega\}$ and $\mathscr{F}_1 = \mathscr{F}$. Then there exist*

– *an extension $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P})$ of Ω,*

– *an extension $\tilde{\mathfrak{F}} = (\tilde{\mathscr{F}}_t)_{t\geqslant 0}$ of $\mathfrak{F}$ on $\tilde{\Omega}$, and*

– *an $\tilde{\mathfrak{F}}$-Brownian motion $\tilde{B} = (\tilde{B}_t)_{t\geqslant 0}$,*

such that for every square integrable $\tilde{\mathfrak{F}}$-martingale $\tilde{M} = (\tilde{M}_t)_{t\geqslant 0}$ there exists an $\tilde{\mathfrak{F}}$-progressively measurable function $f_{\tilde{M}}$: $[0, \infty] \times \tilde{\Omega} \to \boldsymbol{R}$ such that

$$\tilde{M}_t = E\tilde{M}_0 + \int_0^t f_{\tilde{M}}(s)\, d\tilde{B}_s \quad \tilde{P}\text{-a.s.} \quad \text{for all } t \geqslant 0.$$

As an immediate consequence, $\mathfrak{F}$ has a predictable extension.

Proof. (1) First we define $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P})$. We set simply

$$(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P}) := \prod_{k\geqslant 0} (\Omega_k, \mathscr{F}_k, P_k) \times (S, \Sigma, Q) \quad \text{with } \Omega_k = \Omega \text{ for } k \geqslant 0.$$

If we denote by π_k ($k \geqslant 0$), respectively π_S, the canonical projections from $\tilde{\Omega}$ onto Ω_k, respectively S, then we will view $\tilde{\Omega}$ as an extension of Ω relative to the projection $\pi = \pi_0$.

(2) For the definition of $\tilde{\mathfrak{F}}$ we need some preparations.

(i) For every interval $[2^{-(n+1)}, 2^{-n}[$ let $N^n = (N^n_t)_{2^{-(n+1)}\leqslant t<2^{-n}}$ be the martingale defined by the Brownian motion $(B_t)_{2^{-(n+1)}\leqslant t<2^{-n}}$ on S as described in Lemma 2.1. We will identify every martingale N^n on S with its canonical extension $(N^n_t \circ \pi_S)$. It follows easily from Lemma 2.1 that for every N^n the hitting time τ^n of $\{-1, 1\}$ fulfills $\tau^n < 2^{-n}$ a.s. and that for $\varepsilon_n := N^n_{\tau^n}$ we have

$$\tilde{P}\{\varepsilon_n = 1\} = \tilde{P}\{\varepsilon_n = -1\} = 1/2.$$

Moreover, Lemma 2.1 implies that the sequence $(N^n)_{n\geqslant 0}$ is independent, and hence also $(\varepsilon_n)_{n\geqslant 0}$ is independent, i.e. a Bernoulli sequence.

(ii) For the definition of $\tilde{\mathfrak{F}}$ we need also the following sequence $(\psi_n)_{n\geqslant 1}$ of transformations ψ_n: $\tilde{\Omega} \to \tilde{\Omega}$. For every $n \geqslant 1$ and every $\tilde{\omega} = ((\omega_j)_{j\geqslant 0}, s) \in \tilde{\Omega}$ we define

$$\psi_n((\omega_j)_{j\geqslant 0}, s) = ((\omega'_j)_{j\geqslant 0}, s)$$

by setting

$$\omega'_j = \begin{cases} \omega_{j+2^{n-1}} & \text{for } j = (2k)2^{n-1}, \ldots, (2k)2^{n-1}+2^{n-1}-1 \text{ and } k \geqslant 0, \\ \omega_{j-2^{n-1}} & \text{for } j = (2k+1)2^{n-1}, \ldots, (2k+1)2^{n-1}+2^{n-1}-1 \text{ and } k \geqslant 0, \end{cases}$$

i.e. every ψ_n interchanges the $(2k)$-th block of ω_j's of length 2^{n-1} with the $(2k+1)$-st block. Every ψ_n is clearly measurable and $\psi_n \circ \psi_n = \mathrm{Id}_{\tilde{\Omega}}$. Moreover, since $\tilde{P}$ is a product measure and every ψ_n is defined by a permutation of the coordinates, for any random variable $\tilde{X}$ on $\tilde{\Omega}$ the distribution of $\tilde{X}$ is equal to the distribution of $\tilde{X} \circ \psi_n$.

With the aid of the transformations ψ_n we now define by induction for every $n \geq 0$ a family $\mathscr{R}_n$ of random variables on $\tilde{\Omega}$. Let $\tilde{\pi}$ denote the projection from $\tilde{\Omega}$ onto $\prod_{k\geq 0} \Omega_k$. Then we set

$$\mathscr{R}_0 := \{X \in \mathscr{L}^0(\tilde{\Omega}) \mid X = Z \circ \tilde{\pi} \text{ for some } Z \in \mathscr{L}^0(\prod_{k\geq 0} \Omega_k)\}.$$

Suppose that we have already defined $\mathscr{R}_{n-1}$ for $n \geq 1$. Then we set

$$\mathscr{R}_n := \{X \in \mathscr{L}^0(\tilde{\Omega}) \mid X = Y + Y \circ \psi_n \text{ or } X = \varepsilon_{n-1}(Y - Y \circ \psi_n) \text{ for some } Y \in \mathscr{L}^0(\tilde{\Omega}, \sigma(\mathscr{R}_{n-1}))\}.$$

Finally, for every $n \geq 0$ we define

$$\mathscr{H}_n := \sigma(\bigcup_{m \geq n} \mathscr{R}_m).$$

(iii) Now we are ready to define the filtration $\tilde{\mathfrak{F}}$. For all $t \geq 0$ we set $\tilde{\Sigma}_t = \pi_S^{-1}(\Sigma_t)$ and $\tilde{B}_t = B_t \circ \pi_S$, so that $\tilde{B} = (\tilde{B}_t)_{t\geq 0}$ is a $(\tilde{\Sigma}_t)$-Brownian motion on $\tilde{\Omega}$. We set

$$\tilde{\mathscr{F}}_t := \mathscr{H}_0 \vee \tilde{\Sigma}_t \quad \text{for every } t \geq 1,$$

$$\tilde{\mathscr{F}}_t := \mathscr{H}_{n+1} \vee \tilde{\Sigma}_t \quad \text{for } t \in [2^{-(n+1)}, 2^{-n}[\ (n \geq 0),$$

and

$$\tilde{\mathscr{F}}_0 := \bigcap_{t>0} \tilde{\mathscr{F}}_t.$$

Then $\tilde{\mathfrak{F}} = (\tilde{\mathscr{F}}_t)_{t\geq 0}$ is an extension of $\mathfrak{F}$ if $\tilde{\mathscr{F}}_0 \subset \sigma(\tilde{\mathscr{N}})$, where $\tilde{\mathscr{N}}$ denotes the null sets of the $\tilde{P}$-completion of $\tilde{\mathscr{F}}_\infty$. This will be later a consequence of the asserted integral representation.

(3) For the proof of the integral representation we first discuss some essential properties of the filtration $\tilde{\mathfrak{F}}$.

(i) $\tilde{\mathscr{F}}_{2^{-n}} = \tilde{\mathscr{F}}_{2^{-(n+1)}} \vee \tilde{\Sigma}_{2^{-n}}$ *for every* $n \geq 0$.

Proof. By the definition of $\tilde{\mathfrak{F}}$ we have to show that

$$\mathscr{H}_{n+1} \vee \sigma(\mathscr{R}_n) \vee \tilde{\Sigma}_{2^{-n}} = \mathscr{H}_{n+1} \vee \tilde{\Sigma}_{2^{-n}} \quad \text{or} \quad \sigma(\mathscr{R}_n) \subset \mathscr{H}_{n+1} \vee \tilde{\Sigma}_{2^{-n}}.$$

Now, for any $Y \in \mathscr{L}^0(\tilde{\Omega}, \sigma(\mathscr{R}_n))$ the random variables $Y + Y \circ \psi_{n+1}$ and $\varepsilon_n(Y - Y \circ \psi_{n+1})$ are $\mathscr{H}_{n+1}$-measurable by definition and ε_n is $\tilde{\Sigma}_{2^{-n}}$-measurable. Since

$$Y = \tfrac{1}{2}(Y + Y \circ \psi_{n+1}) + \tfrac{1}{2}\varepsilon_n(Y - Y \circ \psi_{n+1})\varepsilon_n,$$

Y is $\mathscr{H}_{n+1} \vee \tilde{\Sigma}_{2^{-n}}$-measurable. ■

(ii) *Denote by* $B^n = (B_t^n)_{t \geq 2^{-(n+1)}}$ *the Brownian motion defined by* $B_t^n = \tilde{B}_t - \tilde{B}_{2^{-(n+1)}}$. *Then* $\tilde{\mathscr{F}}_{2^{-(n+1)}}$ *and* B^n *are independent for every* $n \geq 0$.

Proof. Every $\mathscr{R}_n$ can be written in the form

$$\mathscr{R}_n := \{X \mid X = \tfrac{1}{2}(Y + Y \circ \psi_n) + \tfrac{1}{2}\varepsilon_{n-1}(Y - Y \circ \psi_n) \text{ or}$$
$$X = \tfrac{1}{2}(Y + Y \circ \psi_n) - \tfrac{1}{2}\varepsilon_{n-1}(Y - Y \circ \psi_n) \text{ for } Y \in \mathscr{L}^0(\sigma(\mathscr{R}_{n-1}))\},$$

and it follows that $\sigma(\mathscr{R}_n)$ is ψ_n-invariant. An easy induction argument – using that $\sigma(\mathscr{R}_0)$ is ψ_n-invariant for all n and that $\psi_n \circ \psi_m = \psi_m \circ \psi_n$ for all $n, m \in N$ – implies that $\sigma(\mathscr{R}_n)$ is even ψ_m-invariant for every $m \in N$. By this observation it follows now easily that

$$\tilde{\mathscr{F}}_{2^{-(n+1)}} \subset \sigma(\mathscr{R}_{n+1}) \vee \tilde{\Sigma}_{2^{-(n+1)}} \qquad \text{for every } n \geq 0,$$

and hence it is sufficient to prove that $\sigma(\mathscr{R}_{n+1})$ and B^n are independent. We will even prove by induction that $\sigma(\mathscr{R}_n)$ and $\tilde{\Sigma} = \pi_S^{-1}(\Sigma)$ are independent for all $n \geq 0$. This is clear for $n = 0$. So suppose that we know the independence for n. We introduce the notation

$$Z(Y) = \tfrac{1}{2}(Y + Y \circ \psi_{n+1}) + \tfrac{1}{2}\varepsilon_n(Y - Y \circ \psi_{n+1})$$

and

$$\bar{Z}(Y) = \tfrac{1}{2}(Y + Y \circ \psi_{n+1}) - \tfrac{1}{2}\varepsilon_n(Y - Y \circ \psi_{n+1})$$

for all $Y \in \mathscr{L}^0(\sigma(\mathscr{R}_n))$. Now we take d random variables $Y_1, \ldots, Y_d \in \mathscr{L}^0(\sigma(\mathscr{R}_n))$, a measurable bounded map $F: R^{2d} \to R$, and a $\tilde{\Sigma}$-measurable bounded map $G: \tilde{\Omega} \to R$. For a shorter notation we set

$$\hat{Y} = (Y_1, \ldots, Y_d),$$
$$Z(\hat{Y}) = (Z(Y_1), \ldots, Z(Y_d)) \quad \text{and} \quad \bar{Z}(\hat{Y}) = (\bar{Z}(Y_1), \ldots, \bar{Z}(Y_d)).$$

Then we obtain

$$\begin{aligned}
&E\{F(Z(\hat{Y}), \bar{Z}(\hat{Y})) \cdot G\} \\
&= E\{F(Z(\hat{Y}), \bar{Z}(\hat{Y})) \cdot G \cdot 1_{\{\varepsilon_n = 1\}}\} + E\{F(Z(\hat{Y}), \bar{Z}(\hat{Y})) \cdot G \cdot 1_{\{\varepsilon_n = -1\}}\} \\
&= E\{F(\hat{Y}, \hat{Y} \circ \psi_{n+1}) \cdot G \cdot 1_{\{\varepsilon_n = 1\}}\} + E\{F(\hat{Y} \circ \psi_{n+1}, \hat{Y}) \cdot G \cdot 1_{\{\varepsilon_n = -1\}}\} \\
&= E\{F(\hat{Y}, \hat{Y} \circ \psi_{n+1})\} \cdot E\{G \cdot 1_{\{\varepsilon_n = 1\}}\} + E\{F(\hat{Y} \circ \psi_{n+1}, \hat{Y})\} \cdot E\{G \cdot 1_{\{\varepsilon_n = -1\}}\} \\
&\qquad \text{(by induction hypothesis)} \\
&= E\{F(\hat{Y}, \hat{Y} \circ \psi_{n+1})\} \cdot E\{G\} = E\{F(Z(\hat{Y}), \bar{Z}(\hat{Y}))\} \cdot E\{G\}.
\end{aligned}$$

The last but one equality is valid since

$$F(\hat{Y} \circ \psi_{n+1}, \hat{Y}) \circ \psi_{n+1} = F(\hat{Y}, \hat{Y} \circ \psi_{n+1}),$$

which implies that $F(\hat{Y}, \hat{Y} \circ \psi_{n+1})$ and $F(\hat{Y} \circ \psi_{n+1}, \hat{Y})$ have the same distribution. Since the equation we have just proved is valid for all $d \in N$, all

$Y_1, \ldots, Y_d \in \mathscr{L}^0(\sigma(\mathscr{R}_n))$, and all functions F and G of the above type, we have proved that $\sigma(\mathscr{R}_n)$ and $\tilde{\Sigma}$ are independent for all $n \geqslant 0$. Especially, $\sigma(\mathscr{R}_{n+1})$ and B^n are independent for all $n \geqslant 0$. ■

(4) It follows from (3) that $(\tilde{B}_t)_{t \geqslant r}$ is an $(\tilde{\mathscr{F}}_t)_{t \geqslant r}$-Brownian motion for $r > 0$. The second part of Lemma 2.1 now implies that for every square integrable $\tilde{\mathfrak{F}}$-martingale and every $r > 0$ there exists an $\tilde{\mathfrak{F}}$-progressively measurable function $f_{\tilde{M},r}$: $[r, \infty[\times \tilde{\Omega} \to \boldsymbol{R}$ such that

$$\tilde{M}_t - \tilde{M}_r = \int_r^t f_{\tilde{M},r}(s)\, d\tilde{B}_s \text{ a.s.} \quad \text{for every } t > r.$$

Moreover, it is easy to see that for Lebesgue measure λ

$$f_{\tilde{M},r}|_{[u,\infty[\times\tilde{\Omega}} = f_{\tilde{M},u} \quad (\lambda \otimes \tilde{P})\text{-a.s.} \quad \text{for } u > r.$$

Hence there exists a progressively measurable function $f_{\tilde{M}}$: $[0, \infty[\times \tilde{\Omega} \to \boldsymbol{R}$ such that

$$\tilde{M}_t - \tilde{M}_r = \int_r^t f_{\tilde{M}}(s)\, d\tilde{B}_s \text{ a.s.} \quad \text{for } 0 < r < t.$$

(5) By (4) it remains to prove that for every square integrable $\tilde{\mathfrak{F}}$-martingale $\tilde{M} = (\tilde{M}_t)_{t \geqslant 0}$ the limit $\lim_{r \to 0} \tilde{M}_r$, which exists by the convergence theorem for backward martingales, is necessarily equal to a constant $\tilde{P}$-a.s. Of course, this constant can only be $E\tilde{M}_0$.

Proof. (i) For every $n \geqslant 0$ let $(\tilde{\Sigma}_t^n)_{t \geqslant 2^{-n}}$ be the standard filtration of the Brownian motion $(\tilde{B}_t - \tilde{B}_{2^{-n}})_{t \geqslant 2^{-n}}$. Then we set

$$\mathscr{D}_0 = \tilde{\Sigma}_\infty^0, \quad \mathscr{D}_k = \tilde{\Sigma}_{2^{-(k-1)}},$$

$$\mathscr{C}_n = \mathscr{D}_0 \vee \ldots \vee \mathscr{D}_n, \quad \text{and} \quad \mathscr{B}_n = (\pi_0 \times \ldots \times \pi_{2^n-1})^{-1}(\mathscr{F}_0 \otimes \ldots \otimes \mathscr{F}_{2^n-1}).$$

Therefore we have $\tilde{\mathscr{F}}_\infty = \bigvee_{n \geqslant 0} (\mathscr{B}_n \vee \mathscr{C}_n)$.

(ii) Now we prove that for every $n \geqslant 0$ and every $X \in \mathscr{L}^1(\mathscr{B}_n \vee \mathscr{C}_n)$ the conditional expectation $E\{X | \tilde{\mathscr{F}}_{2^{-n}}\}$ is $\mathscr{B}_n \vee \sigma(\varepsilon_0, \ldots, \varepsilon_{n-1})$-measurable. It is sufficient to prove this for every $X \in \mathscr{L}^1(\mathscr{B}_n \vee \mathscr{C}_n)$ of the form

$$X = Y Z_n \ldots Z_0,$$

where Y is $\mathscr{B}_n$-measurable and the Z_k are $\mathscr{D}_k$-measurable. We prove this by induction. For $n = 0$ the assertion is true since

$$E\{Y Z_0 | \tilde{\mathscr{F}}_{2^{-0}}\} = Y E(Z_0) =: X^{(0)}$$

is $\mathscr{B}_n$-measurable. Suppose that we have already proved that

$$X^{(n-1)} := E\{Y Z_0 \ldots Z_{n-1} | \tilde{\mathscr{F}}_{2^{-(n-1)}}\}$$

is $\mathscr{B}_n \vee \sigma(\varepsilon_0, \ldots, \varepsilon_{n-2})$-measurable. Then we infer that

$$\begin{aligned}
&E\{YZ_0 \ldots Z_n \mid \tilde{\mathscr{F}}_{2^{-n}}\} \\
&= E\{Z_n\, E\{YZ_0 \ldots Z_{n-1} \mid \tilde{\mathscr{F}}_{2^{-(n-1)}}\} \mid \tilde{\mathscr{F}}_{2^{-n}}\} = E\{X^{(n-1)} Z_n \mid \tilde{\mathscr{F}}_{2^{-n}}\} \\
&= E\{\tfrac{1}{2}(X^{(n-1)} + X^{(n-1)} \circ \psi_n) Z_n + \tfrac{1}{2}\varepsilon_{n-1}(X^{(n-1)} - X^{(n-1)} \circ \psi_n)(\varepsilon_{n-1} Z_n) \mid \tilde{\mathscr{F}}_{2^{-n}}\} \\
&= \tfrac{1}{2}(X^{(n-1)} + X^{(n-1)} \circ \psi_n) E(Z_n) \\
&\quad + \tfrac{1}{2}\varepsilon_{n-1}(X^{(n-1)} - X^{(n-1)} \circ \psi_n) E(\varepsilon_{n-1} Z_n) =: X^{(n)}
\end{aligned}$$

is $\mathscr{B}_n \vee \sigma(\varepsilon_0, \ldots, \varepsilon_{n-1})$-measurable. It follows that $X^{(n)}$ is of the form

$$X^{(n)} = \sum_{k=1}^{2^n} Y_k f_k(\varepsilon_0, \ldots, \varepsilon_{n-1}),$$

where every Y_k is $\mathscr{B}_n$-measurable and the f_k are functions on $\{0, 1\}^n$. It is not difficult to derive the exact formula for $X^{(n)}$, but for our aim the above structure is sufficient.

(iii) The proof below is based on the following observation. If Y is $\mathscr{B}_n$-measurable, then

$$Y \circ \psi_{n+1} \text{ is } (\pi_{2^n} \times \ldots \times \pi_{2^{n+1}-1})^{-1} (\mathscr{F}_{2^n} \otimes \ldots \otimes \mathscr{F}_{2^{n+1}-1})\text{-measurable.}$$

For the $X^{(n)}$ above we therefore get

$$\begin{aligned}
E\{X^{(n)} \mid \tilde{\mathscr{F}}_{2^{-(n+1)}}\} &= E\{\tfrac{1}{2}(X^{(n)} + X^{(n)} \circ \psi_{n+1}) + \tfrac{1}{2}\varepsilon_n(X^{(n)} - X^{(n)} \circ \psi_{n+1})\varepsilon_n \mid \tilde{\mathscr{F}}_{2^{-(n+1)}}\} \\
&= \tfrac{1}{2}(X^{(n)} + X^{(n)} \circ \psi_{n+1}) = \sum_{k=1}^{2^n} \big(\tfrac{1}{2}(Y_{k,1} + Y_{k,2})\big) f_k(\varepsilon_0, \ldots, \varepsilon_{n-1}),
\end{aligned}$$

where $Y_{k,1} := Y_k$ and $Y_{k,2} := Y_k \circ \psi_{n+1}$ is independent of $Y_{k,1}$. More generally, one can prove by induction the following structure for

$$X^{(n+m)} := E\{X^{(n)} \mid \tilde{\mathscr{F}}_{2^{-(n+m)}}\} = E\{X \mid \tilde{\mathscr{F}}_{2^{-(n+m)}}\}.$$

For every $k = 1, \ldots, 2^n$ there exists an independent sequence $(Y_{k,j})_{j \geq 1}$ with $Y_{k,1} = Y_k$, such that

$$X^{(n+m)} = \sum_{k=0}^{2^n} \left(\frac{1}{2^m} \sum_{j=1}^{2^m} Y_{k,j} \right) f_k(\varepsilon_0, \ldots, \varepsilon_{n-1}).$$

By the strong law of large numbers we obtain

$$\lim_{m \to \infty} E\{X \mid \tilde{\mathscr{F}}_{2^{-(n+m)}}\} = \sum_{k=0}^{2^n} (EY_k) f_k(\varepsilon_0, \ldots, \varepsilon_{n-1}) \quad \tilde{P}\text{-a.s.},$$

and thus

$$E\{X \mid \tilde{\mathscr{F}}_0\} = \sum_{k=0}^{2^n} E(Y_k) E\big(f_k(\varepsilon_0, \ldots, \varepsilon_{n-1})\big) = \text{const} = EX \quad \tilde{P}\text{-a.s.}$$

for every $X \in \mathscr{L}^1(\mathscr{B}_n \vee \mathscr{C}_n)$. Since $\tilde{\mathscr{F}}_\infty = \bigvee_{n \geq 0} (\mathscr{B}_n \vee \mathscr{C}_n)$, it follows now by a standard argument that

$$E\{X \mid \tilde{\mathscr{F}}_0\} = EX \ \ \tilde{P}\text{-a.s.} \quad \text{for every } X \in \mathscr{L}^1(\tilde{\mathscr{F}}_\infty).$$

Especially, we have proved that, for every square integrable $\tilde{\mathfrak{F}}$-martingale $\tilde{M} = (\tilde{M}_t)_{t \geq 0}$,

$$\tilde{M}_0 = E\tilde{M}_0 \ \ \tilde{P}\text{-a.s.},$$

and the theorem is proved. ∎

Remark 2.3. An inspection of the proof shows that the filtration $\tilde{\mathfrak{F}}$ only depends on the given filtration and the Brownian motion B and not on the special probability measure P. This will be essential in the following.

Suppose that $\mathscr{G}$ and $\mathscr{H}$ are two sub-σ-algebras of $\mathscr{F}$. Let us recall that a *regular conditional probability of* $\mathscr{H}$ *given* $\mathscr{G}$ is defined as a map

$$K: \Omega \times \mathscr{H} \to [0, 1]$$

such that

(i) $K(\omega, \cdot)$ is a probability measure on $\mathscr{H}$ for every $\omega \in \Omega$,

(ii) $K(\cdot, B)$ is $\mathscr{G}$-measurable for every $B \in \mathscr{H}$, and

(iii) $P(A \cap B) = \int 1_A(\omega) K(\omega, B) P(d\omega)$ for $A \in \mathscr{G}$ and $B \in \mathscr{H}$.

If T is a Polish space with Borel field $\mathscr{B}(T)$ and if $\pi: \Omega \to T$ is a map such that $\pi^{-1}(\mathscr{B}(T)) = \mathscr{H}$, then a regular conditional probability of $\mathscr{H}$ given $\mathscr{G}$ exists.

LEMMA 2.4. *For the given probability space* $(\Omega, \mathscr{F}, P)$ *let* $(\bar{\Omega}, \bar{\mathscr{F}})$ *be the measurable space defined by*

$$(\bar{\Omega}, \bar{\mathscr{F}}) = \prod_{k \geq 0} (\Omega_k, \mathscr{F}_k)$$

with $(\Omega_k, \mathscr{F}_k) = (\Omega, \mathscr{F})$ *for every* $k \geq 0$. *As before, we will denote the canonical projections from* $\bar{\Omega}$ *onto* Ω_k *by* π_k. *Let* $\mathscr{G}$ *be a fixed sub-*σ*-algebra of* $\mathscr{F}$. *Then there exists a unique probability measure* $\bar{P}$ *on* $\bar{\Omega}$ *with the following property: For every sub-*σ*-algebra* $\mathscr{H}$ *of* $\mathscr{F}$, *for which there exists a regular conditional probability* $K_{\mathscr{H}}$ *of* $\mathscr{H}$ *given* $\mathscr{G}$, *one has*

$$\bar{P}|_{\mathscr{H}^{\otimes \mathbf{Z}_+}} = K_{\mathscr{H}}(\omega, \cdot)^{\otimes \mathbf{Z}_+} P(d\omega),$$

i.e.

$$\bar{P}\Big(\prod_{k \geq 0} A_k\Big) = \int \prod_{k \geq 0} K_{\mathscr{H}}(\omega, A_k) P(d\omega)$$

for every sequence $(A_k)_{k \geq 0}$ *in* $\mathscr{H}$.

Proof. Let Φ denote the family of all countable subsets of $\mathscr{L}^0(\mathscr{F})$ directed by inclusion. For $\phi = \{X_n \mid n \in \boldsymbol{N}\} \in \Phi$, $\sigma(\phi) = (X_1, X_2, \ldots)^{-1}(\mathscr{B}(\boldsymbol{R}^{\boldsymbol{N}}))$, and hence there exists a regular conditional probability K_ϕ of $\sigma(\phi)$ under $\mathscr{G}$. Furthermore

$$\mathscr{F} = \bigvee_{\phi \in \Phi} \sigma(\phi) = \bigcup_{\phi \in \Phi} \sigma(\phi).$$

If $\phi \subset \psi$, then $K_\phi(\cdot, B) = K_\psi(\cdot, B)$ $\boldsymbol{P}|_{\mathscr{G}}$-a.s. for every $B \in \sigma(\phi)$. It follows that the function

$$\bar{\boldsymbol{P}}\colon \bigcup_{\phi \in \Phi} \sigma(\phi)^{\otimes \boldsymbol{Z}_+} \to [0, 1],$$

given by

$$\bar{\boldsymbol{P}}|_{\sigma(\phi)^{\otimes \boldsymbol{Z}_+}} := K_\phi(\omega, \cdot)^{\otimes \boldsymbol{Z}_+} \boldsymbol{P}(d\omega) \quad \text{for } \phi \in \Phi,$$

is well defined, and it is clear that $\bar{\boldsymbol{P}}$ is finitely additive on the algebra $\mathscr{A} = \bigcup_{\phi \in \Phi} \sigma(\phi)^{\otimes \boldsymbol{Z}_+}$ which generates $\mathscr{F}^{\otimes \boldsymbol{Z}_+}$. To prove that $\bar{\boldsymbol{P}}$ can be (uniquely) extended to a probability measure on $\mathscr{F}^{\otimes \boldsymbol{Z}_+}$ we show that $\bar{\boldsymbol{P}}$ is σ-additive on $\mathscr{A}$. Now, if (B_n) is a decreasing sequence in $\mathscr{A}$ with intersection $\emptyset$, then we may suppose that $B_n \in \sigma(\phi_n)^{\otimes \boldsymbol{Z}_+}$, where (ϕ_n) is an increasing sequence in Φ. But $\psi := \bigcup \phi_n$ is again in Φ, and hence $B_n \in \sigma(\psi)^{\otimes \boldsymbol{Z}_+}$ for all n. Since $\bar{\boldsymbol{P}}$ is a probability measure on $\sigma(\psi)^{\otimes \boldsymbol{Z}_+}$, we have $\lim \bar{\boldsymbol{P}}(B_n) = 0$. This proves that $\bar{\boldsymbol{P}}$ is σ-additive on $\mathscr{A}$.

Now let $\mathscr{H}$ be a sub-σ-algebra of $\mathscr{F}$ for which there exists a regular conditional probability $K_{\mathscr{H}}$ of $\mathscr{H}$ under $\mathscr{G}$. If Ψ denotes the family of all countable subsets of $\mathscr{L}^0(\mathscr{H})$, then $\Psi \subset \Phi$ and, for every $\psi \in \Psi$,

$$K_\psi(\cdot, B) = K_{\mathscr{H}}(\cdot, B) \ \boldsymbol{P}|_{\mathscr{G}}\text{-a.s.},$$

and hence $\bar{\boldsymbol{P}}|_{\mathscr{H}^{\otimes \boldsymbol{Z}_+}} = K_{\mathscr{H}}(\omega, \cdot)^{\otimes \boldsymbol{Z}_+} \boldsymbol{P}(d\omega)$ follows by the definition of $\bar{\boldsymbol{P}}$. ■

Remark. On the probability space $(\bar{\Omega}, \bar{\mathscr{F}}, \bar{\boldsymbol{P}})$ the kernel $K_{\mathscr{H}}(\cdot, \cdot)^{\otimes \boldsymbol{Z}_+}$ is just a regular conditional probability of $\mathscr{H}^{\otimes \boldsymbol{Z}_+}$ under $\mathscr{G}$ if $\mathscr{G}$ is identified with $\pi_0^{-1}(\mathscr{G})$.

THEOREM 2.5. *Suppose that $\mathscr{F}_0$, $\mathscr{F}_1$ are two sub-σ-algebras of $\mathscr{F}$ such that $\mathscr{F}_0 \subset \mathscr{F}_1$. Then for the filtration $\mathfrak{F} = (\mathscr{F}_t)_{t \in \{0,1\}}$ there exists*

- *an extension $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\boldsymbol{P}})$ of Ω,*
- *an extension $\tilde{\mathfrak{F}} = (\tilde{\mathscr{F}}_t)_{t \geq 0}$ of $\mathfrak{F}$ on $\tilde{\Omega}$, and*
- *an $\tilde{\mathfrak{F}}$-Brownian motion $\tilde{B} = (\tilde{B}_t)_{t \geq 0}$,*

such that for every square integrable $\tilde{\mathfrak{F}}$-martingale $\tilde{M} = (\tilde{M}_t)_{t \geq 0}$ there exists an $\tilde{\mathfrak{F}}$-progressively measurable function $f_{\tilde{M}}\colon [0, \infty[\times \tilde{\Omega} \to \boldsymbol{R}$ such that

$$\tilde{M}_t = \tilde{M}_0 + \int_0^t f_{\tilde{M}}(s)\, d\tilde{B}_s \ \ \tilde{\boldsymbol{P}}\text{-a.s.} \quad \text{for every } t \geq 0.$$

As a consequence, $\mathfrak{F}$ has a predictable extension.

Proof. (1) As in Lemma 2.1 let (S, Σ, Q) be a probability space in which there exists a Brownian motion $B = (B_t)_{t \geq 0}$. We denote by $(\Sigma_t)_{t \geq 0}$ the standard filtration generated by B. As in Lemma 2.4 we set

$$(\bar{\Omega}, \bar{\mathscr{F}}) = (\Omega^{\mathbf{Z}_+}, \mathscr{F}^{\otimes \mathbf{Z}_+})$$

and denote by $\bar{P}$ the unique measure on $(\bar{\Omega}, \bar{\mathscr{F}})$ of the structure

$$\bar{P}|_{\mathscr{H}^{\otimes \mathbf{Z}_+}} = K_{\mathscr{H}}(\omega, \cdot)^{\otimes \mathbf{Z}_+} P(d\omega)$$

for every sub-σ-algebra $\mathscr{H} \subset \mathscr{F}$ for which there exists a regular conditional probability of $\mathscr{H}$ under $\mathscr{F}_0$. Then we define

$$(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P}) := (\bar{\Omega} \times S, \bar{\mathscr{F}} \otimes \Sigma, \bar{P} \otimes Q).$$

(2) Let $\tilde{B} = (\tilde{B}_t)_{t \geq 0}$ be the canonical extension of B to $\tilde{\Omega}$, i.e. $\tilde{B}_t = B_t \circ \pi_S$, and denote by $(\tilde{\Sigma}_t)_{t \geq 0}$ the canonical extension of $(\Sigma_t)_{t \geq 0}$ to $\tilde{\Omega}$. For the definition of the filtration $(\tilde{\mathscr{F}}_t)_{t \geq 0}$ below let us reformulate the construction in the proof of Theorem 2.2. For a given σ-algebra $\mathscr{G} \subset \mathscr{F}$ on Ω we first defined by induction a sequence $(\mathscr{R}_n(\mathscr{G}))_{n \geq 0}$ of families of random variables on $\tilde{\Omega}$. Then we defined a sequence $(\mathscr{H}_n(\mathscr{G}))_{n \geq 0}$ of sub-σ-algebras of $\tilde{\mathscr{F}}$ by

$$\mathscr{H}_n(\mathscr{G}) = \sigma\Big(\bigcup_{m \geq n} \mathscr{R}_m(\mathscr{G})\Big).$$

Finally, we defined the filtration $(\mathscr{E}_t(\mathscr{G}))_{t \geq 0}$ – denoted by $(\tilde{\mathscr{F}}_t)_{t \geq 0}$ in Theorem 2.2 – by

$$\mathscr{E}_t(\mathscr{G}) := \mathscr{H}_0(\mathscr{G}) \vee \tilde{\Sigma}_t \quad \text{for } t \geq 1,$$

$$\mathscr{E}_t(\mathscr{G}) := \mathscr{H}_{n+1}(\mathscr{G}) \vee \tilde{\Sigma}_t \quad \text{for } t \in [2^{-(n+1)}, 2^{-n}[\ (n \geq 0),$$

and

$$\mathscr{E}_0(\mathscr{G}) := \bigcap_{t > 0} \mathscr{E}_t(\mathscr{G}).$$

Then it was proved in Theorem 2.2 that $(\mathscr{E}_t(\mathscr{G}))_{t \geq 0}$ is an extension of the filtration $(\mathscr{G}_t)_{t \in \{0,1\}}$, where $\mathscr{G}_0 = \{\emptyset, \Omega\}$ and $\mathscr{G}_1 = \mathscr{G}$.

For the present theorem we now define the filtration $\tilde{\mathfrak{E}}(\mathscr{G}) = (\tilde{\mathscr{E}}_t(\mathscr{G}))_{t \geq 0}$ by $\tilde{\mathscr{E}}_t(\mathscr{G}) = \mathscr{F}_0 \vee \mathscr{E}_t(\mathscr{G})$ for $t \geq 0$, and, finally, $\tilde{\mathfrak{F}} = \tilde{\mathfrak{E}}(\mathscr{F}_1)$.

(3) Now we can prove that $\tilde{B}$ is an $\tilde{\mathfrak{F}}$-Brownian motion and that every square integrable $\tilde{\mathfrak{F}}$-martingale has the asserted integral representation.

(i) $\tilde{B}$ is an $\tilde{\mathfrak{F}}$-Brownian motion.

For the proof, for every $s \geq 0$ we set

$$\mathscr{C}_s := \sigma(B_t - B_s;\ t > s).$$

So we have to prove that $\tilde{\mathscr{F}}_s$ and $\mathscr{C}_s$ are independent for every $s \geq 0$. Let us denote by $\mathscr{R}(\mathscr{F}_0)$ the family of all sub-σ-algebras $\mathscr{H} \subset \mathscr{F}_1$ for which there exists a regular conditional probability $K_{\mathscr{H}}$ of $\mathscr{H}$ given $\mathscr{F}_0$. Then

$$\mathscr{F}_1 = \bigcup \{\mathscr{H} \mid \mathscr{H} \in \mathscr{R}(\mathscr{F}_0)\}$$

and it follows that

$$\tilde{\mathscr{F}}_s = \sigma\big(\tilde{\mathscr{E}}_s(\mathscr{H});\ \mathscr{H} \in \mathscr{R}(\mathscr{F}_0)\big).$$

Hence it is sufficient to prove that $\tilde{\mathscr{E}}_s(\mathscr{H})$ and $\mathscr{C}_s$ are independent for every $s \geqslant 0$. Since $\tilde{\mathscr{E}}_s(\mathscr{H}) = \mathscr{F}_0 \vee \mathscr{E}_s(\mathscr{H})$, it suffices to prove

$$\tilde{P}(A \cap B \cap C) = \tilde{P}(A \cap B)\tilde{P}(C)$$

for all $A \in \mathscr{F}_0$, $B \in \mathscr{E}_s(\mathscr{H})$ and $C \in \mathscr{C}_s$. Let us denote by $\tilde{K}_{\mathscr{H}}(\omega, \cdot)$ the probability measure on $(\tilde{\Omega}, \tilde{\mathscr{F}})$, which is the extension of $K_{\mathscr{H}}(\omega, \cdot)$, i.e.

$$\tilde{K}_{\mathscr{H}}(\omega, \cdot) = K_{\mathscr{H}}(\omega, \cdot)^{\otimes \mathbf{Z}_+} \otimes Q \quad \text{for every } \omega \in \Omega.$$

Now Theorem 2.2 implies that

$$\tilde{K}_{\mathscr{H}}(\omega, B \cap C) = \tilde{K}_{\mathscr{H}}(\omega, B)\tilde{K}_{\mathscr{H}}(\omega, C) = \tilde{K}_{\mathscr{H}}(\omega, B)Q(C') \quad (C' = \pi_S(C)),$$

and from the definition of $\tilde{P}$ we obtain

$$\tilde{P}(A \cap B \cap C) = \int 1_A(\omega) \int 1_{B \cap C}(\tilde{\omega})\tilde{K}_{\mathscr{H}}(\omega, d\tilde{\omega})P(d\omega)$$

$$= \int 1_A(\omega)\tilde{K}_{\mathscr{H}}(\omega, B)P(d\omega)Q(C') = \tilde{P}(A \cap B)\tilde{P}(C),$$

which proves that $\tilde{B}$ is an $\tilde{\mathfrak{F}}$-Brownian motion.

(ii) We will prove that for every $\tilde{X} \in \mathscr{L}^2(\tilde{\mathscr{F}}_\infty)$ there exists an $\tilde{\mathfrak{F}}$-progressively measurable function $f_{\tilde{X}}$: $[0, \infty[\times \tilde{\Omega} \to \mathbf{R}$ such that

$$(*) \qquad E\{\tilde{X} \mid \tilde{\mathscr{F}}_t\} = E\{\tilde{X} \mid \tilde{\mathscr{F}}_0\} + \int_0^t f_{\tilde{X}}(s)\,d\tilde{B}_s \quad \tilde{P}\text{-a.s.}$$

By arguments as in the proof of Theorem 2.2 it is sufficient to prove this for random variables $\tilde{X}$ which are of the special form $\tilde{X} = YZ$, where Y is a bounded $(\pi_0 \times \ldots \times \pi_{2^n-1})^{-1}(\mathscr{F}_1^{\otimes 2^n})$-measurable random variable and Z is a bounded $\mathscr{C}_s$-measurable random variable for some $s > 0$. Since we will work with conditional expectations relative to different probability measures on $(\tilde{\Omega}, \tilde{\mathscr{F}})$, in the following we will write more precisely $E_R\{\cdot \mid \cdot\}$ for the conditional expectation symbol if R is the relevant probability measure on $\tilde{\Omega}$.

Since $\tilde{\mathscr{F}}_t = \sigma\big(\tilde{\mathscr{E}}_t(\mathscr{H});\ \mathscr{H} \in \mathscr{R}(\mathscr{F}_0)\big)$, it is sufficient for the proof of equation $(*)$ to show that, for every $\mathscr{H} \in \mathscr{R}(\mathscr{F}_0)$, every $B \in \mathscr{E}_t(\mathscr{H})$ and $A \in \mathscr{F}_0$,

$$(**) \qquad \int_{A \cap B} E_{\tilde{P}}\{\tilde{X} \mid \tilde{\mathscr{F}}_t\}\,d\tilde{P} = \int_{A \cap B} \Big(\int_0^t f_{\tilde{X}}(s)\,d\tilde{B}_s + E_{\tilde{P}}\{\tilde{X} \mid \tilde{\mathscr{F}}_0\}\Big)\,d\tilde{P}$$

for a certain progressively measurable function $f_{\tilde{X}}$. By the special choice of $\tilde{X}$ we see that Y is $\mathscr{H}_0^{\otimes \mathbf{Z}_+}$-measurable for some $\mathscr{H}_0 \in \mathscr{R}(\mathscr{F}_0)$. So let $\mathscr{H} \in \mathscr{R}(\mathscr{F}_0)$

with $\mathscr{H} \supset \mathscr{H}_0$ be given, and suppose that $A \in \mathscr{F}_0$ and $B \in \mathscr{E}_t(\mathscr{H})$. Then

$$\int_{A\cap B} E_{\tilde{P}}\{\tilde{X} \mid \tilde{\mathscr{F}}_t\}\, d\tilde{P} = \int_{A\cap B} \tilde{X} d\tilde{P} = \int_A \int_B \tilde{X} d\tilde{K}_{\mathscr{H}}(\omega)\, P(d\omega)$$
$$= \int_A \int_B E_{\tilde{K}_{\mathscr{H}}(\omega)}\{\tilde{X} \mid \mathscr{E}_t(\mathscr{H})\}\, d\tilde{K}_{\mathscr{H}}(\omega)\, P(d\omega).$$

Now Theorem 2.2 (with the same measure $\tilde{K}_{\mathscr{H}}(\omega)$ on $\tilde{\Omega}$) yields

$$E_{\tilde{K}_{\mathscr{H}}(\omega)}\{\tilde{X} \mid \mathscr{E}_t(\mathscr{H})\} = \int_0^t f_{\tilde{X}}(s)\, d\tilde{B}_s + E_{\tilde{K}_{\mathscr{H}}(\omega)}(\tilde{X}) \quad \tilde{K}_{\mathscr{H}}(\omega)\text{-a.s.},$$

where $f_{\tilde{X}}$ is $\mathfrak{C}(\mathscr{H})$-progressively measurable. An inspection of the proof of Theorem 2.2 shows also that for every $\mathscr{H}$ one gets the same $f_{\tilde{X}}$. Since

$$E_{\tilde{K}_{\mathscr{H}}(\cdot)}(\tilde{X}) = E_{\tilde{P}}\{\tilde{X} \mid \tilde{\mathscr{F}}_0\} \quad \tilde{P}\text{-a.s.},$$

we have proved (**), and hence (*) for our special $\tilde{X} = YZ$, and standard arguments yield (*) for all $X \in \mathscr{L}^2(\tilde{\mathscr{F}}_\infty)$. This completes the proof of the theorem. ∎

3. The solution in the general case. In this section we will prove that every filtration has a predictable extension. The special case stated in Theorem 2.5 will be used as an important building block for the general construction.

LEMMA 3.1. *Suppose that* $0 < u_0 < \ldots < u_m < \infty$ $(m \geq 1)$ *and that* $\mathfrak{F} = (\mathscr{F}_t)_{t\in\{u_0,\ldots,u_m\}}$ *is the given filtration on* $(\Omega, \mathscr{F}, P)$. *For every* $(k_1, \ldots, k_m) \in \mathbf{Z}_+^m$ *we set*

$$(\Omega^{k_1,\ldots,k_m}, \mathscr{F}^{k_1,\ldots,k_m}) = (\Omega, \mathscr{F})$$

and

$$(\bar{\Omega}^{(u_0,\ldots,u_m)}, \bar{\mathscr{F}}^{(u_0,\ldots,u_m)}) = \prod_{(k_1,\ldots,k_m)\in \mathbf{Z}_+^m} (\Omega^{k_1,\ldots,k_m}, \mathscr{F}^{k_1,\ldots,k_m}).$$

For every $t \in \{u_0, \ldots, u_m\}$ *we denote further by* $\mathscr{F}_t^{k_1,\ldots,k_m}$ *the* σ*-algebra* $\mathscr{F}_t$ *in* $\Omega^{k_1,\ldots,k_m}$. *Suppose that we have already defined the probability measure* $\bar{P}^{(u_0,\ldots,u_{m-1})}$ *on* $\bar{\Omega}^{(u_0,\ldots,u_{m-1})}$, *and that* $\Omega^{k_1,\ldots,k_{m-1}}$ *is identified with* $\Omega^{k_1,\ldots,k_{m-1},0}$. *Denote by*

$$\mathscr{R}\Big(\bigotimes_{(k_1,\ldots,k_{m-1})\in \mathbf{Z}_+^{m-1}} \mathscr{F}_{u_{m-1}}^{k_1,\ldots,k_{m-1},0}\Big)$$

the family of all sub-σ*-algebras* $\mathscr{H}$ *of* $\bigotimes_{(k_1,\ldots,k_{m-1})} \mathscr{F}^{k_1,\ldots,k_{m-1},0}$ *for which there exists a regular conditional probability* $K_{\mathscr{H}}$ *of* $\mathscr{H}$ *given* $\bigotimes \mathscr{F}_{u_{m-1}}^{k_1,\ldots,k_{m-1},0}$. *Then* $\bar{P}^{(u_0,\ldots,u_m)}$ *is defined as the unique probability measure on* $\bar{\Omega}^{(u_0,\ldots,u_m)}$ *such that*

$$\bar{P}^{(u_0,\ldots,u_m)}\big|_{\mathscr{H}\otimes \mathbf{Z}_+} = K_{\mathscr{H}}(\bar{\omega}, \cdot)^{\otimes \mathbf{Z}_+}\, \bar{P}^{(u_0,\ldots,u_{m-1})}(d\bar{\omega})$$

(cf. Lemma 2.4). *Finally, we set*

$$(\tilde{\Omega}^{(u_0,\ldots,u_m)}, \tilde{\mathscr{F}}^{(u_0,\ldots,u_m)}, \tilde{P}^{(u_0,\ldots,u_m)}) := (\bar{\Omega}^{(u_0,\ldots,u_m)}, \bar{\mathscr{F}}^{(u_0,\ldots,u_m)}, \bar{P}^{(u_0,\ldots,u_m)}) \times (S, \Sigma, Q).$$

Then there exists a filtration

$$\tilde{\mathfrak{F}}^{(u_0,\ldots,u_m)} = (\tilde{\mathscr{F}}_t^{(u_0,\ldots,u_m)})_{t\geq 0}$$

on $\tilde{\Omega}^{(u_0,\ldots,u_m)}$, which is an extension of $\mathfrak{F}$, such that for every square integrable $\tilde{\mathfrak{F}}^{(u_0,\ldots,u_m)}$-martingale $\tilde{M} = (\tilde{M}_t)_{t\geq 0}$ there exists a progressively measurable function

$$f_{\tilde{M}}: [u_0, u_m] \times \tilde{\Omega}^{(u_0,\ldots,u_m)} \to R$$

such that, for every $t \in [u_0, u_m]$,

$$\tilde{M}_t - \tilde{M}_{u_0} = \int_{u_0}^{t} f_{\tilde{M}}(s)\, d\tilde{B}_s \quad \tilde{P}^{(u_0,\ldots,u_m)}\text{-a.s.},$$

where $\tilde{B} = (\tilde{B}_t)_{u_0 \leq t \leq u_m}$ is an $(\tilde{\mathscr{F}}_t^{(u_0,\ldots,u_m)})_{u_0 \leq t \leq u_m}$-Brownian motion (as a process, $\tilde{B}$ is just the canonical extension of $(B_t)_{u_0 \leq t \leq u_m}$ to $\tilde{\Omega}^{(u_0,\ldots,u_m)}$). Furthermore, $\tilde{M}_t = \tilde{M}_{u_0}$ for $t \leq u_0$ and $\tilde{M}_t = \tilde{M}_{u_m}$ for $t \geq u_m$.

Proof. The assertions are proved by induction in $m \geq 1$. For $m = 1$ the assertions are essentially proved in Theorem 2.5. The minor modifications will become clear by the proof that the assertions are true for m if they are true for $m-1$. So suppose that the lemma is true for $m-1$ ($m \geq 2$).

(i) Let us first show that the probability space $\tilde{\Omega}^{(u_0,\ldots,u_m)}$ is in fact an extension of $(\Omega, \mathscr{F}, P)$. As a projection map we take the canonical projection

$$\pi: \tilde{\Omega}^{(u_0,\ldots,u_m)} \to \Omega = \Omega^{0,0,\ldots,0} \ (0, 0, \ldots, 0 \ (m+1 \text{ times})).$$

Then π is surely measurable and it remains to prove that $\pi(\tilde{P}^{(u_0,\ldots,u_m)}) = P$. Let $X \in \mathscr{L}^1(\mathscr{F})$ be given. There is a regular conditional probability $K_X(\cdot, \cdot)$ of X given $\mathscr{F}_{u_{m-1}} = \mathscr{F}_{u_{m-1}}^{0,0,\ldots,0}$ and $K_X(\pi(\cdot), \cdot)$ is also a regular conditional probability of $\sigma(X \circ \pi)$ given $\bigotimes \mathscr{F}_{u_{m-1}}^{k_1,\ldots,k_{m-1},0}$. By the definition of $\tilde{P}^{(u_0,\ldots,u_m)}$ we get

$$\begin{aligned} \int X \circ \pi d\tilde{P}^{(u_0,\ldots,u_m)} &= \int\int (X \circ \pi)\, dK_X(\pi(\tilde{\omega}))\, \tilde{P}^{(u_0,\ldots,u_{m-1})}(d\tilde{\omega}) \\ &= \int\int X dK_X(\omega)\, P(d\omega) \quad \text{(by induction hypothesis)} \\ &= \int X dP. \end{aligned}$$

Since X was arbitrary, we have proved $\pi(\tilde{P}^{(u_0,\ldots,u_m)}) = P$.

(ii) Next we define $\tilde{\mathscr{F}}_t^{(u_0,\ldots,u_m)}$ for $t \leq u_{m-1}$ if $\tilde{\mathfrak{F}}^{(u_0,\ldots,u_{m-1})}$ on $\tilde{\Omega}^{(u_0,\ldots,u_{m-1})}$ is given. We denote by

$$\pi_m: \tilde{\Omega}^{(u_0,\ldots,u_m)} \to \tilde{\Omega}^{(u_0,\ldots,u_{m-1})}$$

the projection map defined by

$$\pi_m\big((\omega^{k_1,\ldots,k_m})_{k_1 \geq 0,\ldots,k_m \geq 0}, s\big) = \big((\omega^{k_1,\ldots,k_{m-1},0})_{k_1 \geq 0,\ldots,k_{m-1} \geq 0}, s\big).$$

Then we set

$$\tilde{\mathscr{F}}_t^{(u_0,\ldots,u_m)} := \pi_m^{-1}(\tilde{\mathscr{F}}_t^{(u_0,\ldots,u_{m-1})}) \quad \text{for } t \leq u_{m-1}.$$

(iii) Now, using Theorem 2.5 we define $\tilde{\mathscr{F}}_t^{(u_0,\dots,u_m)}$ for $t > u_{m-1}$. We apply Theorem 2.5 with

$$\Omega' = \tilde{\Omega}^{(u_0,\dots,u_{m-1})}, \quad \mathscr{F}' = \tilde{\mathscr{F}}^{(u_0,\dots,u_{m-1})}, \quad P' = \tilde{P}^{(u_0,\dots,u_{m-1})},$$

$$\mathscr{F}_0' = \bigotimes_{k_1 \geq 0,\dots,k_{m-1} \geq 0} \mathscr{F}_{u_{m-1}}^{k_1,\dots,k_{m-1}}, \quad \mathscr{F}_1' = \bigotimes_{k_1 \geq 0,\dots,k_{m-1} \geq 0} \mathscr{F}_{u_m}^{k_1,\dots,k_{m-1}}$$

instead of $(\Omega, \mathscr{F}, P)$, $\mathscr{F}_0$, $\mathscr{F}_1$. Moreover, instead of the Brownian motion $(B_t)_{0 \leq t \leq 1}$ which we used in the proof of that theorem for the construction of $(\tilde{\mathscr{F}}_t)_{0 \leq t \leq 1}$ and the Brownian motion $\tilde{B}$, we now use the Brownian motion $B' = (B_t')_{0 \leq t \leq 1}$ defined by

$$B_t' = B_{u_{m-1}+t(u_m-u_{m-1})} \quad \text{for } 0 \leq t \leq 1.$$

Then Theorem 2.5 gives an extension $(\tilde{\Omega}', \tilde{\mathscr{F}}', \tilde{P}')$ of $(\Omega', \mathscr{F}', P')$, an extended filtration $(\tilde{\mathscr{F}}_t')_{0 \leq t \leq 1}$ and an $(\tilde{\mathscr{F}}_t')$-Brownian motion $(\tilde{B}_t')_{0 \leq t \leq 1}$ such that the stochastic integral representation holds for square integrable martingales as stated in that theorem. By definition, the probability space $\tilde{\Omega}^{(u_0,\dots,u_m)}$ is the same as $\tilde{\Omega}'$. Hence, if we define

$$\tilde{B}_t = \tilde{B}'_{(t-u_{m-1})/(u_m-u_{m-1})} \quad \text{and} \quad \tilde{\mathscr{F}}_t^{(u_0,\dots,u_m)} = \tilde{\mathscr{F}}'_{(t-u_{m-1})/(u_m-u_{m-1})} \vee \tilde{\Sigma}_{u_{m-1}}$$

for $u_{m-1} \leq t \leq u_m$, then $(\tilde{B}_t)_{u_{m-1} \leq t \leq u_m}$ is an $(\tilde{\mathscr{F}}_t^{(u_0,\dots,u_m)})_{u_{m-1} \leq t \leq u_m}$-Brownian motion and for every square integrable martingale $(\tilde{M}_t)_{u_{m-1} \leq t \leq u_m}$ we have

$$\tilde{M}_t = \tilde{M}_{u_{m-1}} + \int_{u_{m-1}}^{t} f_{\tilde{M}}(s)\, d\tilde{B}_s \quad \tilde{P}^{(u_0,\dots,u_m)}\text{-a.s.}$$

for some progressively measurable function $f_{\tilde{M}}$. Together with the induction hypothesis we have thus proved the assertion of the lemma. ∎

The next step is essential for the final result.

LEMMA 3.2. *The probability space $\tilde{\Omega}^{(u_0,\dots,u_m)}$ is an extension of $\tilde{\Omega}^{(u_1,\dots,u_m)}$, i.e. there exists a measurable map*

$$\phi_m : \tilde{\Omega}^{(u_0,\dots,u_m)} \to \tilde{\Omega}^{(u_1,\dots,u_m)}$$

such that $\phi_m(\tilde{P}^{(u_0,\dots,u_m)}) = \tilde{P}^{(u_1,\dots,u_m)}$. Moreover, for every $\tilde{\mathfrak{F}}^{(u_1,\dots,u_m)}$-martingale $(\tilde{M}_t)_{u_1 \leq t \leq u_m}$ the process $(\tilde{M}_t \circ \phi_m)_{u_1 \leq t \leq u_m}$ is an $(\tilde{\mathscr{F}}_t^{(u_0,\dots,u_m)})_{u_1 \leq t \leq u_m}$-martingale and

$$\tilde{M}_t \circ \phi_m = \tilde{M}_{u_1} \circ \phi_m + \int_{u_1}^{t} f_{\tilde{M}}(s) \circ \phi_m \, d\tilde{B}_s \quad \textit{for } t \in [u_1, u_m].$$

Proof. (i) We identify $(\tilde{\Omega}^{(u_1,\dots,u_m)}, \tilde{\mathscr{F}}_t^{(u_1,\dots,u_m)})$ with the measurable space

$$\prod_{k_2 \geq 0,\dots,k_m \geq 0} (\Omega^{0,k_2,\dots,k_m}, \mathscr{F}^{0,k_2,\dots,k_m}),$$

and define $\phi_m : \tilde{\Omega}^{(u_0,\dots,u_m)} \to \tilde{\Omega}^{(u_1,\dots,u_m)}$ as the canonical projection. Hence ϕ_m is measurable and it remains to show that

$$\phi_m(\tilde{P}^{(u_0,\dots,u_m)}) = \tilde{P}^{(u_1,\dots,u_m)}.$$

It is sufficient to prove

$$\int X d\phi_m(\tilde{P}^{(u_0,\dots,u_m)}) = \int X d\tilde{P}^{(u_1,\dots,u_m)}$$

for all bounded $\tilde{\mathscr{F}}^{(u_1,\dots,u_m)}$-measurable random variables of the special form $X = YZ$, where Y is $\bigotimes_{k_2\geq 0,\dots,k_m\geq 0}\mathscr{F}^{(0,k_2,\dots,k_m)}$-measurable and Z is Σ-measurable. This means that we only have to prove

$$\int (Y\circ\phi_m)\, d\overline{P}^{(u_0,\dots,u_m)} = \int Y d\overline{P}^{(u_1,\dots,u_m)}.$$

Now we may suppose that there is a sub-σ-algebra $\mathscr{H}\subset\mathscr{F}$ for which there exists a regular conditional probability $K_{\mathscr{H}}$ of $\mathscr{H}$ given $\mathscr{F}_{u_{m-1}}$ relative to P, and such that Y is $\bigotimes_{k_2\geq 0,\dots,k_m\geq 0}\mathscr{H}^{(0,k_2,\dots,k_m)}$-measurable. Then by the definition of $P^{(u_0,\dots,u_m)}$ we have

$$\begin{aligned}
E_{\overline{P}^{(u_0,\dots,u_m)}}&\{Y\circ\phi_m \mid \overline{\mathscr{F}}^{(u_0,\dots,u_m)}_{u_{m-1}}\}\left((\omega^{k_1,\dots,k_m-1,0})_{k_1\geq 0,\dots,k_m\geq 0}\right)\\
&= \int Y\circ\phi_m\, d\bigotimes K_{\mathscr{H}}\left((\omega^{k_1,\dots,k_m-1,0})\right)\ (\overline{P}^{(u_0,\dots,u_m)}\text{-a.s.})\\
&= \int Y d\bigotimes K_{\mathscr{H}}\left((\omega^{0,k_2,\dots,k_m-1,0})\right)\\
&= E_{\overline{P}^{(u_1,\dots,u_m)}}\{Y \mid \overline{\mathscr{F}}^{(u_1,\dots,u_m)}_{u_{m-1}}\}\left((\omega^{k_2,\dots,k_m-1,0})\right)\ (\overline{P}^{(u_1,\dots,u_m)}\text{-a.s.}).
\end{aligned}$$

An easy induction shows that for every $j = 1, \dots, m-1$ there exists a measurable function

$$F_j\colon \prod_{k_2\geq 0,\dots,k_j\geq 0} (\Omega^{0,k_2,\dots,k_j}, \mathscr{F}^{0,k_2,\dots,k_j}_{u_j}) \to \boldsymbol{R}$$

such that

$$E_{\overline{P}^{(u_0,\dots,u_m)}}\{Y\circ\phi_m \mid \overline{\mathscr{F}}^{(u_0,\dots,u_m)}_{u_j}\} = F_j \quad \overline{P}^{(u_0,\dots,u_m)}\text{-a.s.}$$

and

$$E_{\overline{P}^{(u_1,\dots,u_m)}}\{Y \mid \overline{\mathscr{F}}^{(u_1,\dots,u_m)}_{u_j}\} = F_j \quad \overline{P}^{(u_1,\dots,u_m)}\text{-a.s.}$$

Now suppose that $\mathscr{G}\subset\mathscr{F}_{u_1}$ is a σ-algebra for which there exists a regular conditional probability $K_{\mathscr{G}}$ of $\mathscr{G}$ given $\mathscr{F}_{u_0}$ such that F_1 is $\mathscr{G}$-measurable. Then from the definition of the measures $\overline{P}^{(u_0,\dots,u_m)}$ we get

$$\begin{aligned}
\int F_1\, d\overline{P}^{(u_0,\dots,u_m)} &= \int F_1\, d\overline{P}^{(u_0,u_1)} = \int\int F_1\, dK_{\mathscr{G}}(\omega,\cdot)\, P(d\omega)\\
&= \int F_1\, dP = \int F_1\, d\overline{P}^{(u_1,\dots,u_m)}
\end{aligned}$$

and it follows that $E_{\overline{P}^{(u_0,\dots,u_m)}}(Y\circ\phi_m) = E_{\overline{P}^{(u_1,\dots,u_m)}}(Y)$. Thus we have proved that $\phi_m(\overline{P}^{(u_0,\dots,u_m)}) = \overline{P}^{(u_1,\dots,u_m)}$.

(ii) For the proof of the asserted stochastic integral representation we proceed as in (i). It is sufficient to consider martingales $(M_t)_{u_1\leq t\leq u_m}$ on $\tilde{\Omega}^{(u_1,\dots,u_m)}$ which are of the form

$$M_t = E_{\tilde{P}^{(u_1,\dots,u_m)}}\{YZ \mid \tilde{\mathscr{F}}^{(u_1,\dots,u_m)}_t\}$$

with Y and Z as in (i). If

$$M_t = M_{u_{m-1}} + \int_{u_{m-1}}^{t} f_M(s)\, d\tilde{B}_s \quad \tilde{P}^{(u_1,\ldots,u_m)}\text{-a.s.}$$

for $t \in [u_{m-1}, u_m]$, then it follows as in (i) that also

$$M_t \circ \phi_m = M_{u_{m-1}} \circ \phi_m + \int_{u_{m-1}}^{t} f_M(s) \circ \phi_m\, d\tilde{B}_s \quad \tilde{P}^{(u_0,\ldots,u_m)}\text{-a.s.}$$

(cf. part (4), (5) of the proof of Theorem 2.2 and part (3) of the proof of Theorem 2.5). Finally, the proof for the case $t \in [u_{j-1}, u_j]$ $(1 < j < m)$ follows in the same way, and the lemma is proved. ∎

THEOREM 3.3. *Let $\mathfrak{F} = (\mathscr{F}_t)_{t\geq 0}$ be a given right-continuous filtration on the probability space $(\Omega, \mathscr{F}, P)$. Then there exists*

- *an extension $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P})$ of $(\Omega, \mathscr{F}, P)$,*
- *an extension $\tilde{\mathfrak{F}} = (\tilde{\mathscr{F}}_t)_{t\geq 0}$ of $\mathfrak{F}$ on $\tilde{\Omega}$, and*
- *an $\tilde{\mathfrak{F}}$-Brownian motion $\tilde{B} = (\tilde{B}_t)_{t\geq 0}$,*

such that for every square integrable $\tilde{\mathfrak{F}}$-martingale $\tilde{M} = (\tilde{M}_t)_{t\geq 0}$ there exists an $\tilde{\mathfrak{F}}$-progressively measurable function $f_{\tilde{M}}\colon [0, \infty[\times \tilde{\Omega} \to \mathbf{R}$ such that

$$M_t = M_0 + \int_0^t f_{\tilde{M}}(s)\, d\tilde{B}_s \quad \tilde{P}\text{-a.s.} \quad \textit{for every } t \geq 0.$$

As a consequence, $\mathfrak{F}$ has a predictable extension on $\tilde{\Omega}$.

Proof. It follows from Proposition 1.1 that we may suppose that $\mathfrak{F}$ is a discrete filtration, i.e. that $\mathfrak{F} = (\mathscr{F}_t)_{t\in D}$ with $D = \{t_n \mid n \in \mathbf{Z}_+\}$, where (t_n) is a decreasing sequence with $\lim_{n\to\infty} t_n = 0$. We set $\mathscr{F}_0 := \bigcap_{t>0} \mathscr{F}_t$.

With the notation of Lemma 3.1 we define

$$(\tilde{\Omega}^{(n)}, \tilde{\mathscr{F}}^{(n)}, \tilde{P}^{(n)}) := (\tilde{\Omega}^{(t_n,\ldots,t_0)}, \tilde{\mathscr{F}}^{(t_n,\ldots,t_0)}, \tilde{P}^{(t_n,\ldots,t_0)}),$$

$$(\bar{\Omega}^{(n)}, \bar{\mathscr{F}}^{(n)}, \bar{P}^{(n)}) := (\bar{\Omega}^{(t_n,\ldots,t_0)}, \bar{\mathscr{F}}^{(t_n,\ldots,t_0)}, \bar{P}^{(t_n,\ldots,t_0)})$$

for every $n \geq 1$. From Lemma 3.2 we know that for every $n \geq 1$ there exists a measurable map $\phi_n\colon \tilde{\Omega}^{(n)} \to \tilde{\Omega}^{(n-1)}$ $(\tilde{\Omega}^{(0)} = \Omega)$ such that $\phi_n(\tilde{P}^{(n)}) = \tilde{P}^{(n-1)}$. If we use the same notation ϕ_n for the restriction of ϕ_n to $\bar{\Omega}^{(n)}$, then also $\phi_n(\bar{P}^{(n)}) = \bar{P}^{(n-1)}$. This means that $((\tilde{\Omega}^{(n)}, \phi_n))_{n\geq 1}$ and $((\bar{\Omega}^{(n)}, \phi_n))_{n\geq 1}$ are both projective families of probability spaces. Now we define $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P})$ and $(\bar{\Omega}, \bar{\mathscr{F}}, \bar{P})$, respectively, as the projective limits of $(\tilde{\Omega}^{(n)})_{n\geq 1}$ and $(\bar{\Omega}^{(n)})_{n\geq 1}$, respectively, in the sense of probability spaces. Again, we will use the same notation $\psi_n\colon \tilde{\Omega} \to \tilde{\Omega}^{(n)}$ and $\psi_n\colon \bar{\Omega} \to \bar{\Omega}^{(n)}$, respectively, for the canonical projections. Then

$$\tilde{\mathscr{F}} = \bigvee_{n\geq 1} \psi_n^{-1}(\tilde{\mathscr{F}}^{(n)}), \qquad \bar{\mathscr{F}} = \bigvee_{n\geq 1} \psi_n^{-1}(\bar{\mathscr{F}}^{(n)}),$$

and $\tilde{P}$ and $\bar{P}$, respectively, are the unique measures such that $\psi_n(\tilde{P}) = \tilde{P}^{(n)}$ and $\psi_n(\bar{P}) = \bar{P}^{(n)}$, respectively. Furthermore, it follows from Lemma 3.2 that

$$(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P}) = (\bar{\Omega}, \bar{\mathscr{F}}, \bar{P}) \times (S, \Sigma, Q).$$

Now the filtration $\tilde{\mathfrak{F}}$ on $\tilde{\Omega}$ is easily defined. For every $t > 0$ we set $n_t := \min\{m \mid t \geqslant t_m\}$ and define

$$\tilde{\mathscr{F}}_t := \bigvee_{n \geqslant n_t} \psi_n^{-1}(\tilde{\mathscr{F}}_t^{(n)}),$$

where $\tilde{\mathfrak{F}}^{(n)} = (\tilde{\mathscr{F}}_t^{(n)})_{t_n \leqslant t \leqslant t_0}$ is the extension of $(\mathscr{F}_t)_{t \in \{t_j | j = 0, \ldots, n\}}$ as defined in Lemma 3.1. For $t = 0$ we set

$$\tilde{\mathscr{F}}_0 := \bigcap_{n \geqslant 1} \psi_n^{-1}(\tilde{\mathscr{F}}_{t_n}^{(n)}).$$

If ψ_0: $\tilde{\Omega} \to \Omega$ denotes the projection of $\tilde{\Omega}$ to Ω, then $\tilde{\mathscr{F}}_0 = \psi_0^{-1}(\mathscr{F}_0)$, since by the definition of the spaces $\tilde{\Omega}^{(n)}$ the σ-algebras $\tilde{\mathscr{F}}_{t_n}^{(n)}$ can be identified with $\mathscr{F}_{t_n}$.

Now we define $\tilde{B} = (\tilde{B}_t)_{t \geqslant 0}$ as the canonical extension of the Brownian motion B defined on S to $\tilde{\Omega}$. Then we know that for any $s > 0$ and $n \geqslant 1$ with $s \geqslant t_n$ the σ-algebras $\psi_n^{-1}(\tilde{\mathscr{F}}_s^{(n)})$ and $\mathscr{C}_s := \sigma(\tilde{B}_t - \tilde{B}_s;\, t \geqslant s)$ are independent. Hence also $\tilde{\mathscr{F}}_s$ and $\mathscr{C}_s$ are independent by the definition of $\tilde{\mathscr{F}}_s$. The independence of $\tilde{\mathscr{F}}_0$ and $\mathscr{C}_0$ is immediately clear. This shows that $\tilde{B}$ is an $\tilde{\mathfrak{F}}$-Brownian motion.

It remains to prove the asserted stochastic integral representation. Since

$$\tilde{\mathscr{F}}_{t_0} = \bigvee_{n \geqslant 1} \psi_n^{-1}(\tilde{\mathscr{F}}_{t_0}^{(n)}),$$

it is sufficient to prove that representation for every martingale $M^X = (M_t^X)_{t \geqslant 0}$ of the form

$$M_t^X = E_{\tilde{P}}\{X \mid \tilde{\mathscr{F}}_t\},$$

where X is bounded and $\psi_n^{-1}(\tilde{\mathscr{F}}_{t_0}^{(n)})$-measurable. If X is bounded and $\psi_n^{-1}(\tilde{\mathscr{F}}_{t_0}^{(n)})$--measurable, then we infer easily from Lemma 3.2 that for $X = Y^n \circ \psi_n$ (Y^n $\tilde{\mathscr{F}}_{t_0}^{(n)}$-measurable)

$$M_t^X = E_{\tilde{P}^{(n)}}\{Y^n \mid \tilde{\mathscr{F}}_t^{(n)}\} \circ \psi_n \quad \tilde{P}\text{-a.s.} \qquad \text{for all } t \in [t_n, t_0].$$

Let f^n: $[t_n, t_0] \times \tilde{\Omega}^{(n)} \to R$ denote the progressively measurable function such that

$$E_{\tilde{P}^{(n)}}\{Y^n \mid \tilde{\mathscr{F}}_t^{(n)}\} = E_{\tilde{P}^{(n)}}\{Y^n \mid \tilde{\mathscr{F}}_{t_n}^{(n)}\} + \int_{t_n}^{t} f^n(s)\, d\tilde{B}_s \qquad \text{for } t \in [t_n, t_0].$$

Now we set $f_X := f^n \circ \psi_n$ on $[t_n, t_0] \times \tilde{\Omega}$. If $m > n$, then Lemma 3.2 shows that

$$f^m \circ \psi_m |_{[t_n, t_0] \times \tilde{\Omega}} = f^n \circ \psi_n,$$

and thus we get a well-defined $\tilde{\mathfrak{F}}$-progressively measurable function f_X: $[0, \infty] \times \tilde{\Omega} \to R$ such that, for $0 < s < t \leqslant t_0$,

$$M_t^X = M_s^X + \int_s^t f_X(s)\, d\tilde{B}_s \quad \tilde{P}\text{-a.s.}$$

From the definition of $\tilde{\mathscr{F}}_0$ we get

$$M_{t_m}^X = E_{\tilde{P}}\{X \mid \psi_m^{-1}(\tilde{\mathscr{F}}_{t_m}^{(m)})\} \to E_{\tilde{P}}\{X \mid \tilde{\mathscr{F}}_0\} = M_0^X \quad \text{as } m \to \infty,$$

and it follows that

$$M_t^X = M_0^X + \int_0^t f_X(s)\, d\tilde{B}_s \quad \tilde{P}\text{-a.s.} \quad \text{for every } t \geqslant 0.$$

Since this holds for every $n \geqslant 1$ and every bounded $X \in \mathscr{L}^0(\psi_n^{-1}(\tilde{\mathscr{F}}_{t_0}^{(n)}))$, we get such a representation for every bounded $\tilde{\mathscr{F}}_{t_0}$-measurable random variable, and the assertion for the general square integrable $\tilde{\mathfrak{F}}$-martingales follows easily. Thus the theorem is proved. ■

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