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MARCINKIEWICZ-TYPE STRONG LAW OF LARGE NUMBERS FOR PAIRWISE INDEPENDENT RANDOM FIELDS

BY

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Abstract. We present the Marcinkiewicz-type strong law of large numbers for random fields $\{X_{\boldsymbol{n}}, \boldsymbol{n} \in Z_+^d\}$ of pairwise independent random variables, where Z_+^d, $d \geqslant 1$, is the set of positive d-dimensional lattice points with coordinatewise partial ordering.

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1. INTRODUCTION

Let Z_+^d, $d \geqslant 1$, be the set of positive integer d-dimensional lattice points. The points in Z_+^d will be denoted by $\boldsymbol{m}$, $\boldsymbol{n}$, etc., or, sometimes, when necessary, more explicitly by $(m_1, m_2, \ldots, m_d)$, $(n_1, n_2, \ldots, n_d)$, etc. Also, for $\boldsymbol{n} = (n_1, \ldots, n_d)$ we define $|\boldsymbol{n}| = \prod_{i=1}^d n_i$. We shall write $\mathbf{0}$ and $\mathbf{1}$ for points $(0, 0, \ldots, 0)$ and $(1, 1, \ldots, 1)$, respectively. The set Z_+^d is partially ordered by stipulating $\boldsymbol{m} \leqslant \boldsymbol{n}$ if $m_i \leqslant n_i$ for each i, $1 \leqslant i \leqslant d$. Furthermore, we shall write $\boldsymbol{m} < \boldsymbol{n}$ if $\boldsymbol{m} \leqslant \boldsymbol{n}$ and $m_i < n_i$ for at least one i, $1 \leqslant i \leqslant d$. In this paper the limit $\boldsymbol{n} \to \infty$ will mean $\max_{1 \leqslant i \leqslant d} n_i \to \infty$.

Let us define

$$d(x) = \operatorname{Card}\{\boldsymbol{n} \in Z_+^d: |\boldsymbol{n}| = [x]\}$$

and

$$M_d(x) = \operatorname{Card}\{\boldsymbol{n} \in Z_+^d: |\boldsymbol{n}| \leqslant [x]\} = M(x),$$

where $[x]$ denotes the greatest integer not exceeding x, $x \in [0, \infty)$. We have, cf. Simithe [6], [7],

$$M_d(n) = n(\log_+ n)^{d-1}/(d-1)! - M_{d-1}(n), \qquad d \geqslant 2, \tag{1.1}$$

where $\log_+ x = \max(1, \log x)$, $x > 0$. Thus, by (1.1),

$$M_d(x) = O\left(x(\log_+ x)^{d-1}\right) \quad \text{as } x \to \infty. \tag{1.2}$$

Furthermore, for every $\delta > 0$,

$$d(x) = o(x^{\delta}) \quad \text{as } x \to \infty. \tag{1.3}$$

Let $\{X_{\boldsymbol{n}}, \boldsymbol{n} \in Z_+^d\}$ be a random field of pairwise independent random variables, defined on a probability space $(\Omega, \mathscr{A}, P)$. For $\boldsymbol{n} \in Z_+^d$ define the partial sum

$$S_{\boldsymbol{n}} = \sum_{\boldsymbol{k} \leqslant \boldsymbol{n}} X_{\boldsymbol{k}}.$$

The aim of this paper is to present the Marcinkiewicz-type strong law of large numbers for random fields $\{X_{\boldsymbol{n}}, \boldsymbol{n} \in Z_+^d\}$ of pairwise independent random variables. The basic assumption we make is that, for some $0 < C < \infty$,

$$\sum_{\boldsymbol{n}:|\boldsymbol{n}|=k} P(|X_{\boldsymbol{n}}| \geqslant t) \leqslant C d(k) P(|X| \geqslant t) \tag{1.4}$$

for all $k \in N$ and every $t > 0$, where X is a random variable. Let us observe that if (1.4) holds, then for all $\boldsymbol{n} \in Z_+^d$ and every $t > 0$

$$\begin{aligned} \sum_{\boldsymbol{k} \leqslant \boldsymbol{n}} P(|X_{\boldsymbol{k}}| \geqslant t) &\leqslant \sum_{\boldsymbol{k}:|\boldsymbol{k}| \leqslant |\boldsymbol{n}|} P(|X_{\boldsymbol{k}}| \geqslant t) = \sum_{i=1}^{|\boldsymbol{n}|} \sum_{\boldsymbol{k}:|\boldsymbol{k}|=i} P(|X_{\boldsymbol{k}}| \geqslant t) \\ &\leqslant C \sum_{i=1}^{|\boldsymbol{n}|} d(i) P(|X| \geqslant t) = C M_d(|\boldsymbol{n}|) P(|X| \geqslant t). \end{aligned} \tag{1.5}$$

Thus, from this point of view, the condition (1.4) seems to be weaker than the following one:

$$\sum_{\boldsymbol{k} \leqslant \boldsymbol{n}} P(|X_{\boldsymbol{k}}| \geqslant t) \leqslant C |\boldsymbol{n}| P(|X| \geqslant t) \tag{1.6}$$

for all $\boldsymbol{n} \in Z_+^d$ and every $t > 0$.

If (1.6) holds, then we sometimes say that the sequence $\{X_{\boldsymbol{n}}, \boldsymbol{n} \in Z_+^d\}$ is *weakly mean dominated* by the random variable X, cf. Fazekas and Tómács [3] (Definition 2.3). In general, in our opinion, the conditions (1.4) and (1.6) are independent. If (1.4) holds, then we have (1.5).

Many authors have investigated the Marcinkiewicz-type strong law of large numbers for random fields $\{X_{\boldsymbol{n}}, \boldsymbol{n} \in Z_+^d\}$ in the case $d = 1$. Etemadi [2] extended the classical law of large numbers for independent and identically distributed random variables to the case where the random variables are pairwise independent and identically distributed. Choi and Sung [1] have shown that if $\{X_n, n \geqslant 1\}$ is a sequence of pairwise independent and dominated in distribution by a random variable X such that $E|X|^p (\log_+ |X|)^2 < \infty$, $1 < p < 2$, then $(S_n - ES_n)/n^{1/p} \to 0$ a.s. as $n \to \infty$. In the case $d = 2$, also Etemadi [2] proved that if $\{X_{\boldsymbol{n}}, \boldsymbol{n} \in Z_+^d\}$ is a sequence of pairwise independent and identically distributed random variables such that $E|X_1|(\log_+ |X_1|) < \infty$, then

$(S_n - ES_n)/|n| \to 0$ a.s. as $n \to \infty$. On the other hand, Hong and Hwang [4] proved that if $\{X_n, n \in Z_+^d\}$ is a double sequence of pairwise independent random variables such that, for every $n \in Z_+^2$ and all $t > 0$,

$$P(|X_n| \geqslant t) \leqslant P(|X| \geqslant t) \tag{1.7}$$

and $E|X|^p (\log_+ |X|)^3 < \infty$, $1 < p < 2$, then

$$(S_n - ES_n)/(|n|)^{1/p} \to 0 \text{ a.s.} \quad \text{as } n \to \infty. \tag{1.8}$$

Furthermore, Hong and Hwang [4] proved that if (1.7) holds with a random variable X such that $E|X|^p(\log_+ |X|) < \infty$, $1 < p < 2$, then

$$(S_n - ES_n)/(|n|)^{1/p} \to 0 \text{ in } L_1 \quad \text{as } n \to \infty. \tag{1.9}$$

This paper contains complements to the results presented by Hong and Hwang [4] and their generalizations. Let us observe that the condition (1.7) implies (1.4). We would also like to note that some calculations given in the paper by Hong and Hwang [4] are not understandable, for example, why

$$\int_0^{(ij)^{2/p}} P\left(t \leqslant |X|^2 < (ij)^{2/p}\right) dt = \int_0^{(ij)^{1/p}} x^2 \, dF(x),$$

where $F(x)$ is the distribution of X. Assume that, for example, $P(X \geqslant 0) = 0$. Then the right-hand side of the last equality equals zero, but the left-hand side can be positive (cf. (2.2), (2.3), (2.10), (2.14)–(2.16) in Hong and Hwang [4]).

Let us observe that the Marcinkiewicz-type strong law of large numbers holds for identically distributed random variables with arbitrary dependence structure if $0 < p < 1$, cf., e.g., Petrov [5], Chapter IV, Theorem 16. Fazekas and Tómács [3] extend this result to the case of weakly mean dominated random fields $\{X_n, n \in Z_+^d\}$. Thus, this paper also contains complements to some results given by Fazekas and Tómács [3] and their generalizations. We present the Marcinkiewicz-type strong law of large numbers for $1 < p < 2$.

2. RESULTS

We can now formulate our main results.

THEOREM 1. *Let $\{X_n, n \in Z_+^d\}$ be a random field of pairwise independent random variables satisfying the condition* (1.4). *If, for some $1 < p < 2$,*

$$E|X|^p (\log_+ |X|)^{d+1} < \infty, \tag{2.1}$$

then

$$(S_n - ES_n)/|n|^{1/p} \to 0 \text{ a.s.} \quad \text{as } n \to \infty. \tag{2.2}$$

THEOREM 2. *Let* $\{X_n, n \in Z_+^d\}$ *be a random field of pairwise independent random variables satisfying the condition* (1.4). *If, for some* $1 < p < 2$,

$$E|X|^p (\log_+ |X|)^{d-1} < \infty, \tag{2.3}$$

then

$$(S_n - ES_n)/|n|^{1/p} \to 0 \quad in\ L_1 \quad as\ n \to \infty. \tag{2.4}$$

3. AUXILIARY LEMMAS

In the proofs of the results stated in Section 2 we need some lemmas, which we present in this section.

Let $\{X_n, n \in Z_+^d\}$ be a random field. Let us put

$$X'_n = X_n I(|X_n| \leqslant |n|^{1/p}), \qquad X''_n = X_n - X'_n,$$

where $1 < p < 2$.

LEMMA 1. *Let* $\{X_n, n \in Z_+^d\}$ *be a random field of random variables satisfying the condition* (1.4). *Then for every* $1 < p < 2$ *there exists a positive constant* C *such that*

$$\sum_{n \in Z_+^d} E(X'_n)^2/(|n|^{2/p}) \leqslant CE|X|^p (\log_+ |X|)^{d-1} \tag{3.1}$$

and

$$\sum_{n \in Z_+^d} E|X''_n|/(|n|)^{1/p} \leqslant CE|X|^p (\log_+ |X|)^{d-1}. \tag{3.2}$$

Let us observe that we do not assume that the random field $\{X_n, n \in Z_+^d\}$ is pairwise independent.

Proof. By (1.4) we have

$$\begin{aligned}
\sum_{n \in Z_+^d} E(X'_n)^2/(|n|)^{2/p} &= \sum_{n \in Z_+^d} |n|^{-2/p} \int_0^\infty P((X'_n)^2 \geqslant t)\, dt \\
&\leqslant \sum_{n \in Z_+^d} |n|^{-2/p} \int_0^{|n|^{2/p}} P(t \leqslant |X_n|^2)\, dt = \sum_{k=1}^\infty k^{-2/p} \int_0^{k^{2/p}} \sum_{n:|n|=k} P(|X_n|^2 \geqslant t)\, dt \\
&\leqslant C \sum_{k=1}^\infty k^{-2/p} d(k) \int_0^{k^{2/p}} P(|X|^2 \geqslant t)\, dt \\
&= C \sum_{k=1}^\infty k^{-2/p} d(k) \int_0^{k^{2/p}} \{P(|X|^2 \geqslant k^{2/p}) + P(t \leqslant |X|^2 < k^{2/p})\}\, dt \\
&= C \sum_{k=1}^\infty d(k) P(|X| \geqslant k^{1/p}) + C \sum_{k=1}^\infty d(k) k^{-2/p} \int_0^{k^{2/p}} P(t \leqslant X^2 < k^{2/p})\, dt.
\end{aligned} \tag{3.3}$$

But, by (1.2), we get

$$(3.4)\qquad \sum_{k=1}^{\infty} d(k)P(|X|\geqslant k^{1/p}) = \sum_{k=1}^{\infty} d(k)\sum_{i=k}^{\infty} P\big(i^{1/p}\leqslant |X| < (i+1)^{1/p}\big)$$

$$= \sum_{i=1}^{\infty}\Big(\sum_{k=1}^{i} d(k)\Big)P\big(i^{1/p}\leqslant |X| < (i+1)^{1/p}\big) = \sum_{i=1}^{\infty} M_d(i)P\big(i^{1/p}\leqslant |X| < (i+1)^{1/p}\big)$$

$$\leqslant C_1\sum_{i=1}^{\infty} i(\log_+ i)^{d-1}P\big(i^{1/p}\leqslant |X| < (i+1)^{1/p}\big) \leqslant C_2 E|X|^p(\log_+|X|)^{d-1},$$

where C_1 and C_2 are absolute constants depending only on p and d. On the other hand,

$$(3.5)\qquad \sum_{k=1}^{\infty} d(k)k^{-2/p}\int_0^{k^{2/p}} P\big(t\leqslant X^2 < k^{2/p}\big)dt$$

$$= \sum_{k=1}^{\infty} k^{-2/p}d(k)\sum_{i=1}^{k}\int_{(i-1)^{2/p}}^{i^{2/p}} P(t\leqslant X^2 < k^{2/p})dt$$

$$\leqslant \sum_{k=1}^{\infty} k^{-2/p}d(k)\sum_{i=1}^{k} P\big((i-1)^{2/p}\leqslant X^2 < k^{2/p}\big)\big(i^{2/p}-(i-1)^{2/p}\big)$$

$$\leqslant \sum_{k=1}^{\infty} k^{-2/p}d(k)\sum_{i=1}^{k} P\big((i-1)^{1/p}\leqslant |X| < k^{1/p}\big)i^{2/p-1}$$

$$= \sum_{k=1}^{\infty} k^{-2/p}d(k)\sum_{i=1}^{k} i^{2/p-1}\sum_{j=i}^{k} P\big((j-1)^{1/p}\leqslant |X| < j^{1/p}\big).$$

But

$$(3.6)\qquad \sum_{i=1}^{k} i^{2/p-1}\sum_{j=i}^{k} P\big((j-1)^{1/p}\leqslant |X| < j^{1/p}\big)$$

$$= \sum_{j=1}^{k} P\big((j-1)^{1/p}\leqslant |X| < j^{1/p}\big)\sum_{i=1}^{j} i^{2/p-1} \leqslant C_4\sum_{j=1}^{k} j^{2/p}P\big((j-1)^{1/p}\leqslant |X| < j^{1/p}\big),$$

where C_4 is an absolute constant. Hence, by (3.5) and (3.6), we have

$$(3.7)\qquad \sum_{k=1}^{\infty} d(k)k^{-2/p}\int_0^{k^{2/p}} P\big(t\leqslant X^2 < k^{2/p}\big)dt$$

$$\leqslant C_4\sum_{k=1}^{\infty} k^{-2/p}d(k)\sum_{j=1}^{k} j^{2/p}\,P\big((j-1)^{1/p}\leqslant |X| < j^{1/p}\big)$$

$$= C_4\sum_{j=1}^{\infty} j^{2/p}P\big((j-1)^{1/p}\leqslant |X| < j^{1/p}\big)\sum_{k=j}^{\infty} k^{-2/p}d(k).$$

But $d(k) = M(k) - M(k-1)$, so that by (1.2) and the mean-value theorem we get

$$\sum_{k=j}^{\infty} k^{-2/p} d(k) \leqslant C_5 \sum_{k=j}^{\infty} k^{-2/p} (\log k)^{d-1} \leqslant C_6 j^{1-2/p} (\log j)^{d-1}, \tag{3.8}$$

where, here and in what follows, C with or without subscripts denotes a positive generic constant.

Consequently, (3.5)–(3.8) yield

$$\sum_{k=1}^{\infty} d(k) k^{-2/p} \int_0^{k^{2/p}} P(t \leqslant X^2 < k^{2/p})\, dt \tag{3.9}$$

$$\leqslant C \sum_{j=1}^{\infty} j (\log j)^{d-1} P\big((j-1)^{1/p} \leqslant |X| < j^{1/p}\big) \leqslant CE|X|^p (\log_+ |X|)^{d-1}.$$

Thus, from (3.3), (3.4) and (3.9) we get (3.1).

Let us observe that, by (1.4),

$$\sum_{n \in Z_+^d} E|X_n''|/(|n|^{1/p}) = \sum_{n \in Z_+^d} |n|^{-1/p} \int_0^{\infty} P(|X_n''| \geqslant t)\, dt \tag{3.10}$$

$$= \sum_{n \in Z_+^d} |n|^{-1/p} \{|n|^{1/p} P(|X_n| > |n|^{1/p}) + \int_{|n|^{1/p}}^{\infty} P(|X_n| \geqslant t)\, dt\}$$

$$= \sum_{k=1}^{\infty} k^{-1/p} \sum_{n:|n|=k} k^{1/p} P(|X_n| > k^{1/p}) + \sum_{k=1}^{\infty} k^{-1/p} \int_{k^{1/p}}^{\infty} \sum_{|n|:|n|=k} P(|X_n| \geqslant t)\, dt$$

$$\leqslant C \sum_{k=1}^{\infty} d(k) P(|X| > k^{1/p}) + C \sum_{k=1}^{\infty} k^{-1/p} d(k) \int_{k^{1/p}}^{\infty} P(|X| \geqslant t)\, dt.$$

Furthermore, by (1.2), we get

$$\sum_{k=1}^{\infty} d(k) P(|X| > k^{1/p}) = \sum_{k=1}^{\infty} d(k) \sum_{j=k}^{\infty} P\big(k^{1/p} < |X| \leqslant (k+1)^{1/p}\big) \tag{3.11}$$

$$= \sum_{j=1}^{\infty} P\big(j^{1/p} < |X| \leqslant (j+1)^{1/p}\big) \sum_{k=1}^{j} d(k)$$

$$\leqslant C \sum_{j=1}^{\infty} j (\log j)^{d-1} P\big(j^{1/p} < |X| \leqslant (j+1)^{1/p}\big) \leqslant CE|X|^p (\log_+ |X|)^{d-1}.$$

On the other hand, we have

$$\sum_{k=1}^{\infty} k^{-1/p} d(k) \int_{k^{1/p}}^{\infty} P(|X| \geqslant t)\, dt = \sum_{k=1}^{\infty} k^{-1/p} d(k) \sum_{j=k}^{\infty} \int_{j^{1/p}}^{(j+1)^{1/p}} P(|X| \geqslant t)\, dt \tag{3.12}$$

$$\leqslant \sum_{k=1}^{\infty} k^{-1/p} d(k) \sum_{j=k}^{\infty} P(|X| \geqslant j^{1/p})\, j^{1/p-1}$$

$$= \sum_{k=1}^{\infty} k^{-1/p} d(k) \sum_{j=k}^{\infty} j^{1/p-1} \sum_{i=j}^{\infty} P\big(i^{1/p} \leqslant |X| < (i+1)^{1/p}\big)$$

$$= \sum_{k=1}^{\infty} k^{-1/p} d(k) \sum_{i=k}^{\infty} P\big(i^{1/p} \leqslant |X| < (i+1)^{1/p}\big) \sum_{j=k}^{i} j^{1/p-1}$$

$$\leqslant C_1 \sum_{k=1}^{\infty} k^{-1/p} d(k) \sum_{i=k}^{\infty} i^{1/p} P\big(i^{1/p} \leqslant |X| < (i+1)^{1/p}\big)$$

$$= \sum_{i=1}^{\infty} i^{1/p} P\big(i^{1/p} \leqslant |X| < (i+1)^{1/p}\big) \sum_{k=1}^{i} k^{-1/p} d(k)$$

$$\leqslant C_2 \sum_{i=1}^{\infty} i (\log i)^{d-1} P\big(i^{1/p} \leqslant |X| < (i+1)^{1/p}\big) \leqslant C_3 E|X|^p (\log_+ |X|)^{d-1}.$$

Thus, taking into account (3.10)–(3.12) we easily get (3.2), and this completes the proof of Lemma 1.

LEMMA 2. *Let $\{X_{\boldsymbol{n}}, \boldsymbol{n} \in Z_+^d\}$ be a random field of pairwise independent random variables such that $EX_{\boldsymbol{n}} = 0$, $\boldsymbol{n} \in Z_+^d$. Then there exists a positive constant C such that*

$$E\big(\max_{\boldsymbol{1} \leqslant \boldsymbol{k} \leqslant \boldsymbol{n}} |S_{\boldsymbol{k}}|\big)^2 \leqslant C |\log_+ \boldsymbol{n}|^2 \sum_{\boldsymbol{k} \leqslant \boldsymbol{n}} EX_{\boldsymbol{k}}^2, \tag{3.13}$$

where, for $\boldsymbol{n} = (n_1, \ldots, n_d)$, $|\log_+ \boldsymbol{n}| = \prod_{i=1}^{d} \log_+ n_i$.

Proof. If $\boldsymbol{n} = \mathbb{1}$, then (3.13) holds. We now turn to the case $\mathbb{1} < \boldsymbol{n} = (n_1, \ldots, n_d) \in Z_+^d$. Let $\boldsymbol{s} = \boldsymbol{s}(\boldsymbol{n}) = (s_1, \ldots, s_d)$, where s_i, $1 \leqslant i \leqslant d$, are integers such that $2^{s_i - 1} < n_i \leqslant 2^{s_i}$ if $n_i > 1$ and $s_i = 0$ if $n_i = 1$, i.e., $s_i = \lceil \log_2 n_i \rceil = \min\{k \geqslant 0\colon \log_2 n_i \leqslant k\}$. Set $X_{\boldsymbol{k}}^* = X_{\boldsymbol{k}}$ if $\boldsymbol{k} \leqslant \boldsymbol{n}$ and $X_{\boldsymbol{k}}^* = 0$ otherwise. We obviously have

$$\sum_{\boldsymbol{k} \leqslant 2^{\boldsymbol{s}}} X_{\boldsymbol{k}}^* = \sum_{\boldsymbol{k} \leqslant \boldsymbol{n}} X_{\boldsymbol{k}}, \qquad S_{\boldsymbol{k}} = S_{\boldsymbol{k}}^*,\ \boldsymbol{k} \leqslant \boldsymbol{n},$$

where $2^{\boldsymbol{s}} = (2^{s_1}, \ldots, 2^{s_d})$ and $S_{\boldsymbol{k}}^* = \sum_{\boldsymbol{i} \leqslant \boldsymbol{k}} X_{\boldsymbol{i}}^*$.

Let us divide every interval $(0, 2^{s_i}]$ into $(0, 2^{s_i - 1}]$ and $(2^{s_i - 1}, 2^{s_i}]$ and each of these two intervals into two halves, and so on. Thus, the elements of the j_i-th partition are of length $2^{s_i - j_i}$, $j_i = 0, 1, \ldots, s_i$, so that we obtain the $\boldsymbol{j} = (j_1, j_2, \ldots, j_d)$-th partition $P_{\boldsymbol{j}} = P_{j_1, \ldots, j_d}$ of $(0, 2^{s_1}] \times \ldots \times (0, 2^{s_d}]$ by the j_i-th partition of $(0, 2^{s_i}]$, $1 \leqslant i \leqslant d$. Furthermore, let us observe that for every $\boldsymbol{k} = (k_1, \ldots, k_d) \in Z_+^d$, $\boldsymbol{k} \leqslant \boldsymbol{n}$, the set $(\boldsymbol{0}, \boldsymbol{k}] = (0, k_1] \times \ldots \times (0, k_d]$ is the sum of at most $|\boldsymbol{s} + \mathbb{1}| = \prod_{i=1}^{d} (s_i + 1)$ disjoint sets each of which belongs to a different partition. Thus, we can write

$$S_{\boldsymbol{k}} = \sum_{\boldsymbol{0} \leqslant \boldsymbol{l} \leqslant \boldsymbol{s}} Y_{\boldsymbol{l}, \boldsymbol{k}},$$

where $Y_{l,k}$ is the sum of all random variables X_r^* belonging to the set $(a_1, b_1] \times \ldots \times (a_d, b_d]$, $b_i - a_i = 2^{l_i}$, $1 \leqslant i \leqslant d$, and such that $r \leqslant k$, where $l = (l_1, \ldots, l_d)$.

Let

$$T_j = \sum_{k \leqslant 2^j} |Y_k|^2, \qquad T = \sum_{0 \leqslant r \leqslant s} T_r,$$

where Y_k is the sum of all random variables X_r^* which belong to the k-element of P_j and $2^j = (2^{j_1}, \ldots, 2^{j_d})$. By the Schwarz inequality we have

$$(3.14) \qquad |S_k|^2 \leqslant |s+1| \sum_{0 \leqslant l \leqslant s} |Y_{l,k}|^2 \leqslant |s+1| \, T,$$

where $s+1 = (s_1+1, \ldots, s_d+1)$. On the other hand,

$$(3.15) \qquad ET_j \leqslant \sum_{k \leqslant n} E|X_k|^2$$

and

$$(3.16) \qquad ET \leqslant |s+1| \sum_{k \leqslant n} E|X_k|^2.$$

Thus, by (3.14)–(3.16), we get

$$E(\max_{1 \leqslant k \leqslant n} |S_k|)^2 \leqslant |s+1|^2 \sum_{k \leqslant n} E|X_k|^2 \leqslant |\lceil \log_2 n \rceil + 1|^2 \sum_{k \leqslant n} E|X_k|^2,$$

where $\lceil \log_2 n \rceil = (\lceil \log_2 n_1 \rceil, \ldots, \lceil \log_2 n_d \rceil)$.

This last inequality implies (3.13) and completes the proof of Lemma 2.

Lemma 2 is a d-dimensional version of Lemma 2.2 presented by Hong and Hwang [4].

4. PROOFS OF THEOREMS

The symbol C, with or without subscripts, denotes a positive generic constant.

Proof of Theorem 1. Let us put

$$X'_n = X_n I(|X_n| \leqslant |n|^{1/p}), \qquad X''_n = X_n - X''_n,$$

$$S_n = \sum_{k \leqslant n} X_k, \qquad S'_n = \sum_{k \leqslant n} X'_k.$$

Then, by (1.4), we get

$$(4.1)\quad \sum_{n\in Z_+^d} P(X_n \neq X_n') = \sum_{k=1}^{\infty} \sum_{n:|n|=k} P(|X_n| > |n|^{1/p}) \leqslant C \sum_{k=1}^{\infty} d(k) P(|X| \geqslant k^{1/p})$$

$$= C \sum_{k=1}^{\infty} d(k) \sum_{j=k}^{\infty} P\big(j^{1/p} \leqslant |X| < (j+1)^{1/p}\big)$$

$$= C \sum_{j=1}^{\infty} P\big(j^{1/p} \leqslant |X| < (j+1)^{1/p}\big) M(j)$$

$$\leqslant C_1 \sum_{j=1}^{\infty} j(\log_+ j)^{d-1} P\big(j^{1/p} \leqslant |X| < (j+1)^{1/p}\big) \leqslant C_2 E|X|^p (\log_+ |X|)^{d-1} < \infty.$$

Thus, by (4.1) and the Borel–Cantelli lemma, we obtain

$$(4.2)\quad (S_n - S_n')/|n|^{1/p} \to 0 \text{ a.s.} \quad \text{as } n \to \infty.$$

On the other hand, we have

$$(4.3)\quad (S_n - ES_n)/|n|^{1/p} = (S_n - S_n')/|n|^{1/p} + (S_n' - ES_n')/|n|^{1/p} + (ES_n' - ES_n)/|n|^{1/p}$$

and

$$(4.4)\quad |ES_n' - ES_n|/|n|^{1/p} \leqslant \sum_{k\leqslant n} E|X_k''|/|n|^{1/p}.$$

Moreover, by (3.2) in Lemma 1, we have

$$(4.5)\quad \sum_{k\in Z_+^d} \Big(\sum_{i\leqslant 2^k} E|X_i''|\Big)/|2^k|^{1/p}$$
$$\leqslant C \sum_{i\in Z_+^d} E|X_i''|/|i|^{1/p} \leqslant C_1 E|X|^p (\log_+ |X|)^{d-1} < \infty,$$

where, for $k = (k_1, \ldots, k_d)$, here and subsequently $2^k = (2^{k_1}, \ldots, 2^{k_d})$. We conclude from (4.5) that

$$(4.6)\quad \sum_{k\leqslant 2^n} E|X_k''|/|2^n|^{1/p} \to 0 \quad \text{as } n \to \infty.$$

Moreover, for every $n \in Z_+^d$ such that $2^k < n < 2^{k+1}$, we have

$$(4.7)\quad \sum_{i\leqslant 2^k} E|X_i''|/|2^{k+1}|^{1/p} \leqslant \sum_{i\leqslant n} E|X_i''|/|n|^{1/p} \leqslant \sum_{i\leqslant 2^{k+1}} E|X_i''|/|2^k|^{1/p}$$

and $|2^{k+1}| = 2^d |2^k|$ for all $k \in Z_+^d$. By using now (4.4) combined with (4.5)–(4.7), we find

$$(4.8)\quad |ES_n' - ES_n|/|n|^{1/p} \to 0 \quad \text{as } n \to \infty.$$

Now we prove that

$$(4.9)\qquad (S'_{\mathbf{n}}-ES'_{\mathbf{n}})/|\mathbf{n}|^{1/p}\to 0 \text{ a.s.} \quad \text{as } \mathbf{n}\to\infty .$$

By Chebyshev's inequality and (3.1) in Lemma 1, we get

$$(4.10)\qquad \sum_{\mathbf{k}\in Z_+^d} P\big(|S'_{2^{\mathbf{k}}}-ES'_{2^{\mathbf{k}}}|\geqslant \varepsilon|2^{\mathbf{k}}|^{1/p}\big)\leqslant \varepsilon^{-2}\sum_{\mathbf{k}\in Z_+^d} E(S'_{2^{\mathbf{k}}}-ES'_{2^{\mathbf{k}}})|2^{\mathbf{k}}|^{-2/p}$$

$$\leqslant \varepsilon^{-2}\sum_{\mathbf{k}\in Z_+^d}(|2^{\mathbf{k}}|^{-2/p})\sum_{\mathbf{i}\leqslant 2^{\mathbf{k}}} E(X'_{\mathbf{i}})^2\leqslant C\varepsilon^{-2}\sum_{\mathbf{n}\in Z_+^d} E(X'_{\mathbf{n}})^2/|\mathbf{n}|^{2/p}$$

$$\leqslant C\varepsilon^{-2}E|X|^p(\log_+|X|)^{d-1}<\infty .$$

Thus, by the Borel–Cantelli lemma and (4.10), we have

$$(4.11)\qquad (S'_{2^{\mathbf{n}}}-ES'_{2^{\mathbf{n}}})/|2^{\mathbf{n}}|^{1/p}\to 0 \text{ a.s.} \quad \text{as } \mathbf{n}\to\infty .$$

On the other hand, if $2^{\mathbf{k}}<\mathbf{n}<2^{\mathbf{k}+1}$, then

$$(4.12)\qquad |S'_{\mathbf{n}}-ES'_{\mathbf{n}}|/|\mathbf{n}|^{1/p}\leqslant |S'_{2^{\mathbf{k}}}-ES'_{2^{\mathbf{k}}}|/|\mathbf{n}|^{1/p}+\Big|\sum_{2^{\mathbf{k}}<\mathbf{i}\leqslant\mathbf{n}}(X'_{\mathbf{i}}-EX'_{\mathbf{i}})\Big|/|\mathbf{n}|^{1/p}$$

$$\leqslant |S'_{2^{\mathbf{k}}}-ES'_{2^{\mathbf{k}}}|/|2^{\mathbf{k}}|^{1/p}+\max_{2^{\mathbf{k}}<\mathbf{i}<2^{\mathbf{k}+1}}|T(\mathbf{i},\mathbf{k})|/|2^{\mathbf{k}}|^{1/p},$$

where

$$T(\mathbf{i},\mathbf{k})=\sum_{2^{\mathbf{k}}<\mathbf{l}\leqslant\mathbf{i}}(X'_{\mathbf{l}}-EX'_{\mathbf{l}}).$$

Now, by using Lemma 2, easy computations lead to

$$(4.13)\qquad \sum_{\mathbf{k}\in Z_+^d} P\Big(\max_{2^{\mathbf{k}}<\mathbf{i}<2^{\mathbf{k}+1}}|T(\mathbf{i},\mathbf{k})|\geqslant\varepsilon|2^{\mathbf{k}}|^{1/p}\Big)$$

$$\leqslant \varepsilon^{-2}\sum_{\mathbf{k}\in Z_+^d} E\Big(\max_{2^{\mathbf{k}}<\mathbf{i}<2^{\mathbf{k}+1}}|T(\mathbf{i},\mathbf{k})|^2\Big)/|2^{\mathbf{k}}|^{2/p}$$

$$\leqslant C\varepsilon^{-2}\sum_{\mathbf{k}\in Z_+^d}|2^{\mathbf{k}}|^{-2/p}|\log_+(2^{\mathbf{k}+1}-2^{\mathbf{k}})|^2\sum_{2^{\mathbf{k}}<\mathbf{i}<2^{\mathbf{k}+1}}E(X'_{\mathbf{i}})^2$$

$$\leqslant C_1\varepsilon^{-2}\sum_{\mathbf{k}\in Z_+^d}|\mathbf{k}|^2|2^{\mathbf{k}}|^{-2/p}\sum_{2^{\mathbf{k}}<\mathbf{i}<2^{\mathbf{k}+1}}E(X'_{\mathbf{i}})^2$$

$$\leqslant C_2\varepsilon^{-2}\sum_{\mathbf{k}\in Z_+^d}(\log_{2+}|\mathbf{k}|)^2|\mathbf{k}|^{-2/p}E(X'_{\mathbf{k}})^2,$$

where, if $\mathbf{k}=(k_1,\ldots,k_d)$, then

$$(2^{\mathbf{k}+1}-2^{\mathbf{k}})=(2^{k_1+1}-2^{k_1},\ldots,2^{k_d+1}-2^{k_d})=2^{\mathbf{k}}.$$

Moreover, as in the proof of Lemma 1 ((3.3)–(3.9)), we get

(4.14)
$$\sum_{k\in Z_+^d} (\log_{2+}|k|)^2 |k|^{-2/p} E(X_k')^2$$
$$\leqslant C\sum_{k=1}^{\infty} (\log_+ k)^2 k^{-2/p} \int_0^k \sum_{n:|n|=k} P(|X_n|^2 \geqslant t)\,dt$$
$$\leqslant C\sum_{k=1}^{\infty} (\log_+ k)^2 k^{-2/p} d(k) \int_0^{k^{2/p}} P(|X|^2 \geqslant t)\,dt$$
$$\leqslant C\sum_{k=1}^{\infty} (\log_+ k)^2 d(k) P(|X| \geqslant k^{1/p})$$
$$+C\sum_{k=1}^{\infty} (\log_+ k)^2 d(k) k^{-2/p} \int_0^{k^{2/p}} P(t \leqslant X^2 < k^{2/p})\,dt$$
$$\leqslant C\sum_{i=1}^{\infty} \Big(\sum_{k=1}^{i} d(k)(\log_+ k)^2\Big) P\big(i^{1/p} \leqslant |X| < (i+1)^{1/p}\big)$$
$$+C\sum_{k=1}^{\infty} k^{-2/p} d(k)(\log_+ k)^2 \sum_{i=1}^{k} \int_{(i-1)^{2/p}}^{i^{2/p}} P(t \leqslant X^2 < k^{2/p})\,dt$$
$$\leqslant C\sum_{i=1}^{\infty} (\log_+ i)^2 M(i) P\big(i^{1/p} \leqslant |X| < (i+1)^{1/p}\big)$$
$$+C\sum_{k=1}^{\infty} k^{-2/p} d(k)(\log_+ k)^2 \sum_{i=1}^{k} P\big((i-1)^{2/p} \leqslant X^2 < k^{2/p}\big)\big(i^{2/p}-(i-1)^{2/p}\big)$$
$$\leqslant C_1 E|X|(\log_+|X|)^{d+1}$$
$$+C_2\sum_{k=1}^{\infty} k^{-2/p} d(k)(\log_+ k)^2 \sum_{i=1}^{k} i^{2/p-1} P\big((i-1)^{1/p} \leqslant |X| < k^{1/p}\big)$$
$$= C_1 E|X|(\log_+|X|)^{d+1}$$
$$+C_2\sum_{k=1}^{\infty} k^{-2/p} d(k)(\log_+ k)^2 \sum_{i=1}^{k} i^{2/p-1} \sum_{j=i}^{k} P\big((j-1)^{1/p} \leqslant |X| < j^{1/p}\big)$$
$$\leqslant C_1 E|X|(\log_+|X|)^{d+1}$$
$$+C_3\sum_{k=1}^{\infty} k^{-2/p} d(k)(\log_+ k)^2 \sum_{j=1}^{k} j^{2/p} P\big((j-1)^{1/p} \leqslant |X| < j^{1/p}\big)$$
$$\leqslant C_1 E|X|(\log_+|X|)^{d+1}$$
$$+C_4\sum_{j=1}^{\infty} j^{2/p} P(j-1 \leqslant |X|^p < j) \sum_{k=j}^{\infty} k^{-2/p} d(k)(\log_+ k)^2$$
$$\leqslant C_1 E|X|(\log_+|X|)^{d+1} + C_5\sum_{j=1}^{\infty} j(\log_+ j)^{d+1} P(j-1 \leqslant |X|^p < j)$$
$$\leqslant C_6 E|X|(\log_+|X|)^{d+1}.$$

Finally, (4.9) follows by combining (4.11) with (4.12)–(4.14) and the Borel–Cantelli lemma. We complete the proof of Theorem 1 by using (4.3) together with (4.2), (4.9) and (4.8).

Proof of Theorem 2. It follows that

$$(4.15)\quad E|S_n-ES_n|/|n|^{1/p} \leqslant E|S_n-S'_n|/|n|^{1/p}+E|S'_n-ES'_n|/|n|^{1/p}+|ES'_n-ES_n|/|n|^{1/p},$$

$$(4.16)\quad E|S_n-S'_n|/|n|^{1/p} \leqslant \sum_{k\leqslant n} E|X''|/|n|^{1/p},$$

$$(4.17)\quad E|S'_n-ES'_n|/|n|^{1/p} \leqslant \{E(S'_n-ES'_n)^2\}/|n|^{1/p} \leqslant \{\sum_{k\leqslant n} E(X'_k)^2/|n|^{2/p}\}^{1/2}$$

and, by (4.8),

$$(4.18)\quad |ES'_n-ES_n|/|n|^{1/p}\to 0 \quad \text{as } n\to\infty.$$

Notice that (4.8) is a consequence of (2.3). Moreover, (2.3) also implies that (4.6) and (4.7) hold. Thus, by (2.3), we also get

$$(4.19)\quad \sum_{k\leqslant n} E|X''_k|/|n|^{1/p}\to 0 \quad \text{as } n\to\infty.$$

On the other hand, by (3.1) in Lemma 1, we get

$$(4.20)\quad \sum_{k\in Z^d_+}\Big(\sum_{i\leqslant 2^k} E(X'_i)^2\Big)/|2^k|^{2/p} \leqslant C\sum_{k\in Z^d_+} E(X'_i)^2/|i|^{2/p} \leqslant C_1 E|X|(\log_+|X|)^{d-1}.$$

Hence, by (4.20),

$$(4.21)\quad \sum_{k\leqslant 2^n} E(X'_k)^2/|2^n|^{2/p}\to 0 \quad \text{as } n\to\infty.$$

Next, by using the fact that for every $2^k<n<2^{k+1}$

$$\sum_{i\leqslant 2^k} E(X'_i)^2/|2^{k+1}|^{2/p} \leqslant \sum_{i\leqslant n} E(X'_i)^2/|n|^{2/p} \leqslant \sum_{i\leqslant 2^{k+1}} E(X'_i)^2/|2^k|^{2/p},$$

and since $|2^{k+1}|=2^d|2^k|$, we obtain

$$(4.22)\quad \sum_{k\leqslant n} E(X'_k)^2/|n|^{2/p}\to 0 \quad \text{as } n\to\infty.$$

Finally, (2.4) follows by combining (4.15) with (4.16)–(4.19) and (4.22).

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