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ON STABILITY OF TRIMMED SUMS

BY

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Abstract. Let $\{X_n, n \geqslant 1\}$ be a sequence of i.i.d. random variables and let $\{a_n, n \geqslant 1\}$ and $\{b_n, n \geqslant 1\}$ be sequences of constants where $0 < b_n \uparrow \infty$. Let $X_n^{(1)}, X_n^{(2)}, \ldots, X_n^{(n)}$ be a rearrangement of $X_1, \ldots, X_n$ such that $|X_n^{(1)}| \geqslant |X_n^{(2)}| \geqslant \ldots \geqslant |X_n^{(n)}|$. Consider the sequence of weighted sums $T_n = \sum_{i=1}^n a_i X_i$, $n \geqslant 1$, and, for fixed $r \geqslant 1$, set $T_n^{(r)} = \sum_{i=1}^n a_i X_i I(|X_i| \leqslant |X_n^{(r+1)}|)$, $n \geqslant r+1$; i.e., $T_n^{(r)}$ is the sum T_n minus the sum of the $X_n^{(k)}$'s multiplied by their corresponding coefficients for $k = 1, \ldots, r$. The main results provide sufficient and, separately, necessary conditions for $b_n^{-1} T_n^{(r)} - k_n \to 0$ almost surely for some sequence of centering constants $\{k_n, n \geqslant 1\}$. The current work extends that of Mori [14], [15] wherein $a_n \equiv 1$.

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1. INTRODUCTION

If $\{X_n, n \geqslant 1\}$ is a sequence of independent and identically distributed (i.i.d.) random variables which are not necessarily integrable and $S_n = \sum_{i=1}^n X_i$ denotes the n-th partial sum, it is known under suitable conditions that the sum S_n is essentially dominated by the contribution of a small number of its extreme terms, the remainder being asymptotically negligible. The almost sure (a.s.) asymptotic behavior of the partial sums S_n is strongly influenced by the effect of the most extreme terms of the sample $\{|X_1|, |X_2|, \ldots, |X_n|\}$. Even though limit laws such as the strong law of large numbers (SLLN) and the law of the iterated logarithm (LIL) can fail for the partial sums S_n, it has been shown by various authors that if the most extreme terms in the sample $\{|X_1|, |X_2|, \ldots, |X_n|\}$ are removed from S_n, then versions of those limit laws can indeed prevail. These limit laws which then hold when extreme terms are removed are often referred to as "improved" versions of the classical limit laws. The partial sum S_n with the extreme terms removed from it is referred to as a *trimmed sum*. Trimmed sums are used for reducing variability in connection with some statistical inference procedures; see Barnett and Lewis [2], Section 3.2.1.

The influence of extreme terms on the limiting behavior of S_n was apparently first noticed by Lévý [11] over sixty five years ago. More recently, Feller [6] and Mori [14], [15] obtained improved versions of the LIL and SLLN, respectively, by removing the most extreme terms from the partial sums. In other words, Feller [6] and Mori [14], [15] established a LIL and SLLN, respectively, for trimmed sums of i.i.d. random variables.

Let $X_n^{(1)}, X_n^{(2)}, \ldots, X_n^{(n)}$ be a rearrangement of $X_1, X_2, \ldots, X_n$ in decreasing order of absolute magnitude; i.e., $|X_n^{(1)}| \geqslant |X_n^{(2)}| \geqslant \ldots \geqslant |X_n^{(n)}|$, $n \geqslant 1$. For $1 \leqslant r \leqslant n$, let us set $S_n^{(r)} = S_n - X_n^{(1)} - X_n^{(2)} - \ldots - X_n^{(r)}$. Thus $S_n^{(r)}$ is the partial sum S_n with the r largest (in absolute value) summands removed. Since r is fixed, $S_n^{(r)}$ is a so-called *lightly* trimmed sum. Pioneering work of Mori [14] on the SLLN problem for lightly trimmed sums $S_n^{(r)}$ included the following elegant analogue of the SLLN. For fixed $r \geqslant 1$,

$$S_n^{(r)}/n - k_n \to 0 \text{ a.s.}$$

for some sequence of constants $\{k_n, n \geqslant 1\}$ if and only if $\int_0^\infty x^r \mathfrak{I}(x)^{r+1}\, dx < \infty$, where $\mathfrak{I}(x) = P(|X_1| > x)$, $x \geqslant 0$. This shows that by trimming off a fixed number of extreme terms, the a.s. limiting behavior of S_n can be improved since we may have $\int_0^\infty x^r \mathfrak{I}(x)^{r+1}\, dx < \infty$ when $E|X_1| = \infty$. Mori [15] generalized his work in [14] to allow for a more general norming sequence $\{b_n, n \geqslant 1\}$ instead of only the choice $b_n = n$, $n \geqslant 1$.

Following the work of Mori [14], [15], there has been a large literature of investigation of the strong and weak limiting behavior of trimmed sums; see, for example, the work of Maller [12], [13], Mori [16], Einmahl and Haeusler [5], Kesten and Maller [8]–[10], Kesten [7], Pozdnyakov [18], and Csörgő and Simons [4]. In all of these papers, the summands $\{X_n, n \geqslant 1\}$ are i.i.d.

In the current paper, we extend the work of Mori [14], [15] to the case of lightly trimmed *weighted sums* of i.i.d. random variables. We are unaware of any previous work on the SLLN problem for trimmed sums where the summands are independent but are not identically distributed. We will show that we can also improve the a.s. limiting behavior of weighted sums by trimming off a fixed number of extreme terms. Our methods are analogous to those of Mori [14], [15]; the authors take great pleasure in acknowledging that the current work owes much to that of Mori [14], [15].

Assume that $\{a_n, n \geqslant 1\}$ and $\{b_n, n \geqslant 1\}$ are sequences of constants where $0 < b_n \uparrow \infty$. Let

$$T_n = \sum_{i=1}^{n} a_i X_i, \qquad T_n^{(0)} = T_n,$$

and

$$T_n^{(r)} = \sum_{i=1}^{n} a_i X_i I(|X_i| \leqslant |X_n^{(r+1)}|) = T_n - \sum_{i=1}^{n} a_i X_i I(|X_i| > |X_n^{(r+1)}|), \qquad n \geqslant 1;$$

i.e., $T_n^{(r)}$ is equal to T_n minus the sum of the $X_n^{(k)}$'s multiplied by their corresponding coefficients for $k = 1, 2, \ldots, r$. Adler and Rosalsky [1] provided necessary and/or sufficient conditions for $\{a_n X_n, n \geqslant 1\}$ to obey the general SLLN with norming constants $\{b_n, n \geqslant 1\}$; that is, for the normed weighted sum T_n/b_n to converge a.s. to 0.

For the sequences $\{c_n, n \geqslant 1\}$ and $\{q_n, n \geqslant 1\}$ defined by

$$c_n = \frac{b_n}{\max_{k \leqslant n} |a_k|}, \qquad q_n = \frac{b_n}{\min_{k \leqslant n} |a_k|}, \qquad n \geqslant 1,$$

two main results will be presented. Based on the convergence of

$$\int_0^\infty x^r \mathfrak{I}(C(x))^{r+1}\, dx \quad \text{and} \quad \int_0^\infty x^r \mathfrak{I}(Q(x))^{r+1}\, dx,$$

where $C(x)$ and $Q(x)$ are extensions of $\{c_n, n \geqslant 1\}$ and $\{q_n, n \geqslant 1\}$, respectively, we shall provide a sufficient condition in Theorem 1 and a necessary condition in Theorem 2, respectively, for the stability of the sequence of normed trimmed sums $T_n^{(r)}/b_n$; that is, for $b_n^{-1} T_n^{(r)} - k_n \to 0$ a.s. for some sequence of centering constants $\{k_n, n \geqslant 1\}$. Results of Mori [14], [15] are the special case $a_n \equiv 1$ of Corollary 1. An example is also given where the conditions of Theorem 2 of Adler and Rosalsky [1] fail but the conditions of our Theorem 1 hold.

2. PRELIMINARIES

Throughout this paper, let $\{X_n, n \geqslant 1\}$ be a sequence of i.i.d. random variables with common distribution function F and let $\mathfrak{I}(x) = P(|X_1| > x)$, $x \geqslant 0$. For $r \geqslant 1$ and $n \geqslant r$, let $X_n^{(r)} = X_j$ if $|X_j|$ is the r-th largest of $|X_1|, |X_2|, \ldots, |X_n|$. More precisely, let $M_n(j)$, $n \geqslant 1$, $1 \leqslant j \leqslant n$, be the number of X_i's satisfying either $|X_i| > |X_j|$, $1 \leqslant i \leqslant n$, or $|X_i| = |X_j|$, $1 \leqslant i \leqslant j$, and let $X_n^{(r)} = X_j$ if $M_n(j) = r$. Let $\{a_n, n \geqslant 1\}$ be a sequence of constants and, for $n \geqslant 1$, set

$$T_n = \sum_{i=1}^{n} a_i X_i, \qquad T_n^{(0)} = T_n, \qquad \text{and} \qquad T_n^{(r)} = \sum_{i=1}^{n} a_i X_i I(|X_i| \leqslant |X_n^{(r+1)}|).$$

Some preliminary lemmas will be established before stating the main results. The first two lemmas are due to Mori [14] and [15] who stated them without proof. For completeness we present their proofs.

LEMMA 1 (Mori [14]). *If* $0 < p_n < 1$, $n \geqslant 1$, *and* $\lim_{n\to\infty} np_n = 0$, *then for all fixed integers* $r \geqslant 0$

$$\sum_{k=r}^{n} \binom{n}{k} p_n^k (1-p_n)^{n-k} \sim \frac{1}{r!}(np_n)^r \quad \text{as } n \to \infty.$$

Proof. Note that since $np_n \to 0$, for arbitrary $0 < \varepsilon < 1$ and all large n we have $np_n < \varepsilon$ and

$$\frac{\binom{n}{k+1} p_n^{k+1}(1-p_n)^{n-k-1}}{\binom{n}{k} p_n^k (1-p_n)^{n-k}} = \frac{(n-k)p_n}{(k+1)(1-p_n)} \leqslant \varepsilon$$

for $0 \leqslant k \leqslant n$. From Stirling's formula

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

it follows that

$$\begin{aligned} \binom{n}{r} p_n^r (1-p_n)^{n-r} &= \frac{n!}{r!(n-r)!} p_n^r (1-p_n)^{n-r} \\ &\sim \frac{(n/e)^n \sqrt{2\pi n}}{r!((n-r)/e)^{n-r}\sqrt{2\pi(n-r)}} p_n^r (1-p_n)^{n-r} \\ &= \frac{(np_n)^r}{r!e^r}\left(\frac{n(1-p_n)}{n-r}\right)^{n-r}\sqrt{\frac{n}{n-r}} \sim \frac{(np_n)^r}{r!}. \end{aligned}$$

Then for all large n

$$\begin{aligned} \sum_{k=r}^{n} \binom{n}{k} p_n^k (1-p_n)^{n-k} &\leqslant \sum_{k=r}^{n} \binom{n}{r} p_n^r (1-p_n)^{n-r} \varepsilon^{k-r} \\ &= (1+o(1))\frac{(np_n)^r}{r!}\sum_{k=r}^{n} \varepsilon^{k-r} = \frac{(1+o(1))(np_n)^r}{(1-\varepsilon)r!}. \end{aligned}$$

Since $0 < \varepsilon < 1$ is arbitrary,

$$\limsup_{n\to\infty} \frac{\sum_{k=r}^{n} \binom{n}{k} p_n^k (1-p_n)^{n-k}}{(np_n)^r/r!} \leqslant 1.$$

On the other hand,

$$\liminf_{n\to\infty} \frac{\sum_{k=r}^{n} \binom{n}{k} p_n^k (1-p_n)^{n-k}}{(np_n)^r/r!} \geqslant \lim_{n\to\infty} \frac{\binom{n}{r} p_n^r (1-p_n)^{n-r}}{(np_n)^r/r!} = 1$$

and the result follows. ∎

Let $\{c_n, n \geqslant 1\}$ be a sequence of positive constants satisfying for some $\alpha \in (0, 2)$

$$\text{(1)} \qquad \frac{c_n}{n^{1/\alpha}} \uparrow \quad \text{and} \quad \sup_{n \geqslant 1} \frac{c_{2n}}{c_n} < \infty$$

and set $c_0 = 0$, $h_0 = 0$, $h_n = c_n/n^{1/\alpha}$, $n \geqslant 1$. Let $H(x)$ be the continuous extension of the $\{h_n, n \geqslant 0\}$ defined by linear interpolation between integers; that is,

$$H(x) = (h_{n+1} - h_n)(x - n) + h_n \quad \text{for } 0 \leqslant n \leqslant x < n+1$$

and define

$$\text{(2)} \qquad C(x) = x^{1/\alpha} H(x), \quad x \geqslant 0.$$

Properties of the function $C(x)$ are spelled out by the ensuing lemma.

LEMMA 2 (Mori [15]). *Let $\{c_n, n \geqslant 1\}$ be a sequence of positive constants satisfying* (1) *for some $\alpha \in (0, 2)$ and define the function C as in* (2). *Then C is an absolutely continuous strictly increasing function on $[0, \infty)$ with $C(0) = 0$, $C(n) = c_n$, $n \geqslant 1$, and*

$$\text{(3)} \qquad \frac{C(x)}{x^{1/\alpha}} \text{ is nondecreasing on } (0, \infty) \quad \text{and} \quad \sup_{x > 0} \frac{C(2x)}{C(x)} < \infty.$$

Proof. Only the second half of (3) needs verification. For $x \geqslant 1$, let n be such that $n \leqslant x < n+1$. Then

$$\frac{C(2x)}{C(x)} \leqslant \frac{C(2n+2)}{C(n)} \leqslant \frac{C(2n+2)}{C(n+1)} \cdot \frac{C(2n)}{C(n)},$$

and so, by (1),

$$\sup_{x \geqslant 1} \frac{C(2x)}{C(x)} < \infty.$$

It is easy to see that $\sup_{0 < x \leqslant 1} C(2x)/C(x) < \infty$, and hence

$$\sup_{x > 0} C(2x)/C(x) < \infty. \blacksquare$$

For a sequence of positive constants $\{c_n, n \geqslant 1\}$ satisfying (1) for some $\alpha \in (0, 2)$, throughout the rest of this paper we define the function $C(x)$ as in (2).

LEMMA 3 (Mori [15]). *Let $\{c_n, n \geqslant 1\}$ be a sequence of positive constants satisfying* (1). *Then for all fixed integers $r \geqslant 0$ and for every $\varepsilon > 0$*

$$P\{|X_n^{(r+1)}| > c_n \varepsilon \ i.o.(n)\} = 0 \text{ or } 1$$

according as the integral $\int_0^\infty x^r \mathfrak{I}(C(x))^{r+1} dx$ converges or diverges.

In Lemmas 4, 7, and 8 we impose the condition

(4) $\qquad \mathfrak{I}$ is positive and differentiable on $[0, \infty)$.

Theorem 1 will initially be proved assuming (4) but then this assumption will be removed.

Suppose $\{c_n, n \geqslant 1\}$ satisfies (1). Then, according to Lemma 2, the function $C(x)$ is absolutely continuous and strictly increasing on $[0, \infty)$ with $C(0) = 0$, $C(n) = c_n$ for $n \geqslant 1$ and satisfies (3). Since $C(\infty) = \infty$, the inverse function D of C is also absolutely continuous and strictly increasing on $[0, \infty)$ with $D(0) = 0$ and $D(\infty) = \infty$.

Let $\psi(x) = \sqrt{D(x)/\mathfrak{I}(x)}$, $x \geqslant 0$. Under the assumption (4), ψ is absolutely continuous and strictly increasing with $\psi(0) = 0$ and $\psi(\infty) = \infty$. Hence the inverse function φ of ψ is also absolutely continuous and strictly increasing with $\varphi(0) = 0$ and $\varphi(\infty) = \infty$. If $\int_0^\infty x^r \mathfrak{I}(C(x))^{r+1} dx < \infty$, then since $\mathfrak{I}(C(x))\downarrow$, we have

$$\left(\frac{x}{2}\right)^{r+1} \mathfrak{I}(C(x))^{r+1} = \left(\frac{x}{2}\right)^{r} \mathfrak{I}(C(x))^{r+1}\left(x - \frac{x}{2}\right)$$
$$\leqslant \int_{x/2}^{x} t^r \mathfrak{I}(C(t))^{r+1} dt \to 0 \quad \text{as } x \to \infty,$$

and so

(5) $$x\mathfrak{I}(C(x)) \to 0 \quad \text{as } x \to \infty$$

implying by replacing x by $D(x)$ that

$$\frac{\psi(x)}{D(x)} = \frac{1}{\sqrt{D(x)\mathfrak{I}(x)}} \to \infty \quad \text{as } x \to \infty.$$

Hence we have

(6) $$\frac{\varphi(x)}{C(x)} = \frac{y}{C(\psi(y))} = \frac{C(D(y))}{C(\psi(y))} \leqslant \left(\frac{D(y)}{\psi(y)}\right)^{1/\alpha} \to 0 \quad \text{as } x \to \infty,$$

where $y = \varphi(x) \to \infty$. Note that the inequality holds by the first half of (3).

The next lemma is a slight modification of Lemma 7 of Mori [15], and its proof can be omitted.

LEMMA 4. *Assume that the condition* (4) *holds and the sequence* $\{c_n, n \geqslant 1\}$ *satisfies* (1). *Let* $\varphi(x)$ *be defined as above and let* $N_m = \#\{j\colon \varphi(2^m) \leqslant |X_j|, j \leqslant 2^{m+1}\}$, $m \geqslant 1$. *If*

(7) $$\int_0^\infty x^r \mathfrak{I}(C(x))^{r+1} dx < \infty \quad \textit{for some integer } r \geqslant 0,$$

then $P(N_m \geqslant 2r+2 \ \textit{i.o.}(m)) = 0$.

In the next lemma, let $\mu(S_n)$ and $\mu(M_k)$ denote any median of S_n and M_k, respectively.

LEMMA 5 (Stout [20], p. 158). *Let $\{Y_n, n \geqslant 1\}$ be a sequence of independent random variables and $\{b_n, n \geqslant 1\}$ be a sequence of constants with $0 < b_n \uparrow \infty$. Suppose there exist constants c and d such that $1 < c < d < \infty$ and $c \leqslant b_{2^{k+1}}/b_{2^k} \leqslant d$, $k \geqslant 1$. Let $S_n = \sum_{i=1}^n Y_i$, $n \geqslant 1$, and*

$$M_k = (S_{2^k} - S_{2^{k-1}})/b_{2^k}, \quad k \geqslant 1.$$

Then

$$\frac{S_n - \mu(S_n)}{b_n} \to 0 \text{ a.s.}$$

if and only if

$$\sum_{k=1}^{\infty} P\big(|M_k - \mu(M_k)| > \varepsilon\big) < \infty \quad \text{for all } \varepsilon > 0.$$

LEMMA 6 (Prokhorov's inequality [19]). *Let $Y_1, Y_2, \ldots, Y_n$ be independent mean 0 random variables such that $|Y_k| \leqslant c$, $1 \leqslant k \leqslant n$, for some constant $c < \infty$ and let $S = \sum_{k=1}^n Y_k$ and $\sigma^2 = \sum_{k=1}^n EY_k^2$. Then for all $\varepsilon > 0$*

$$P(S \geqslant \varepsilon) \leqslant \exp\left(-\frac{\varepsilon}{2c}\sinh^{-1}\frac{\varepsilon c}{2\sigma^2}\right).$$

LEMMA 7 (Mori [15]). *Assume that (4) holds and that the sequence $\{c_n, n \geqslant 1\}$ satisfies (1). Let φ be defined as above. If (7) holds, then*

$$\int_0^{\infty} x^{-2r-3} D\big(\varphi(x)\big)^{2r+2} dx = \frac{1}{r+1}\int_0^{\infty} x^r \mathfrak{I}\big(C(x)\big)^{r+1} dx.$$

Throughout the rest of the paper, let $\{b_n, n \geqslant 1\}$ be a sequence of constants with $0 < b_n \uparrow \infty$ and define the sequence $\{c_n, n \geqslant 1\}$ by $c_n = b_n/\max_{k \leqslant n}|a_k|$, $n \geqslant 1$, and the sequence of blocks of positive integers $\{I_m, m \geqslant 0\}$ by $I_m = \{n\colon 2^m \leqslant n < 2^{m+1}\}$, $m \geqslant 0$.

LEMMA 8. *Assume that (4) holds and the sequence $\{c_n, n \geqslant 1\}$ satisfies (1). Suppose that there exist constants c and d such that $1 < c < d < \infty$ and $c \leqslant b_{2^{k+1}}/b_{2^k} \leqslant d$, $k \geqslant 1$. If (7) holds, then there exists a sequence of constants $\{k_n, n \geqslant 1\}$ such that*

$$\frac{1}{b_n}\sum_{j=1}^n a_j X_j I\big(|X_j| \leqslant \varphi(j)\big) - k_n \to 0 \text{ a.s.}$$

Proof. Let X_j^s be a symmetrized version of $X_j I\big(|X_j| \leqslant \varphi(j)\big)$, $j \geqslant 1$. Then $EX_j^s = 0$ and $|X_j^s| \leqslant 2\varphi(j)$ and it suffices to show that

$$\frac{1}{b_n}\sum_{j=1}^n a_j X_j^s \to 0 \text{ a.s.}$$

By Lemma 5, this is equivalent to

$$\sum_{m=1}^{\infty} P(\sum_{j\in I_{m-1}} a_j X_j^s > \varepsilon b_{2^m}) < \infty \qquad \text{for all } \varepsilon > 0$$

since the $\{X_j^s, j \geqslant 1\}$ are symmetric.

For $j \in I_{m-1}$, we have $E(X_j^s)^2 \leqslant 2E\big(X_j^2 I(|X_j| \leqslant \varphi(j))\big)$ and

$$\frac{D(\varphi(2^m))}{2^m (\max_{j\leqslant 2^m}|a_j|)^2 \varphi(2^m)^2} \sum_{j\in I_{m-1}} E(a_j X_j^s)^2 \leqslant \frac{2D(\varphi(2^m))}{2^m \varphi(2^m)^2} \sum_{j\in I_{m-1}} (-\int_0^{\varphi(2^m)} t^2\, d\mathfrak{I}(t))$$

$$= \frac{2D(\varphi(2^m))}{2^m \varphi(2^m)^2} \cdot 2^{m-1} (-\int_0^{\varphi(2^m)} t^2\, d\mathfrak{I}(t)) = \frac{D(\varphi(2^m))}{\varphi(2^m)^2} (-t^2 \mathfrak{I}(t)|_0^{\varphi(2^m)} + 2\int_0^{\varphi(2^m)} t\,\mathfrak{I}(t)\, dt)$$

$$= \frac{D(\varphi(2^m))}{\varphi(2^m)^2} \big(-\varphi(2^m)^2 \mathfrak{I}(\varphi(2^m)) + 2\int_0^{\varphi(2^m)} t\,\mathfrak{I}(t)\, dt\big)$$

$$= -D(\varphi(2^m))\mathfrak{I}(\varphi(2^m)) + \frac{2D(\varphi(2^m))}{\varphi(2^m)^2} \int_0^{\varphi(2^m)} t\mathfrak{I}(t)\, dt = o(1) + \frac{2D(\varphi(2^m))}{\varphi(2^m)^2} \int_0^{\varphi(2^m)} t\mathfrak{I}(t)\, dt$$

by (5) with x replaced by $D(t)$. Note that since $t/D(t)^{1/\alpha}$ is nondecreasing, we have $D(\varphi(2^m))/\varphi(2^m)^\alpha \leqslant D(t)/t^\alpha$ whenever $0 < t \leqslant \varphi(2^m)$. Moreover, for arbitrary $\delta > 0$, condition (5) (with x replaced by $D(t)$) ensures that there exists $t_0 > 0$ such that $D(t)\mathfrak{I}(t) \leqslant \delta$ for $t \geqslant t_0$. Hence, for all large m,

$$\frac{2D(\varphi(2^m))}{\varphi(2^m)^2} \int_0^{\varphi(2^m)} t\mathfrak{I}(t)\, dt \leqslant \frac{2}{\varphi(2^m)^2} \int_0^{\varphi(2^m)} \left(\frac{\varphi(2^m)}{t}\right)^\alpha tD(t)\,\mathfrak{I}(t)\, dt$$

$$\leqslant \frac{2}{\varphi(2^m)^{2-\alpha}} [D(t_0) \int_0^{t_0} t^{1-\alpha}\, dt + \delta \int_{t_0}^{\varphi(2^m)} t^{1-\alpha}\, dt] \leqslant o(1) + \frac{2\delta}{2-\alpha}.$$

Since $\delta > 0$ is arbitrary, we have

$$\frac{2D(\varphi(2^m))}{\varphi(2^m)^2} \int_0^{\varphi(2^m)} t\mathfrak{I}(t)\, dt \to 0,$$

and hence

$$\sum_{j\in I_{m-1}} E(a_j X_j^s)^2 = o\left(\frac{2^m \varphi(2^m)^2 (\max_{j\leqslant 2^m}|a_j|)^2}{D(\varphi(2^m))}\right). \tag{8}$$

By Lemma 6, for $m \geqslant 1$

$$P(\sum_{j\in I_{m-1}} a_j X_j^s > \varepsilon b_{2^m}) \leqslant$$

$$\leqslant \exp\left(-\frac{\varepsilon b_{2^m}}{2\cdot 2\varphi(2^m)\cdot \max_{j\leqslant 2^m}|a_j|}\sinh^{-1}\frac{\varepsilon b_{2^m}\cdot 2\varphi(2^m)\cdot \max_{j\leqslant 2^m}|a_j|}{2\sum_{j\in I_{m-1}} E(a_j X_j^s)^2}\right)$$

$$= \exp\left(-\frac{\varepsilon C(2^m)}{2\cdot 2\varphi(2^m)}\sinh^{-1}\frac{\varepsilon b_{2^m}\cdot 2\varphi(2^m)\cdot \max_{j\leqslant 2^m}|a_j|}{2\sum_{j\in I_{m-1}} E(a_j X_j^s)^2}\right).$$

(i) Suppose, for every $m \geqslant 1$,

$$\sinh^{-1}\frac{\varepsilon b_{2^m}\cdot 2\varphi(2^m)\cdot \max_{j\leqslant 2^m}|a_j|}{2\sum_{j\in I_{m-1}} E(a_j X_j^s)^2} \geqslant \frac{4}{\varepsilon}.$$

Then we easily see that

$$\sum_{m=1}^{\infty}\exp\left(-\frac{\varepsilon C(2^m)}{2\cdot 2\varphi(2^m)}\sinh^{-1}\frac{\varepsilon b_{2^m}\cdot 2\varphi(2^m)\cdot \max_{j\leqslant 2^m}|a_j|}{2\sum_{j\in I_{m-1}} E(a_j X_j^s)^2}\right)$$

$$\leqslant \sum_{m=1}^{\infty}\exp\left(-\frac{C(2^m)}{\varphi(2^m)}\right) \leqslant \sum_{m=1}^{\infty}\left(\frac{\varphi(2^m)}{C(2^m)}\right)^{2\alpha(r+1)} = \sum_{m=1}^{\infty}\sum_{n\in I_{m-1}}\frac{1}{2^{m-1}}\left(\frac{\varphi(2^m)}{C(2^m)}\right)^{2\alpha(r+1)}$$

$$\leqslant \text{const}\cdot\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\varphi(n)}{C(n)}\right)^{2\alpha(r+1)} \leqslant \text{const}\cdot\int_0^{\infty}\frac{1}{x}\left(\frac{\varphi(x)}{C(x)}\right)^{2\alpha(r+1)}dx$$

$$\leqslant \text{const}\cdot\int_0^{\infty}\frac{1}{x}\left(\frac{D(\varphi(x))}{x}\right)^{(1/\alpha)\cdot 2\alpha(r+1)}dx \qquad \text{(by (6))}$$

$$= \text{const}\cdot\int_0^{\infty} x^{-2r-3}D(\varphi(x))^{2r+2}dx < \infty \qquad \text{(by (7) and Lemma 7).}$$

(ii) Suppose, for every $m \geqslant 1$,

$$\sinh^{-1}\frac{\varepsilon b_{2^m}\cdot 2\varphi(2^m)\cdot \max_{j\leqslant 2^m}|a_j|}{2\sum_{j\in I_{m-1}} E(a_j X_j^s)^2} < \frac{4}{\varepsilon}.$$

Now, for any constant $a > 0$ there exists a constant k such that $\sinh^{-1}x > kx$ for $0 < x < a$, and so

$$\sinh^{-1}\frac{\varepsilon b_{2^m}\cdot 2\varphi(2^m)\cdot \max_{j\leqslant 2^m}|a_j|}{2\sum_{j\in I_{m-1}} E(a_j X_j^s)^2} > k\frac{\varepsilon b_{2^m}\cdot 2\varphi(2^m)\cdot \max_{j\leqslant 2^m}|a_j|}{2\sum_{j\in I_{m-1}} E(a_j X_j^s)^2}$$

$$\geqslant k\frac{\varepsilon b_{2^m}\cdot 2\varphi(2^m)\cdot \max_{j\leqslant 2^m}|a_j|}{2\cdot 2^m\varphi(2^m)^2(\max_{j\leqslant 2^m}|a_j|)^2/D(\varphi(2^m))} \qquad \text{(by (8))}$$

$$= k\varepsilon\frac{C(2^m)D(\varphi(2^m))}{2^m\varphi(2^m)}.$$

Thus

$$\sum_{m=1}^{\infty} \exp\left(-\frac{\varepsilon C(2^m)}{2\cdot 2\varphi(2^m)}\sinh^{-1}\frac{\varepsilon b_{2^m}\cdot 2\varphi(2^m)\cdot \max_{j\leqslant 2^m}|a_j|}{2\sum_{j\in I_{m-1}} E(a_j X_j^s)^2}\right)$$

$$\leqslant \sum_{m=1}^{\infty} \exp\left(-\frac{\varepsilon C(2^m)}{2\cdot 2\varphi(2^m)}\cdot k\varepsilon\frac{C(2^m)D(\varphi(2^m))}{2^m\varphi(2^m)}\right)$$

$$= \sum_{m=1}^{\infty} \exp\left[\frac{k\varepsilon^2}{4}\left(-\frac{C(2^m)^2 D(\varphi(2^m))}{2^m\varphi(2^m)^2}\right)\right]$$

$$\leqslant \left(\frac{k\varepsilon^2}{4}\right)^{2\alpha(r+1)/(2-\alpha)} \sum_{m=1}^{\infty}\left(\frac{2^m\varphi(2^m)^2}{C(2^m)^2 D(\varphi(2^m))}\right)^{2\alpha(r+1)/(2-\alpha)}$$

$$= \text{const}\cdot \sum_{m=1}^{\infty}\left(\frac{2^m\varphi(2^m)^2}{C(2^m)^2 D(\varphi(2^m))}\right)^{2\alpha(r+1)/(2-\alpha)}$$

$$\leqslant \text{const}\cdot \sum_{m=1}^{\infty}\sum_{n\in I_{m-1}}\frac{1}{n}\left(\frac{n\varphi(n)^2}{C(n)^2 D(\varphi(n))}\right)^{2\alpha(r+1)/(2-\alpha)}$$

$$\leqslant \text{const}\cdot \int_0^{\infty}\frac{1}{x}\left(\frac{x}{D(\varphi(x))}\right)^{2\alpha(r+1)/(2-\alpha)}\left(\frac{\varphi(x)}{C(x)}\right)^{2\cdot 2\alpha(r+1)/(2-\alpha)} dx$$

$$\leqslant \text{const}\cdot \int_0^{\infty}\frac{1}{x}\left(\frac{x}{D(\varphi(x))}\right)^{2\alpha(r+1)/(2-\alpha)}\left(\frac{D(\varphi(x))}{x}\right)^{(2/\alpha)\cdot 2\alpha(r+1)/(2-\alpha)} dx \qquad \text{(by (6))}$$

$$\leqslant \text{const}\cdot \int_0^{\infty} x^{-2r-3} D(\varphi(x))^{2r+2}\,dx < \infty \qquad \text{(by (7) and Lemma 7).}$$

Consequently, we have

$$\sum_{m=1}^{\infty} P\Big(\sum_{j\in I_{m-1}} a_j X_j^s > \varepsilon b_{2^m}\Big)$$

$$\leqslant \sum_{m=1}^{\infty}\left(\frac{\varphi(2^m)}{C(2^m)}\right)^{2\alpha(r+1)} + \text{const}\cdot \sum_{m=1}^{\infty}\left(\frac{2^m\varphi(2^m)^2}{C(2^m)^2 D(\varphi(2^m))}\right)^{2\alpha(r+1)/(2-\alpha)} < \infty,$$

completing the proof of the lemma. ■

LEMMA 9. *Let $\{\zeta_n, n\geqslant 1\}$ be a sequence of i.i.d. symmetric random variables with $E|\zeta_1|^\alpha < \infty$ for some $\alpha\in(0, 2)$. Let $\{a_n, n\geqslant 1\}$ and $\{b_n, n\geqslant 1\}$ be sequences of constants satisfying $0 < b_n\uparrow\infty$ and*

(9) $$a_n/b_n = O(n^{-1/\alpha}).$$

Then

$$\frac{1}{b_n}\sum_{j=1}^{n} a_j \zeta_j \to 0 \ a.s. \tag{10}$$

Proof. Apply the same argument that Adler and Rosalsky [1] used to prove (20) of their Theorem 2. ∎

3. THE MAIN RESULTS

With the preliminaries accounted for, the first main result, Theorem 1, may be established.

THEOREM 1. *Let* $\{a_n, n \geqslant 1\}$ *and* $\{b_n, n \geqslant 1\}$ *be sequences of constants where* $0 < b_n \uparrow \infty$ *and* $b_{2n} = O(b_n)$. *Let* $c_n = b_n/\max_{k \leqslant n}|a_k|$, $n \geqslant 1$, *and suppose that* $\{c_n, n \geqslant 1\}$ *satisfies* (1). *If*

$$\int_0^\infty x^r \Im(C(x))^{r+1} dx < \infty \quad \text{for some integer } r \geqslant 0, \tag{11}$$

then there exists a sequence of constants $\{k_n, n \geqslant 1\}$ *such that*

$$T_n^{(r)}/b_n - k_n \to 0 \ a.s. \tag{12}$$

Proof. Note at the outset that the first half of (1) ensures that

$$\inf_{n \geqslant 1} b_{2n}/b_n > 1.$$

Hence in view of $b_{2n} = O(b_n)$ we have

$$1 < \inf_{n \geqslant 1} b_{2n}/b_n \quad \text{and} \quad \sup_{n \geqslant 1} b_{2n}/b_n < \infty. \tag{13}$$

We initially assume that (4) holds. By (11) and Lemma 3,

$$P(|X_n^{(r+1)}| > c_n \varepsilon \text{ i.o.}(n)) = 0 \quad \text{for every } \varepsilon > 0.$$

Then, with probability 1, for all large n

$$\left|\frac{T_n^{(r)}}{b_n} - \frac{1}{b_n}\sum_{j=1}^{n} a_j X_j I(|X_j| \leqslant c_n \varepsilon)\right| \leqslant \frac{r \max_{j \leqslant n}|a_j| \cdot c_n \varepsilon}{b_n} = r\varepsilon. \tag{14}$$

Now, for every $\varepsilon > 0$ it follows from (6) that we can choose $l_1 = l_1(\varepsilon)$ such that $\varphi(n) \leqslant c_n \varepsilon$ for $n \geqslant 2^{l_1}$. Then for all large n there exists an integer $m > l_1$ such that $n \in I_m$ and $\{\varphi(0), \varphi(1), \varphi(2), \ldots, \varphi(2^i), \ldots, \varphi(2^m), c_n \varepsilon\}$ induces a partition of $(0, c_n \varepsilon]$. Thus, for $j = 1, 2, \ldots, n$, we have

$$X_j I(|X_j| \leqslant c_n \varepsilon) = X_j\Big(I(0 \leqslant |X_j| < \varphi(1)) + \sum_{i=1}^{m} I(\varphi(2^{i-1}) \leqslant |X_j| < \varphi(2^i)) + I(\varphi(2^m) \leqslant |X_j| < c_n \varepsilon)\Big).$$

Hence

$$\left|\frac{1}{b_n}\sum_{j=1}^{n} a_j X_j I(|X_j| \leqslant c_n \varepsilon) - \frac{1}{b_n}\sum_{j=1}^{n} a_j X_j I(|X_j| \leqslant \varphi(j))\right|$$
$$\leqslant \frac{\max_{j\leqslant n}|a_j|}{b_n} c_n \varepsilon \cdot \#\{j\colon\ \varphi(2^m) \leqslant |X_j| \leqslant c_n \varepsilon,\ \varphi(j) \leqslant |X_j|\}$$
$$+\sum_{i=1}^{m} \frac{\max_{j\leqslant n}|a_j|}{b_n} \varphi(2^i)\cdot \#\{j\colon\ \varphi(2^{i-1}) \leqslant |X_j| < \varphi(2^i),\ \varphi(j) \leqslant |X_j|\}$$
$$\leqslant \frac{\max_{j\leqslant n}|a_j|}{b_n} c_n \varepsilon \cdot N_m + \sum_{i=1}^{m} \frac{\max_{j\leqslant n}|a_j|}{b_n} \varphi(2^i)\cdot N_{i-1}.$$

By Lemma 4 we see that $P\big(N_i \geqslant 2r+2 \text{ i.o.}(i)\big) = 0$. Then, with probability 1, there exists a (random) integer l_2 such that $N_i \leqslant 2r+2$ for all $i \geqslant l_2$. Let $l = \max\{l_1, l_2\}$. Then, with probability 1, for all large n

$$\left|\frac{1}{b_n}\sum_{j=1}^{n} a_j X_j I(|X_j| \leqslant c_n \varepsilon) - \frac{1}{b_n}\sum_{j=1}^{n} a_j X_j I(|X_j| \leqslant \varphi(j))\right|$$
$$\leqslant \frac{\max_{j\leqslant n}|a_j|}{b_n} c_n \varepsilon (2r+2) + \sum_{i=l}^{m} \frac{\max_{j\leqslant n}|a_j|}{b_n} \varphi(2^i)\,(2r+2) + \sum_{i=1}^{l-1} \frac{\max_{j\leqslant n}|a_j|}{b_n} \varphi(2^i)\cdot 2^i$$
$$\leqslant \varepsilon(2r+2) + (2r+2)\sum_{i=l}^{m} \frac{\varphi(2^i)}{C(2^m)} + 2^l \frac{\sum_{i=1}^{l-1}\varphi(2^i)}{C(2^m)}$$
$$\leqslant (2r+2)\,\varepsilon\cdot\left(1 + \frac{C(2^m)}{C(2^m)} + \frac{C(2^{m-1})}{C(2^m)} + \ldots + \frac{C(2^l)}{C(2^m)}\right) + 2^l \frac{\sum_{i=1}^{l-1}\varphi(2^i)}{C(2^m)},$$

where, recalling (1),

$$\frac{C(n)}{C(2n)} \leqslant \left(\frac{n}{2n}\right)^{1/\alpha} = 2^{-1/\alpha} < \frac{3}{4}.$$

Choosing the (random) integer m such that $C(2^m) \geqslant 2^l \sum_{i=1}^{l-1} \varphi(2^i)/(2r+2)\,\varepsilon$, we obtain with probability 1, for all large n,

$$(15)\quad \left|\frac{1}{b_n}\sum_{j=1}^{n} a_j X_j I(|X_j| \leqslant c_n \varepsilon) - \frac{1}{b_n}\sum_{j=1}^{n} a_j X_j I(|X_j| \leqslant \varphi(j))\right|$$
$$\leqslant (2r+2)\,\varepsilon\left(1 + 1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \ldots + \left(\frac{3}{4}\right)^{m-l}\right) + \frac{1}{C(2^m)} 2^l \sum_{i=1}^{l-1} \varphi(2^i)$$
$$\leqslant 6\,(2r+2)\,\varepsilon.$$

Thus, since ε is arbitrary, (14) and (15) yield

$$\frac{T_n^{(r)}}{b_n} - \frac{1}{b_n}\sum_{j=1}^{n} a_j X_j I(|X_j| \leqslant \varphi(j)) \to 0 \text{ a.s.}$$

Now, by (13) and Lemma 8, there exists a sequence of constants $\{k_n, n \geqslant 1\}$ such that

$$\frac{1}{b_n}\sum_{j=1}^{n} a_j X_j I(|X_j| \leqslant \varphi(j)) - k_n \to 0 \text{ a.s.}$$

Thus we have proved assuming (4) that the conclusion (12) holds.

We now remove the assumption (4). This will be accomplished by purturbing each X_j by an independent continuous random variable ζ_j, applying the part of the theorem already proved with the assumption (4) to then arrive at (16), and finally verifying in (20) that the ζ_j have a negligible effect. Again, by (11) and Lemma 3, for every $\varepsilon > 0$, $P(|X_n^{(r+1)}| > c_n \varepsilon \text{ i.o.}(n)) = 0$. Let us set

$$\zeta_j = \xi_j^{1/\alpha} I(\xi_j \geqslant 0) - |\xi_j|^{1/\alpha} I(\xi_j < 0), \qquad j \geqslant 1,$$

where $\alpha \in (0, 2)$ and $\{\xi_n, n \geqslant 1\}$ is a sequence of i.i.d. normal random variables with $E\xi_1 = 0$, $E\xi_1^2 = 1$ and such that $\{\xi_n, n \geqslant 1\}$ is independent of $\{X_n, n \geqslant 1\}$. Then we see that each ζ_j is symmetric with $E|\zeta_j|^\alpha = E(|\xi_j|^{1/\alpha})^\alpha = E|\xi_j| < \infty$. Let $Y_j = X_j + \zeta_j$, $j \geqslant 1$. Put $\Gamma(x) = P(|Y_j| > x)$, $x \geqslant 0$, and $U_n = \sum_{j=1}^{n} a_j Y_j$, $n \geqslant 1$, and define $U_n^{(r)}$ in a similar way to $T_n^{(r)}$. Since $E|\zeta_j|^\alpha < \infty$, we have $\sum_{j=1}^{\infty} P(|\zeta_j| > j^{1/\alpha}\varepsilon) < \infty$ for all $\varepsilon > 0$, which ensures by the Borel–Cantelli lemma and (1) that for all $\varepsilon > 0$ there exists a random integer n_0 such that $|\zeta_j| \leqslant c_1 j^{1/\alpha}\varepsilon \leqslant c_n \varepsilon$ for all $n \geqslant j \geqslant n_0$. Thus, there exists a random integer $N_0 \geqslant n_0$ such that $|\zeta_j| \leqslant c_n \varepsilon$ for all $n \geqslant N_0$ and all $1 \leqslant j \leqslant n$. Hence, if $|Y_n^{(r+1)}| > 2c_n \varepsilon$ for some $n \geqslant N_0$, i.e., if $|Y_j| > 2c_n \varepsilon$ for at least $r+1$ integers $j \in [1, n]$, where $n \geqslant N_0$, then we have $|X_n^{(r+1)}| > c_n \varepsilon$. Thus

$$P(|Y_n^{(r+1)}| > 2c_n \varepsilon \text{ i.o.}(n)) \leqslant P(|X_n^{(r+1)}| > c_n \varepsilon \text{ i.o.}(n)) = 0$$

and $\int_0^\infty x^r \Gamma(C(x))^{r+1} dx < \infty$ follows from Lemma 3.

Since Γ satisfies (4), we have by the portion of Theorem 1 already proved that

$$U_n^{(r)}/b_n - k_n \to 0 \text{ a.s.} \tag{16}$$

for some sequence of constants $\{k_n, n \geqslant 1\}$. Moreover,

$$\begin{aligned} \frac{1}{b_n}|U_n^{(r)} - T_n^{(r)}| &\leqslant \frac{1}{b_n}|U_n - T_n| + \frac{1}{b_n}|(U_n - U_n^{(r)}) - (T_n - T_n^{(r)})| \\ &= \left|\frac{1}{b_n}\sum_{j=1}^{n} a_j \zeta_j\right| + \frac{1}{b_n}|(U_n - U_n^{(r)}) - (T_n - T_n^{(r)})|. \end{aligned} \tag{17}$$

Since, recalling (1),

$$\frac{|a_n|}{b_n} \leqslant \frac{\max_{j \leqslant n}|a_j|}{b_n} = O(n^{-1/\alpha}),$$

it follows from Lemma 9 that

(18) $$\frac{1}{b_n}\sum_{j=1}^{n} a_j\zeta_j \to 0 \text{ a.s.}$$

Note that

$$T_n - T_n^{(r)} = \sum_{j=1}^{n} a_j X_j I(|X_j| > |X_n^{(r+1)}|)$$

and

$$U_n - U_n^{(r)} = \sum_{j=1}^{n} a_j Y_j I(|Y_j| > |Y_n^{(r+1)}|).$$

Now for all large n and $1 \leqslant j \leqslant n$:

(i) If $Y_j = X_j + \zeta_j = Y_n^{(k)}$, where $1 \leqslant k \leqslant r$ and $X_j = X_n^{(h)}$ with $h \geqslant r+1$, then

$$|Y_n^{(k)}| \leqslant |X_n^{(r+1)}| + |\zeta_j| \leqslant 2c_n\varepsilon,$$

whence

$$\left|a_j Y_j I(|Y_j| > |Y_n^{(r+1)}|) - a_j X_j I(|X_j| > |X_n^{(r+1)}|)\right| \leqslant 2|a_j| c_n \varepsilon.$$

(ii) If $Y_j = X_j + \zeta_j = Y_n^{(k)}$, where $1 \leqslant k \leqslant r$ and $X_j = X_n^{(h)}$ with $1 \leqslant h \leqslant r$, then

$$\left|a_j Y_j I(|Y_j| > |Y_n^{(r+1)}|) - a_j X_j I(|X_j| > |X_n^{(r+1)}|)\right| = |a_j||\zeta_j| \leqslant |a_j| c_n \varepsilon.$$

(iii) If $Y_j = X_j + \zeta_j = Y_n^{(k)}$, where $k \geqslant r+1$, then $|X_j| \leqslant |Y_n^{(r+1)}| + |\zeta_j| \leqslant 2c_n\varepsilon$, whence

$$\left|a_j Y_j I(|Y_j| > |Y_n^{(r+1)}|) - a_j X_j I(|X_j| > |X_n^{(r+1)}|)\right| \leqslant 2|a_j| c_n \varepsilon.$$

Since

$$\left|a_j Y_j I(|Y_j| > |Y_n^{(r+1)}|) - a_j X_j I(|X_j| > |X_n^{(r+1)}|)\right| > 0$$

for at most $2r$ values of $j \in \{1, 2, \ldots, n\}$, we have for all large n

$$\frac{1}{b_n}|(U_n - U_n^{(r)}) - (T_n - T_n^{(r)})|$$

$$\leqslant \frac{(2r)(2\max_{j\leqslant n}|a_j| c_n\varepsilon)}{b_n} + \frac{(2r)\max_{j\leqslant n}|a_j| c_n\varepsilon}{b_n} + \frac{(2r)(2\max_{j\leqslant n}|a_j| c_n\varepsilon)}{b_n} = 10r\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary,

(19) $$b_n^{-1}|(U_n - U_n^{(r)}) - (T_n - T_n^{(r)})| \to 0 \text{ a.s.}$$

Thus, by (17)–(19),

$$b_n^{-1}|U_n^{(r)}-T_n^{(r)}|\to 0 \text{ a.s.} \tag{20}$$

Hence by (16) and (20) the conclusion (12) holds. ∎

Remark. Since $C(x)/x^{1/\alpha}$ is nondecreasing, $C(x)\leqslant \varepsilon C(x/\varepsilon^\alpha)$ for $x\geqslant 0$ and $0<\varepsilon<1$. Then, by (11),

$$\int_0^\infty x^r\mathfrak{I}\big(\varepsilon C(x/\varepsilon^\alpha)\big)^{r+1}dx\leqslant \int_0^\infty x^r\mathfrak{I}\big(C(x)\big)^{r+1}dx<\infty,$$

and so, by a change of variables,

$$\int_0^\infty x^r\mathfrak{I}\big(\varepsilon C(x)\big)^{r+1}dx<\infty.$$

Then, arguing as in the proof of (5), we obtain $x\mathfrak{I}(\varepsilon C(x))\to 0$ as $x\to\infty$. Then, by applying Lemma 1, we obtain for all $k\geqslant 1$ and $0<\varepsilon<1$

$$P(|X_n^{(k)}|>c_n\varepsilon)=\sum_{j=k}^n\binom{n}{j}\mathfrak{I}(c_n\varepsilon)^j\big(1-\mathfrak{I}(c_n\varepsilon)\big)^{n-j}\sim\frac{1}{k!}\big(n\mathfrak{I}(c_n\varepsilon)\big)^k\to 0 \quad \text{as } n\to\infty.$$

Thus $|X_n^{(k)}|/c_n\xrightarrow{P}0$ as $n\to\infty$ for all $k\geqslant 1$, and therefore

$$\left|\frac{1}{b_n}\sum_{j=1}^n a_jX_j-k_n\right|\leqslant\left|\frac{T_n^{(r)}}{b_n}-k_n\right|+\frac{\max_{j\leqslant n}|a_j|}{b_n}\sum_{k=1}^r|X_n^{(k)}|$$
$$=\left|\frac{T_n^{(r)}}{b_n}-k_n\right|+\frac{1}{c_n}\sum_{k=1}^r|X_n^{(k)}|\xrightarrow{P}0.$$

Hence, recalling (1), the sequence $\{k_n,\ n\geqslant 1\}$ can be chosen as

$$k_n=\frac{1}{b_n}\sum_{j=1}^n a_jEX_jI(|X_j|\leqslant b_n/a_j),\qquad n\geqslant 1$$

(see Chow and Teicher [3], p. 356). In particular, if $E|X_1|<\infty$, then we can choose

$$k_n=\frac{EX_1}{b_n}\sum_{j=1}^n a_j,\qquad n\geqslant 1.$$

The next main result, Theorem 2, is a converse to Theorem 1. Let

$$q_n=\frac{b_n}{\min_{k\leqslant n}|a_k|},\qquad n\geqslant 1.$$

Now, if for some $\alpha\in(0,2)$ $\{q_n,\ n\geqslant 1\}$ satisfies

$$\frac{q_n}{n^{1/\alpha}}\uparrow\quad\text{and}\quad\sup_{n\geqslant 1}\frac{q_{2n}}{q_n}<\infty, \tag{21}$$

then as in (2) and Lemma 2 we can define an absolutely continuous strictly increasing function Q on $[0, \infty)$ with $Q(n) = q_n$ for $n = 1, 2, \ldots$ and satisfying for some $\alpha \in (0, 2)$

$$(22) \qquad \frac{Q(x)}{x^{1/\alpha}} \text{ is nondecreasing and } \sup_{x>0} \frac{Q(2x)}{Q(x)} < \infty .$$

Note that if the first half of (1) holds, then the first half of (21) also holds.

THEOREM 2. *Let $\{a_n, n \geqslant 1\}$ and $\{b_n, n \geqslant 1\}$ be sequences of constants such that $c_n = b_n/\max_{k \leqslant n} |a_k| \uparrow \infty$ and $q_n = b_n/\min_{k \leqslant n} |a_k|$, $n \geqslant 1$, satisfies* (21). *If*

$$(23) \qquad T_n^{(r)}/b_n - k_n \to 0 \ \text{a.s.}$$

for some integer $r \geqslant 0$ and some sequence of constants $\{k_n, n \geqslant 1\}$, then

$$(24) \qquad \int_0^\infty x^r \mathfrak{I}(Q(x))^{(r+1)} dx < \infty .$$

Proof. If $r = 0$, then since $c_n \uparrow \infty$, we have

$$\left|\frac{1}{b_n}\Big(\sum_{j=1}^{n} a_j X_j - \sum_{j=1}^{n-1} a_j X_j\Big)\right| = \left|\frac{a_n}{b_n} X_n\right| \leqslant \frac{1}{c_n} |X_n| \xrightarrow{P} 0 .$$

Hence, by (23) (with $r = 0$) and $b_n \uparrow$,

$$k_n - k_{n-1} \frac{b_{n-1}}{b_n} = \left(k_n - \frac{T_n}{b_n}\right) + \frac{b_{n-1}}{b_n}\left(\frac{T_{n-1}}{b_{n-1}} - k_{n-1}\right) + \frac{T_n - T_{n-1}}{b_n} \xrightarrow{P} 0 .$$

Thus $k_n - k_{n-1} b_{n-1}/b_n \to 0$ and again using (23) we obtain

$$\frac{a_n}{b_n} X_n = \frac{T_n - T_{n-1}}{b_n} = \frac{T_n}{b_n} - k_n - \frac{b_{n-1}}{b_n}\left(\frac{T_{n-1}}{b_{n-1}} - k_{n-1}\right) + k_n - k_{n-1} \frac{b_{n-1}}{b_n} \to 0 \ \text{a.s.}$$

Then

$$\sum_{n=1}^{\infty} P(|X_n| > b_n/a_n) < \infty$$

follows from the Borel–Cantelli lemma. Moreover,

$$\sum_{n=1}^{\infty} P\left(|X_n| > \frac{b_n}{\min_{k \leqslant n} |a_k|}\right) \leqslant \sum_{n=1}^{\infty} P\left(|X_n| > \frac{b_n}{a_n}\right) < \infty ,$$

and then, by the Borel–Cantelli lemma and Lemma 3 (with $\{c_n, n \geqslant 1\}$ replaced by $\{q_n, n \geqslant 1\}$ and $C(x)$ replaced by $Q(x)$), we have

$$\int_0^\infty \mathfrak{I}(Q(x))\, dx < \infty .$$

Next, suppose $r \geqslant 1$. Then

(25)
$$\max_{k\leqslant n+1} |a_k|\cdot \min\{|X_{n+1}|, |X_n^{(r)}|\} \geqslant |T_{n+1}^{(r)}-T_n^{(r)}| \geqslant \min_{k\leqslant n+1} |a_k|\cdot \min\{|X_{n+1}|, |X_n^{(r)}|\}.$$

Hence

(26)
$$b_{n+1}^{-1}|T_{n+1}^{(r)}-T_n^{(r)}| \xrightarrow{P} 0.$$

It follows from (23) and (26) that

$$k_{n+1}-k_n\frac{b_n}{b_{n+1}} \to 0.$$

Applying this to (23) again gives

(27)
$$\frac{1}{b_{n+1}}|T_{n+1}^{(r)}-T_n^{(r)}| \to 0 \text{ a.s.}$$

Suppose that $\int_0^\infty x^r \mathfrak{I}(Q(x))^{(r+1)}dx = \infty$. Then, by Lemma 3,

$$P(|X_n^{(r+1)}| > Q(n)\varepsilon \text{ i.o.}(n)) = 1.$$

Let $r+1 \leqslant n_1 \leqslant n_2 \leqslant \ldots$ be successive indices n with $|X_{n+1}^{(r+1)}| > |X_n^{(r+1)}|$. It is easy to see that $|X_{n_j+1}| > |X_{n_j}^{(r+1)}|$ and

$$|X_{n_j+1}^{(r+1)}| = \min(|X_{n_j+1}|, |X_{n_j}^{(r)}|).$$

Furthermore, $[|X_n^{(r+1)}| > Q(n)\varepsilon \text{ i.o.}(n)] = [|X_{n_j+1}^{(r+1)}| > Q(n_j+1)\varepsilon \text{ i.o.}(j)]$, and hence $P(|X_{n_j+1}^{(r+1)}| > Q(n_j+1)\varepsilon \text{ i.o.}(j)) = 1$. Thus, recalling (25), with probability 1, for infinitely many j we have

$$\frac{1}{b_{n_j+1}}|T_{n_j+1}^{(r)}-T_{n_j}^{(r)}| \geqslant \frac{\min_{k\leqslant n_j+1}|a_k|}{b_{n_j+1}}|X_{n_j+1}^{(r+1)}| \geqslant \frac{1}{Q(n_j+1)}Q(n_j+1)\varepsilon = \varepsilon.$$

Hence $P(|T_{n+1}^{(r)}-T_n^{(r)}| \geqslant b_{n+1}\varepsilon \text{ i.o.}(n)) = 1$, which contradicts (27). ∎

Combining Theorems 1 and 2 yields the following corollary.

Corollary 1. *Let $\{a_n, n \geqslant 1\}$ be a sequence of constants such that $|a_n|$ is bounded from above and bounded away from 0 and let $\{b_n, n \geqslant 1\}$ be a sequence of positive constants such that for some $\alpha \in (0, 2)$*

(28)
$$\frac{b_n}{n^{1/\alpha}}\uparrow \quad \textit{and} \quad \sup_{n\geqslant 1}\frac{b_{2n}}{b_n} < \infty.$$

Let $B(x)$ be defined in a way determined in (2). Furthermore, let $c_n = b_n/\max_{k\leqslant n}|a_k|$, $n \geqslant 1$, be such that $c_n/n^{1/\alpha}\uparrow$. Let r be a nonnegative integer. Then there exists a sequence of constants $\{k_n, n \geqslant 1\}$ such that

(29)
$$T_n^{(r)}/b_n - k_n \to 0 \textit{ a.s.}$$

if and only if

$$\int_0^\infty x^r \Im(B(x))^{r+1}\,dx < \infty. \tag{30}$$

Proof. It follows from (28) that $b_{2n}/b_n \geqslant (2n/n)^{1/\alpha} = 2^{1/\alpha} > 1$, $n \geqslant 1$. Note that for all $\varepsilon > 0$ the condition (30) is equivalent to

$$\int_0^\infty x^r \Im(B(x)\varepsilon)^{r+1}\,dx < \infty \tag{31}$$

since by (28) and Lemma 2 we have $B(\varepsilon^\alpha x) \leqslant \varepsilon B(x) \leqslant B(x)$ if $0 < \varepsilon \leqslant 1$ and $B(x) \leqslant \varepsilon B(x) \leqslant B(\varepsilon^\alpha x)$ if $\varepsilon \geqslant 1$. Suppose $0 < m \leqslant |a_k| \leqslant M < \infty$, $k \geqslant 1$. Then, for all $n \geqslant 1$,

$$c_n = \frac{b_n}{\max_{k \leqslant n} |a_k|} \geqslant \frac{b_n}{M} \quad \text{and} \quad q_n = \frac{b_n}{\min_{k \leqslant n} |a_k|} \leqslant \frac{b_n}{m}.$$

Moreover, $c_n/n^{1/\alpha}\uparrow$ and (28) ensure that $\{c_n, n \geqslant 1\}$ satisfies (1) and $\{q_n, n \geqslant 1\}$ satisfies (21). Since (31) with $\varepsilon = 1/M$ guarantees that

$$\int_0^\infty x^r \Im(C(x))^{r+1}\,dx \leqslant \int_0^\infty x^r \Im\left(\frac{B(x)}{M}\right)^{r+1} dx < \infty,$$

the sufficiency part follows from Theorem 1.

Conversely, suppose that (29) holds. Then, by Theorem 2,

$$\int_0^\infty x^r \Im\left(\frac{B(x)}{m}\right)^{r+1} dx \leqslant \int_0^\infty x^r \Im(Q(x))^{r+1}\,dx < \infty,$$

proving (31) with $\varepsilon = 1/m$, and hence (30) holds. ∎

The work of Mori in [14] and [15] follows from Corollary 1 by taking $a_n \equiv 1$.

In Example 3 of Adler and Rosalsky [1], it is shown for an arbitrary sequence of i.i.d. random variables $\{X_n, n \geqslant 1\}$ with $E|X_1| < \infty$ that if $a_n = 1$ or $1/n$ according to whether n is odd or even and $b_n = n$, $n \geqslant 1$, then

$$\frac{1}{b_n} \sum_{j=1}^{n} a_j (X_j - EX_1) \to 0 \text{ a.s.}$$

Note that in this example we have $c_n = n$, $n \geqslant 1$, and so (1) holds with $\alpha = 1$ and $b_{2n} = O(b_n)$ also holds. Suppose

$$\Im(x) = \frac{e}{x(\log x)^\beta}, \quad x \geqslant e, \quad \text{where } 0 < \beta \leqslant 1.$$

Then $E|X_1| = \infty$ (hence Example 3 of Adler and Rosalsky [1] is not applicable), but $\int_0^\infty x^r \Im(x)^{r+1}\,dx < \infty$ for all integers $r > \beta^{-1} - 1$. Applying Theorem 1 we thus obtain, for all integers $r > \beta^{-1} - 1$,

$$T_n^{(r)}/b_n - k_n \to 0 \text{ a.s.}$$

for some sequence of constants $\{k_n, n \geqslant 1\}$. This example shows that Theorem 1 can indeed be applicable when $E|X_1| = \infty$ thereby characterizing the a.s. limiting behavior of the lightly trimmed sums $T_n^{(r)}$. Moreover, it follows from Theorem 2 with $r = 0$ that there does not exist a sequence of constants $\{k_n, n \geqslant 1\}$ such that $b_n^{-1} T_n - k_n \to 0$ a.s.

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