PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 23, Fasc. 1 (2003), pp. 105–138

GAUSSIAN SEMIPARAMETRIC ESTIMATION FOR RANDOM FIELDS WITH SINGULAR SPECTRUM

BY

DOMENICO MARINUCCI* (ROMA)

Abstract. We analyze the asymptotic behaviour of the tapered discrete Fourier transforms for random fields with singular spectrum. The results are used to establish consistency and asymptotic normality for semiparametric estimates of the singularity parameter under broad conditions.

AMS Subject Classification: 60G60, 62M15, 62G20.

Key words and phrases: Discrete Fourier transforms, random fields, singular spectrum, Whittle semiparametric estimates.

1. INTRODUCTION

Let $U(t_1, t_2, t_3) = (U_1(\cdot), \ldots, U_p(\cdot))'$ be a wide-sense homogeneous and isotropic random field defined on the lattice Z^3 and taking values in R^p, $p \in N$. Let $\{f_{gh}(\lambda, \mu, \nu)\}_{g,h=1,\ldots,p}$ denote the spectral density matrix. In this paper, we shall be concerned with random fields such that the component functions $f_{gh}(\cdot)$ satisfy, for some $\varepsilon > 0$,

$$(1.1) \quad f_{gh}(\omega) = L_{gh}(\|\omega\|)\|\omega\|^{\alpha_g+\alpha_h} \quad \text{for } \|\omega\| \leqslant \varepsilon, \ -3/2 < \alpha_g, \ \alpha_h \leqslant 3/2,$$

where $\omega = (\lambda, \mu, \nu)$, $\|\cdot\|$ denotes Euclidean norm, and $L_{gh}(\|\omega\|)$ is a complex-valued scalar function whose modulus is bounded and bounded away from zero at the origin. Throughout the paper, the exact form of $L_{gh}(\|\omega\|)$ is assumed to be unknown. We write α_{gh} for $\alpha_g + \alpha_h$; thus

$$\lim_{\|\omega\| \to 0} f_{gg}(\omega) = \begin{cases} 0 & \text{for } \alpha_{gg} > 0, \\ L_{gg}(0) > 0 & \text{for } \alpha_{gg} = 0, \\ \infty & \text{for } \alpha_{gg} < 0. \end{cases}$$

* Research supported by MURST. The author is grateful to the referee for useful comments.

Hence we say that the random field $U(t_1, t_2, t_3)$ has *singular spectrum*, i.e. it is long-range dependent for $\alpha_{gg} \neq 0$ and some g; the condition $\alpha_{gg} > -3$ is needed to ensure integrability of the spectral density, and hence wide-sense homogeneity of the field. $U(\cdot)$ may represent a scalar density field for $p = 1$ or a vector velocity field for $p = 3$. The equality (1.1) implies that $U(\cdot)$ is isotropic, which is a standard assumption in the vast majority of physical applications.

Random fields with spectral singularities are now known to arise in many cases of interest. Albeverio et al. [1] and many subsequent authors have studied the asymptotic behaviour of solutions for Burgers' nonlinear differential equation in three dimensions, as motivated in particular by the analysis of the distribution of self-gravitating matter in the late stages of the Universe. In particular, Albeverio et al. [1] have shown heuristically that in the absence of long-range dependent behaviour these models imply that the density of matter is asymptotically uniform — an implication utterly contradicted by astronomical data (see Shandarin and Zeldovich [22] and the references therein), where the presence of large voids and intermittent structures (Voronoi tessellations) is firmly established (see also Funaki et al. [5], Molchanov et al. [17], Leonenko and Woyczynski [13]. In fact, in the astrophysical literature on matter distribution (1.1) with $p = 1$ and α close to unity is often taken for granted, for instance by the highly popular Harrison–Zeldovich model (Peebles [18]). In the same context, Shandarin and Zeldovich [22], p. 205, mention six alternative proposals for $f(\omega)$, all of them satisfying (1.1) with $\alpha > 0$. Many other stochastic models outside Burgers' turbulence can produce long-range dependent behaviour in random fields; for instance, fractional and non-fractional diffusion-wave equations producing spectral singularities are considered by Anh et al. [2], Anh and Leonenko [3], [4] and others, with applications including wave diffusions in porous media, nonlinear acoustic shock waves and other types of irrotational flow. In all these cases the exact form of the spectral density can be quite complicated or even not yet known, depending on as many as sixteen parameters in some cosmological models, however typically (1.1) does hold around the origin.

In the time series case, statistical inference and its mathematical foundations in the presence of long-range dependence have now been investigated in great detail, under both parametric and semiparametric conditions. On the other hand, although the probabilistic literature in the random field case has now reached a high level of sophistication, statistical inference procedure have not been developed to the same extent as for time series. The main references are Heyde and Gay [8], Leonenko and Woyczynski [14], [15], Ludena and Lavielle [16], each of these authors considering, under different assumptions, a fully parametric specification over the whole frequency band. Therefore, it seems that semiparametric procedures, which impose only the milder condition (1.1), i.e. which make assumptions only on an arbitrary small neighbourhood around zero frequency, have largely been neglected in the random field case.

Nevertheless, data sets which are candidates for long-range dependent behaviour (like catalogs of galaxy redshifts in the astronomical context) are often characterized by an extremely high number of observations. Hence it can be computationally very hard to implement fully parametric estimates, which are typically not available in closed form and require lengthy iterations. More important, a full-band model entails by necessity a number of assumptions and approximations whose validity can be questioned, while many of them need not be necessary for the analysis of the behaviour of the system at the largest scale. The presence of observational error, moreover, can add a white noise additive component in the spectral density of the observables, so that a full band model may be misspecified, whereas (1.1) may still be valid, at least for negative α. It is also important to remark how most physical models are developed for continuous parameter fields, whereas observations are usually available on a lattice like Z^3. Discretization procedures have a complicated nonlinear effect on the spectral density, which is often difficult to pin down exactly, especially as data collection is in many cases beyond the control of the statistician. The most common discretization procedures, however, such as neighbourhood smoothing or grid sampling, do not have effects at zero frequency, except at most some rescaling in the constants, and this provides in our opinion one further reason to favour local-to-zero specifications such as (1.1). The parameter α is often of considerable interest by itself; for instance, many geometric functionals of random fields commonly used as model checking devices are well-known to have asymptotic distribution depending only on α and $L(0)$, see Ivanov and Leonenko [9]. Finally, estimates of α can be used as the bench--mark to discriminate between alternative models, such as different inflationary scenarios for the very early Universe (Kolb and Turner [10]).

The purpose of this paper is to develop a semiparametric procedure for statistical inference on the long-range dependence parameters α, imposing only local-to-zero conditions. Our basic idea is to extend to the random field case the Whittle semiparametric procedure considered for long-range dependent time series by Künsch [11] and Robinson [21]. As many semiparametric methods, Whittle estimates rely only on the information at the smallest frequencies, and therefore have asymptotic efficiency zero with respect to procedures based on a correctly specified parametric model. In the presence of misspecification of the high-frequency component, however, a parametric model will generally lead to inconsistent estimates, whereas semiparametric procedures have robustness properties that seem desirable. Moreover, the loss of asymptotic efficiency seems acceptable in many random fields contexts, where data sets candidate for long-range dependent behaviour are often characterized by an extremely high number of observations (e.g. the ongoing Sloan Digital Survey on stellar distribution aims at mapping the position of more than 10^6 galaxies). Finally, from the computational point of view the procedure we advocate seems extremely convenient, requiring minimization of a globally concave univariate function, a task which can be easily accomplished by several well-known optimization routines.

The plan of the paper is as follows. In Section 2 we establish some results of independent interest on the asymptotic behaviour of the discrete Fourier transform of the vector field $U(\cdot)$. This material extends to the random field case analogous results by Robinson [20] and Velasco [24], and we believe it may find applications in other semiparametric inference procedures in the presence of spectral singularities. In Section 3 we focus more directly on statistical inference; in particular, we apply the results of Section 2 to the analysis of the Whittle semiparametric estimates, for which we prove consistency and asymptotic Gaussianity. Most proofs are rather lengthy and thus collected separately in Sections 4 and 5. In the sequel, we use C to denote a generic constant whose value may vary from line to line.

2. ASYMPTOTIC BEHAVIOUR OF THE DISCRETE FOURIER TRANSFORMS

For technical reasons, we need to strengthen (1.1) slightly and impose some additional smoothness condition; more precisely, we shall assume that

ASSUMPTION A. There exist $\varepsilon > 0$ such that, for $\|\omega\| < \varepsilon$, (1.1) holds, where $L_{gg}(\|\omega\|)$ is differentiable with derivatives Hölder continuous of degree $\beta - 1$ for $1 < \beta \leqslant 2$, $g = 1, \ldots, p$.

Assumption A is a mild local smoothness condition which covers most parametric models so far considered in the applications; note that we are only considering the terms on the main diagonal of the spectral density matrix, which are real valued. We stress that, as in Robinson [20], [21], no condition whatsoever is imposed throughout the paper on $f(\omega)$ outside a neigbourhood of the origin, except of course integrability which is implied by wide-sense homogeneity. The condition $\beta \leqslant 2$ is only convenient for notation, as any function such that Assumption A holds with $\beta > 2$ would obviously satisfy Assumption A with $\beta = 2$ also.

Define by

$$R_{gh}(\|\omega\|) = \frac{f_{gh}(\omega)}{\sqrt{f_{gg}(\omega) f_{hh}(\omega)}}$$

the coherency of the field. For some results in the sequel, we need a further regularity condition on the behaviour of the cross-spectral density at the origin, namely,

ASSUMPTION A′. Assumption A holds and there exists $\varepsilon > 0$ such that, for $\|\omega\| < \varepsilon$,

$$\left| \frac{d \operatorname{Re} R_{gh}(\|\omega\|)}{d\|\omega\|} + i \frac{d \operatorname{Im} R_{gh}(\|\omega\|)}{d\|\omega\|} \right| = O(\|\omega\|^{-1}). \tag{2.1}$$

By the Cauchy–Schwarz inequality, $R_{gh}(\|\omega\|)$ is bounded in modulus by unity, so Assumption A′ is requiring little more than differentiability in a neighbourhood of the origin; note that (2.1) holds trivially for $g = h$, as in this case the left-hand side is identically zero.

Now assume we have a sample of dimension $n_1 \times n_2 \times n_3$ from the random field $U(\cdot)$, i.e. we have observed $U_g(t_1, t_2, t_3)$ for $1 \leqslant t_i \leqslant n_i$, $i = 1, 2, 3$, $g = 1, \ldots, p$. Define the row vectors $t = (t_1, t_2, t_3)$, $\omega_{jkl} = (\lambda_j, \mu_k, \nu_l)$, where λ_j, μ_k, ν_l represent Fourier frequencies, i.e. (omitting for notational simplicity any reference to n_1, n_2, n_3)

$$\lambda_j = \frac{2\pi j}{n_1}, \qquad \mu_k = \frac{2\pi k}{n_2}, \qquad \nu_l = \frac{2\pi l}{n_3};$$

and the tapered discrete Fourier transforms

$$w_g^T(\omega_{jkl}) = H^{-1} \sum_{t_1, t_2, t_3 = 1}^{n_1, n_2, n_3} h_t U_g(t) \exp\{it\omega'_{jkl}\}, \tag{2.2}$$

$$H = [(2\pi)^3 H_{n_1} H_{n_2} H_{n_3}]^{1/2}, \qquad H_n = \sum_{u=1}^{n} h_u^2, \tag{2.3}$$

$h_t = h_{t_1} h_{t_2} h_{t_3}$ representing the taper (or convergence factor); the untapered discrete Fourier transform clearly corresponds to $h_{t_j} \equiv 1$, $j = 1, 2, 3$. Likewise, we define the tapered (cross-) periodogram

$$I_{gh}^T(\omega_{jkl}) = w_g^T(\omega_{jkl}) \bar{w}_h^T(\omega_{jkl}),$$

the bar denoting complex conjugation, whence

$$\begin{aligned} EI_{gh}^T(\omega_{jkl}) &= Ew_g^T(\omega_{jkl}) \bar{w}_h^T(\omega_{jkl}) \\ &= H^{-2} \sum_{t_1, t_2, t_3}^{n_1, n_2, n_3} \sum_{s_1, s_2, s_3}^{n_1, n_2, n_3} \int_T \exp\{i(t-s)(\omega_{jkl}-\omega)'\} f_{gh}(\omega)\, d\omega \\ &= \int_T K_n^T(\omega_{jkl}-\omega) f_{gh}(\omega)\, d\omega \end{aligned}$$

for $T = [-\pi, \pi]^3$, $s = (s_1, s_2, s_3)$, $K_n^T(\omega_{jkl}-\omega) = K_{n_1}^T(\lambda_j-\lambda) K_{n_2}^T(\mu_k-\mu) K_{n_3}^T(\nu_l-\nu)$, and

$$K_n^T(\xi) = \frac{1}{2\pi H_n} \Big| \sum_{u=1}^{n} h_u e^{iu\xi} \Big|^2.$$

Tapering is a well-known bias reduction technique, which is useful, but not strictly necessary, in the frequency domain analysis of long-range dependent time series. It becomes mandatory when random fields are considered, indeed it is known that, due to edge effects, the untapered periodogram is second-order biased even in the independent and identically distributed (i.i.d.) case, see Guyon

[6] or Ludena and Lavielle [16] for details. In this paper, following Velasco [24] and for convenience of computations we shall always use the asymmetric cosine bell taper, i.e.

$$h_u = \frac{1}{2}\left(1-\cos\left[\frac{2\pi u}{n}\right]\right), \quad H_n = \frac{3n}{8}.$$

With this choice $K_n^T(\xi)$ integrates to one, it is even, positive and satisfies (Hannan [7], p. 265)

$$\sup_{\xi} |K_n^T(\xi)| = O(\min\{n, n^{-5}\xi^{-6}\}), \quad -\pi \leqslant \xi \leqslant \pi. \tag{2.4}$$

Other choices of tapers satisfying (2.4) are possible, but the cosine bell taper has the nice property to factorize the discrete Fourier transform as

$$w_g^T(\omega_{jkl}) = \sum_{|j'-j|,|k'-k|,|l'-l| \leqslant 1} \left(-\tfrac{1}{2}\right)^{3+|(j,k,l)-(j',k',l')|} w_g(\omega_{j'k'l'}), \tag{2.5}$$

where $w_g(\omega_{jkl})$ corresponds to the discrete Fourier transform in the untapered case, and we define

$$|(j, k, l)-(j', k', l')| = |j'-j|+|k'-k|+|l'-l|.$$

Hence the cosine bell taper has the orthogonality property when at least one of the coordinates is two or more fundamental frequencies further away, which is clearly convenient for many proofs.

In the sequel, we shall need three user-chosen bandwidth parameters m_i, i.e. three positive integers $m_i = m(n_i)$, $i = 1, 2, 3$, non-decreasing with n_i; we write $N = (n_1, n_2, n_3)$, $M = (m_1, m_2, m_3)$, and we assume that

ASSUMPTION B. $\|N\| \to \infty$ in such a way that $0 < \kappa_1 \leqslant m_i/\|M\|$, $n_i/\|N\|$, $i = 1, 2, 3$.

Assumption B imposes a mild restriction on the degree of elongation of the observed range of (t_1, t_2, t_3), with a corresponding constraint on the user--chosen bandwidth parameters m_i. In some authors' terminology, it relates to the notion of going to infinity in the Fischer sense, see for instance Ivanov and Leonenko [9].

The first result of this paper relates to the bias of the tapered periodogram at very low frequencies. For convenience, we write $M_1 = \max(j, k, l)$ and $N_1 = \min(j, k, l)$.

THEOREM 1. *Under Assumptions* A *and* B, *for* $-3 < \alpha_{gg} \leqslant 3$, $g = 1, \ldots, p$,

$$EI_{gg}^T(\omega_{jkl}) - L_{gg}(0)\|\omega_{jkl}\|^{\alpha_{gg}} = O\left(\left\{M_1^{-2}+N_1^{-5}+\left\{\frac{M_1}{\|N\|}\right\}^{\beta}\right\}\|\omega_{jkl}\|^{\alpha_{gg}}\right), \tag{2.6}$$

and assuming further that A′ *holds, for* $g,\ h = 1, 2, \ldots, p$ *we have*

$$|EI_{gh}^T(\omega_{jkl}) - f_{gh}(\omega_{jkl})| = O(\{M_1^{-1} + N_1^{-5}\}\, \|\omega_{jkl}\|^{\alpha_{gh}}). \tag{2.7}$$

Theorem 1, as Theorems 2–4 to follow, can be seen as an extension to the random field case of analogous results for time series by Robinson [20] and Velasco [24] in the untapered and tapered cases, respectively. Note that our method of proof for (2.6) would allow some slight improvement over the Velasco result [24] even in the time series case, as the exponent of M_1 is -2, irrespective of the value of β. The result for (2.7) is in a sense not so sharp as (2.6), as a consequence of the fact that Assumption A relates only to terms on the main diagonal of the spectral density matrix. Although it would be straightforward to extend Assumption A to cover non-diagonal components, the present formulation of Theorem 1 is sufficient for our purposes and it affords the greatest generality of a priori conditions. The following result concerns the cross-products of two transforms.

THEOREM 2. *Under Assumptions* A′ *and* B, *for* $g, h = 1, \ldots, p$,

$$|Ew_g^T(\omega_{jkl})\, w_h^T(\omega_{jkl})| = O\left(\frac{\|\omega_{jkl}\|^{\alpha_{gh}}}{N_1^8}\right).$$

For the results to follow, we define

$$M_2 = \max(j_1, j_2, k_1, k_2, l_1, l_2), \qquad N_2 = \min(j_1, j_2, k_1, k_2, l_1, l_2).$$

THEOREM 3. *Under Assumptions* A′ *and* B, *and if*

$$\min\{|j_1 - j_2|,\ |k_1 - k_2|,\ |l_1 - l_2|\} \geqslant 3,$$
$$\min\{n_1 - |j_1 - j_2|,\ n_2 - |k_1 - k_2|,\ n_3 - |l_1 - l_2|\} \geqslant 3,$$

we have

$$|Ew_g^T(\omega_{j_1k_1l_1})\, \bar{w}_h^T(\omega_{j_2k_2l_2})|$$
$$= O\left(\|\omega_{j_1k_1l_1}\|^{\alpha_{gh}/2}\, \|\omega_{j_2k_2l_2}\|^{\alpha_{gh}/2} \prod_{a=j,k,l} \{\max(|a_2 - a_1|, 1)\}^{-2} \frac{M_2^{1-\alpha_{gh}/2}}{N_2^{2-\alpha_{gh}/2}}\right)$$

for $\alpha_{gh} < 0$, *whereas for* $\alpha_{gh} \geqslant 0$

$$|Ew_g^T(\omega_{j_1k_1l_1})\, \bar{w}_h^T(\omega_{j_2k_2l_2})|$$
$$= O\left(\|\omega_{j_1k_1l_1}\|^{\alpha_{gh}/2}\, \|\omega_{j_2k_2l_2}\|^{\alpha_{gh}/2} \prod_{a=j,k,l} \{\max(|a_2 - a_1|, 1)\}^{-2} \frac{M_2^{\alpha_{gh}/2-1}}{N_2^{\alpha_{gh}/2}}\right).$$

THEOREM 4. *Under Assumptions* A′ *and* B, *for* $\alpha_{gh} < 0$,

$$|Ew_g^T(\omega_{j_1k_1l_1})\, w_h^T(\omega_{j_2k_2l_2})| = O\left(\frac{\|\omega_{j_1k_1l_1}\|^{\alpha_{gh}/2}\, \|\omega_{j_2k_2l_2}\|^{\alpha_{gh}/2}}{N_2^9}\right), \tag{2.8}$$

and for $\alpha_{gh} \geqslant 0$

$$(2.9)\qquad |Ew_g^T(\omega_{j_1k_1l_1})\, w_h^T(\omega_{j_2k_2l_2})| = O\left(\frac{\|\omega_{j_1k_1l_1}\|^{\alpha_{gh}/2}\,\|\omega_{j_2k_2l_2}\|^{\alpha_{gh}/2}}{N_2^5}\right).$$

In the proofs of Theorems 1–4 we borrow several ideas from the arguments of Robinson [20] and Velasco [24]. The random field setting, however, poses considerable extra difficulties, because in the three-dimensional case the Fourier frequencies satisfy only partial order relationships, and this introduces major technical problems, as confirmed by a careful analysis of the proofs in Section 4. It is also important to stress that the error term which emerges in Theorem 1 when we approximate $f(\omega)$ by $L_{gg}(0)\|\omega\|^{\alpha}$ is of the order $(\|M\|/\|N\|)^{\beta}$, i.e. it is not improved in the random field case with respect to time series circumstances; on the other hand, the rates of convergence of our estimates, as we shall see in Section 3, is of the order $\|M\|^{3/2}$, i.e. "much faster" (in a loose sense – a strict comparison is clearly meaningless) than the rate for the time series case, which in our notation would be $\|M\|^{1/2}$. In terms of applications, this implies that for our arguments in Section 3 to go through we need to provide a bound for the variance of a sum of discrete Fourier transforms, rather than more simply (but less efficiently) bounding each of the elements of the sums itself (see Lemma 3 for more details). This is the main difference between our arguments and the arguments of Robinson [20], [21] and Velasco [24] for the time series case, and also the main motivation for the presence of the terms of order $\prod_{a=j,k,l}(a_2 - a_1)^{-2}$ in Theorem 3; such terms would not contribute to the bounds in any positive way for a fixed distance between a_2 and a_1, but their role is crucial as we sum over these same indexes. For related reasons, we had to strengthen the smoothness condition in Assumption A to $\beta > 1$, whereas $\beta > 0$ can suffice in time series circumstances.

3. ESTIMATION OF THE SINGULARITY PARAMETER

The main purpose of the present paper is to analyze a semiparametric procedure for estimation and inference on the parameters α_{gh}. For simplicity, and because this is by far the most relevant case for applications, we shall concentrate in this section on the case $p = 1$, i.e. the scalar environment (density fields, say); here it is also convenient to denote by α_0 the "true" value of the (unique) memory parameter of interest, and by α any generic value. It is worthwhile to remark that even if we focus on the scalar case, cross-periodogram terms do arise in the arguments to follow, so that, for instance, the "multivariate" result (2.7) is required explicitly in the proof of Lemma 3 below.

We focus on an extension to random fields of the Whittle semiparametric procedure introduced by Künsch [11] and developed by Robinson [21]. Write

$$\hat{\sum}_{jkl} = \sum_j \sum_k \sum_l,$$

(3.1) $$j = r_1, r_1+3, \ldots, m_1, \quad k = r_2, r_2+3, \ldots, m_2, \quad l = r_3, r_3+3, \ldots, m_3,$$

(3.2) $$\tilde{m}_1 = \frac{m_1 - r_1}{3}, \quad \tilde{m}_2 = \frac{m_2 - r_2}{3}, \quad \tilde{m}_3 = \frac{m_3 - r_3}{3},$$

where we assume for simplicity that $(m_i - r)/3$ is integer-valued, $i = 1, 2, 3$; the integers r_1, r_2, r_3 are user-chosen trimming numbers, motivated by the need to drop very low frequencies for the bound in Theorem 3 to become effective; trimming of low frequencies is imposed (with a different motivation) in Robinson [20] and Velasco [24]. The consideration of only one fundamental frequency ω_{jkl} out of three with respect to j, k, l is motivated by (2.5), which implies that with this device we are able to retain orthogonality among the tapered discrete Fourier transforms. Now take

(3.3) $$Q(L, \alpha) = \frac{1}{\tilde{m}_1 \tilde{m}_2 \tilde{m}_3} \hat{\sum}_{jkl} \left\{ \log L \|\omega_{jkl}\|^{\alpha} + \frac{I^T(\omega_{jkl})}{L \|\omega_{jkl}\|^{\alpha}} \right\},$$

(3.4) $$(\hat{L}, \hat{\alpha}) = \arg \min_{\substack{0 < L < \infty \\ \alpha \in \Theta}} Q(L, \alpha)$$

for $\|M\|/\|N\| \to 0$, and Θ a compact subset of $(-3, 3]$. Compactness is needed for the minimum to exist: in the absence of any a priori information on the range of values of α, Θ can be chosen to be $[-3+\delta, 3]$ for an arbitrary small, positive δ. The quantity $Q(G, \alpha)$ can be viewed as the Whittle approximation to the Gaussian likelihood, considered at the only frequencies where the parametric model $f(\|\omega\|) = L(0)\|\omega\|^{\alpha_0}$ is relied upon, i.e. on a degenerating band around the origin.

Standard manipulations give

$$\hat{\alpha} = \arg\min_{\alpha \in \Theta} Z(\alpha), \qquad Z(\alpha) = \log \hat{G}(\alpha) + \alpha \frac{1}{\tilde{m}_1 \tilde{m}_2 \tilde{m}_3} \hat{\sum}_{jkl} \log \|\omega_{jkl}\|,$$

$$\hat{G}(\alpha) = \frac{1}{\tilde{m}_1 \tilde{m}_2 \tilde{m}_3} \hat{\sum}_{jkl} \frac{I^T(\omega_{jkl})}{\|\omega_{jkl}\|^{\alpha}}.$$

It follows immediately from Proposition (2.15) in Vajda [23] that $Z(\alpha)$ is strictly concave, and this is very convenient for the derivation of $\hat{\alpha}$, which of course must be obtained through a numerical optimization procedure. On the other hand, a rigorous analysis of the asymptotic behaviour of $\hat{\alpha}$ requires some further assumptions.

Assumption C. The field $U(t)$ is Gaussian.

Gaussianity is needed in our argument to follow in order to bound conveniently the fourth-order cumulant of normalized discrete Fourier transforms. In the time series context, Robinson [21] relaxes Gaussianity to a linear process condition driven by martingale difference innovations. Although it seems possible to extend his argument to the present setting, a linear representation with martingale difference innovations appears of little practical significance in a context without a temporal ordering such as the one we are considering.

As mentioned above, it is very hard to justify linearity as a primitive condition for a random field; a linear representation for $U(\cdot)$, however, can indeed be derived as a nice consequence of Assumption C. More precisely, by the Wold representation theorem for random fields, we can write, as in Leonenko and Woyczynski [14],

$$U(t) = \sum_{i_1, i_2, i_3 = 1}^{\infty} a(i)\,\varepsilon(t-i),$$

where $i = (i_1, i_2, i_3)$ and the $\varepsilon(\cdot, \cdot, \cdot)$ are mean-zero, uncorrelated innovations with constant variance $E\varepsilon^2 = \sigma_\varepsilon^2$, and hence by Gaussianity i.i.d. Now, clearly,

$$f(\omega) = \frac{\sigma_\varepsilon^2}{(2\pi)^3} |\hat{a}(\omega)|^2 \quad \text{for } \hat{a}(\omega) = \sum_{\tau_1, \tau_2, \tau_3 = 1}^{\infty} a(\tau) \exp\{i\tau\omega'\}, \ \tau = (\tau_1, \tau_2, \tau_3).$$

In the sequel, we shall write for brevity (see (2.2) and (2.3))

$$I_{jkl}^T = I^T(\omega_{jkl}), \qquad I_{\varepsilon jkl}^T = I_\varepsilon^T(\omega_{jkl}) = w_\varepsilon^T(\omega_{jkl})\,\bar{w}_\varepsilon^T(\omega_{jkl}),$$

$$w_\varepsilon^T(\omega_{jkl}) = H^{-1} \sum_{t_1, t_2, t_3 = 1}^{n_1, n_2, n_3} h_t\,\varepsilon(t) \exp\{it\omega_{jkl}'\},$$

$$f_{jkl} = f(\omega_{jkl}) = \frac{\sigma_\varepsilon^2}{(2\pi)^3} |\hat{a}_{jkl}|^2, \qquad \hat{a}_{jkl} = \hat{a}(\omega_{jkl}).$$

We now need to impose some conditions on the rate of increase of the trimming coefficients r and the bandwidth parameters m. In the sequel, we let $R = (r_1, r_2, r_3)$.

Assumption D. As $\|N\| \to \infty$,

(3.5) $$r_i/r_j = O(1) \quad \text{for any } i, j = 1, 2, 3,$$

(3.6) $$\lim_{\|N\| \to \infty} \frac{m_2/n_2}{m_1/n_1} = \lim_{\|N\| \to \infty} \frac{m_3/n_3}{m_1/n_1} = 1,$$

and

(3.7) $$\frac{\|R\|}{\|M\|} + \frac{\|M\|^{3+2\beta}}{\|N\|^{2\beta}} + \frac{\|M\|^{1+\max(0, -\alpha_0/2)} \log^2 \|N\|}{\|R\|^{2+\max(0, -\alpha_0/2)}} \to 0.$$

The conditions (3.5) and (3.6) seem natural in view of Assumption B and symmetry in law of the field with respect to the coordinates (recall that, of course, trimming and bandwidth parameters are user-chosen). (3.6) is not strictly necessary for the argument to follow, but it allows the derivation of a much neater expression for the asymptotic variance. (3.7) imposes a mild lower bound on the rate of increase of $\|M\|$ with $\|N\|$, a significant upper bound on the rate of increase of the same $\|M\|$, and a significant lower bound on the rate of increase of $\|R\|$. Again, the condition $\|R\|/\|M\| = o(1)$ is not strictly needed for our argument, but if the trimming rate grows more slowly than $\|M\|$, asymptotic efficiency is unaffected. (3.7) is close to Assumptions 6 and A4′ in Robinson [20] and [21], respectively; for instance, it is nice to remark from our arguments to follow how the factor 3 in the numerator of the third summand of (3.7) would correspond to unity in the time series case. A bandwidth choice satisfying Assumption D is granted by, for instance,

$$m_1 = \left[\frac{n_1^{\varrho}}{\log n_1}\right], \quad m_2 = m_1\left[\frac{n_2}{n_1}\right], \quad m_3 = m_1\left[\frac{n_3}{n_1}\right], \quad \varrho = \frac{2\beta}{3+2\beta} \in \left(\frac{2}{5}, \frac{4}{7}\right),$$

where $[\cdot]$ denotes integer part. In practice β is unknown; a practitioner, however, can choose ϱ on the basis of a priori assumptions on the smoothness of the spectral density of the field of interest around the origin, the values of ϱ closer to $\frac{2}{5}$ ensuring more robustness, the values closer to $\frac{4}{7}$ entailing more efficient estimates. On the other hand, the results of this paper do not imply that trimming (i.e. $\|R\| > 0$) is necessary for the asymptotic theory to go through, indeed in the time series case trimming has recently been proved to be unnecessary. As discussed above, however, trimming does not effect the asymptotic variance under Assumption D, whereas a careful inspection of the proofs to follow suggests that very low frequency periodogram ordinates are typically heavily biased, so that dropping such frequencies seems in any case desirable (see Robinson [20]).

Now let us put

$$\Lambda_{jkl} = \log\|\omega_{jkl}\| - \frac{1}{\tilde{m}_1 \tilde{m}_2 \tilde{m}_3} \widehat{\sum}_{jkl} \log\|\omega_{jkl}\|;$$

we are now in the position to provide the main result of this paper, which is the following

THEOREM 5. *Under Assumptions* A, B, C *and* D, *as* $\|N\| \to \infty$, *for* $\hat{\alpha}$ *defined by* (3.3) *and* (3.4) *we have*

$$\{\tilde{m}_1 \tilde{m}_2 \tilde{m}_3\}^{1/2} (\hat{\alpha} - \alpha_0) \xrightarrow{d} N(0, \Phi^{-1}),$$

where, for $x = (x_1, x_2, x_3)$,

$$\Phi = \int_{[0,1]^3} \log^2\|x\|\,dx - \Big\{\int_{[0,1]^3} \log\|x\|\,dx\Big\}^2 \simeq 0.125.$$

Proof. The proof is similar to the argument in Robinson [21]; we consider only the case $n_1 = n_2 = n_3 = n$, $\tilde{m}_1 = \tilde{m}_2 = \tilde{m}_3 = m$, $r_1 = r_2 = r_3 = r$, say, the general circumstances being only notationally more complicated under Assumption B. Consistency of $\hat{\alpha}$ is established in Lemma 5 below, by a long but standard argument for extremum estimates. Then, by the Mean Value Theorem,

$$0 = \frac{dZ(\alpha_0)}{d\alpha} + \frac{d^2 Z(\tilde{\alpha})}{d\alpha^2}(\hat{\alpha} - \alpha_0),$$

i.e.

$$m^{3/2}(\hat{\alpha} - \alpha_0) = -\left\{\frac{d^2 Z(\tilde{\alpha})}{d\alpha^2}\right\}^{-1} \left\{m^{3/2}\frac{dZ(\alpha_0)}{d\alpha}\right\} \tag{3.8}$$

for $|\tilde{\alpha} - \alpha_0| \leqslant |\hat{\alpha} - \alpha_0|$, provided the second derivative is non-zero. Now

$$\frac{dZ(\alpha)}{d\alpha} = \frac{\hat{G}_1(\alpha)}{\hat{G}_0(\alpha)} + \frac{1}{m^3}\widehat{\sum}_{jkl} \log \|\omega_{jkl}\|, \tag{3.9}$$

$$\frac{d^2 Z(\alpha)}{d\alpha^2} = \frac{\hat{G}_2(\alpha)\hat{G}_0(\alpha) - \hat{G}_1^2(\alpha)}{\hat{G}_0^2(\alpha)}, \tag{3.10}$$

where we define

$$\hat{G}_a(\alpha) = \frac{(-1)^a}{m^3}\widehat{\sum}_{jkl} \log^a \|\omega_{jkl}\| \frac{I_{jkl}}{\|\omega_{jkl}\|^{\alpha}}, \qquad a = 0, 1, 2.$$

Concerning (3.10), the consistency of $\hat{\alpha}$, Slutsky's theorem and Lemma 4 below imply that, as $n, m \to \infty$,

$$\frac{d^2 Z(\tilde{\alpha})}{d\alpha^2} = \frac{\hat{G}_2(\tilde{\alpha})\hat{G}_0(\tilde{\alpha}) - \hat{G}_1^2(\tilde{\alpha})}{\hat{G}_0^2(\tilde{\alpha})}$$

$$\xrightarrow{p} \frac{1}{m^3}\widehat{\sum}_{jkl} \log^2 \|\omega_{jkl}\| - \left\{\frac{1}{m^3}\widehat{\sum}_{jkl} \log \|\omega_{jkl}\|\right\}^2$$

$$= \frac{1}{m^3}\widehat{\sum}_{jkl} \log^2 \left\|\left(\frac{j}{m}, \frac{k}{m}, \frac{l}{m}\right)\right\| - \left\{\frac{1}{m^3}\widehat{\sum}_{jkl} \log \left\|\left(\frac{j}{m}, \frac{k}{m}, \frac{l}{m}\right)\right\|\right\}^2 \to \Phi > 0,$$

from integral approximation. Concerning (3.9), it is immediate to see that $\hat{G}_0(\alpha_0) \xrightarrow{p} L(0)$. Hence to conclude the analysis of (3.8) it is enough to focus on the asymptotic distribution of

$$\frac{1}{m^{3/2}}\widehat{\sum}_{jkl} \Lambda_{jkl} \frac{I_{jkl}^T}{L(0)\|\omega_{jkl}\|^{\alpha_0}},$$

which equals

$$\frac{1}{m^{3/2}}\hat{\sum}_{jkl}\left(\frac{f_{jkl}}{L(0)\,\|\omega_{jkl}\|^{\alpha_0}}-1\right)\Lambda_{jkl}\frac{I^T_{jkl}}{f_{jkl}}+\frac{1}{m^{3/2}}\hat{\sum}_{jkl}\Lambda_{jkl}\frac{I^T_{jkl}-|a_{jkl}|^2\,I^T_{\varepsilon jkl}}{f_{jkl}}$$
$$+\frac{1}{m^{3/2}}\hat{\sum}_{jkl}\Lambda_{jkl}\left(\frac{(2\pi)^3\,I^T_{\varepsilon jkl}}{\sigma^2_\varepsilon}-1\right),$$

because

$$\hat{\sum}_{jkl}\Lambda_{jkl}=0.$$

We then need to show that, as $n\to\infty$,

$$\frac{1}{m^{3/2}}\hat{\sum}_{jkl}\Lambda_{jkl}\left(\frac{(2\pi)^3\,I^T_{\varepsilon jkl}}{\sigma^2_\varepsilon}-1\right)\xrightarrow{d}N(0,\,\Phi),\tag{3.11}$$

$$\frac{1}{m^{3/2}}\hat{\sum}_{jkl}\left(\frac{f_{jkl}}{L(0)\,\|\omega_{jkl}\|^{\alpha_0}}-1\right)\Lambda_{jkl}\frac{I^T_{jkl}}{f_{jkl}}\xrightarrow{p}0,\tag{3.12}$$

$$\frac{1}{m^{3/2}}\hat{\sum}_{jkl}\Lambda_{jkl}\frac{I^T_{jkl}-|a_{jkl}|^2\,I^T_{\varepsilon jkl}}{f_{jkl}}\xrightarrow{p}0.\tag{3.13}$$

The proofs of (3.11), (3.12) and (3.13) are consequences of Lemmas 2, 3 and 4 (respectively), all three collected in the Appendix. ■

Theorem 5 appears qualitatively analogous to Theorem 2 in Robinson [21]; to aid comparison, we note that our α corresponds to $2d$ in the time series circumstances of Robinson [21], whose main result can be presented as

$$m^{1/2}(2\hat{d}-2d)\xrightarrow{d}N(0,\,1)\quad\text{as } n\to\infty.$$

Note that in the time series case the inverse of the asymptotic variance is indeed provided by

$$\int_0^1\log^2 x_1\,dx_1-\Big\{\int_0^1\log x_1\,dx_1\Big\}^2=1.$$

The limiting result in Theorem 5 is expressed in a form as compact as possible; an alternative formulation is

$$\frac{\{\tilde{m}_1\,\tilde{m}_2\,\tilde{m}_3\}^{-1/2}}{\tilde{\Phi}^{1/2}}\hat{\sum}_{jkl}\Lambda_{jkl}\left(\frac{(2\pi)^3\,I^T_{\varepsilon jkl}}{\sigma^2_\varepsilon}-1\right)\xrightarrow{d}N(0,\,1),\tag{3.14}$$

where

$$\tilde{\Phi}=\{\tilde{m}_1\,\tilde{m}_2\,\tilde{m}_3\}^{-1}\hat{\sum}_{jkl}\log^2\|\omega_{jkl}\|-\Big\{\{\tilde{m}_1\,\tilde{m}_2\,\tilde{m}_3\}^{-1}\hat{\sum}_{jkl}\log\|\omega_{jkl}\|\Big\}^2,$$

and $\lim_{\|N\|\to\infty}\tilde{\Phi}=\Phi$. It is easy to see from our argument in the proof of Lemma 2 that (3.14) and (5.1) are indeed equivalent; for any triplets $(\tilde{m}_1,\tilde{m}_2,\tilde{m}_3)$, (n_1,n_2,n_3), the functional $\tilde{\Phi}$ is bounded and bounded away from zero and can be immediately computed. We conjecture that (3.14) may provide a better approximation to asymptotic distribution in finite samples.

4. PROOFS FOR SECTION 2

Proof of Theorem 1. We start with the proof of (2.6), and we drop the subscripts from $I_{gg}^T(\cdot)$, $f_{gg}(\cdot)$, $L_{gg}(0)$ and α_{gg} for notational simplicity. Consider first $\alpha < 0$ and (without loss of generality) $j \leqslant k \leqslant l$. We have

$$EI^T(\omega_{jkl}) - L(0)\,\|\omega_{jkl}\|^\alpha = EI^T(\omega_{jkl}) - f(\omega_{jkl}) + f(\omega_{jkl}) - L(0)\,\|\omega_{jkl}\|^\alpha,$$

where, for $\|N\|$ large enough

$$f(\omega_{jkl}) - L(0)\,\|\omega_{jkl}\|^\alpha = \{L(\|\omega_{jkl}\|) - L(0)\}\,\|\omega_{jkl}\|^\alpha = O(\|\omega_{jkl}\|^{\beta+\alpha}),$$

as in Robinson [20] and Velasco [24].

Also,

$$EI^T(\omega_{jkl}) - f(\omega_{jkl}) = \int_T K_n^T(\omega_{jkl} - \omega)\{f(\omega) - f(\omega_{jkl})\}\,d\omega.$$

From (2.4), for $\|\omega\| > \varepsilon$, we have easily the bound

$$\int_{\|\omega\|>\varepsilon} K_n^T(\omega_{jkl} - \omega)\{f(\omega) - f(\omega_{jkl})\}\,d\omega$$
$$\leqslant C\,\|N\|^{-3}\left(\frac{\varepsilon}{\sqrt{3}}\right)^{-6}\Big\{\int_{\|\omega\|>\varepsilon} f(\omega)\,d\omega + f(\omega_{jkl})\Big\}$$
$$= O(\|N\|^{-3}\{1 + \|\omega_{jkl}\|^\alpha\}) = O(M_1^{-3}\,\|\omega_{jkl}\|^\alpha),$$

because $\|\omega\| > \varepsilon$ implies $\max\{\lambda, \mu, \nu\} > \varepsilon/\sqrt{3}$, and for any $\alpha \leqslant 3$

$$\|N\|^{-3} = O(\|N\|^{-3+\alpha}\,\|\omega_{jkl}\|^\alpha\, l^{-\alpha}) = O\left(M_1^{-3}\frac{M_1^{3-\alpha}}{\|N\|^{3-\alpha}}\|\omega_{jkl}\|^\alpha\right) = O(M_1^{-3}\,\|\omega_{jkl}\|^\alpha).$$

Clearly, $\|\omega\| < \varepsilon$ implies $-\varepsilon \leqslant \nu \leqslant \varepsilon$, which can be decomposed in the four regions

$$\{-\varepsilon \leqslant \nu \leqslant -\nu_l/2\} \cup \{-\nu_l/2 \leqslant \nu \leqslant \nu_l/2\} \cup \{\nu_l/2 \leqslant \nu \leqslant 3\nu_l/2\} \cup \{3\nu_l/2 \leqslant \nu \leqslant \varepsilon\}.$$

We have

$$\int_{\|\omega\|\leqslant\varepsilon,\,\nu\leqslant-\nu_l/2} K_n^T(\omega_{jkl} - \omega)\{f(\omega) - f(\omega_{jkl})\}\,d\omega$$
$$\leqslant \left\{\max_{\|\omega\|\leqslant\varepsilon,\,\nu\leqslant-\nu_l/2}\frac{f(\omega)}{|\nu|^{(\alpha+3)/2}}\right\}\int_{-\varepsilon}^{-\nu_l/2}|\nu|^{(\alpha+3)/2}K_{n_3}^T(\nu_l - \nu)\,d\nu + f(\omega_{jkl})\int_{-\varepsilon}^{-\nu_l/2}K_{n_3}^T(\nu_l - \nu)\,d\nu$$
$$= O\Big(\|\omega_{jkl}\|^\alpha n_3^{-5}\int_{-\varepsilon}^{-\nu_l/2}|\nu|^{-6}\,d\nu + \nu_l^{(\alpha-3)/2}n_3^{-5}\int_{-\varepsilon}^{-\nu_l/2}|\nu|^{(\alpha-9)/2}\,d\nu\Big)$$
$$= O(\|\omega_{jkl}\|^\alpha\, l^{-5}) = O(\|\omega_{jkl}\|^\alpha M_1^{-5}).$$

Exactly the same argument can be used for $\{3\nu_l/2 \leqslant \nu \leqslant \varepsilon\}$. Then we consider $\{-\nu_l/2 \leqslant \nu \leqslant \nu_l/2\}$, subpartitioned as follows:

$$\int\limits_{-\lambda_j/2}^{\lambda_j/2}\int\limits_{-\mu_k/2}^{\mu_k/2}\int\limits_{-\nu_l/2}^{\nu_l/2}+\int\limits_{-\lambda_j/2}^{\lambda_j/2}\int\limits_{|\mu|>\mu_k/2}\int\limits_{-\nu_l/2}^{\nu_l/2}+\int\limits_{|\lambda|>\lambda_j/2}\int\limits_{-\mu_k/2}^{\mu_k/2}\int\limits_{-\nu_l/2}^{\nu_l/2}+\int\limits_{|\lambda|>\lambda_j/2}\int\limits_{|\mu|>\mu_k/2}\int\limits_{-\nu_l/2}^{\nu_l/2}.$$

The first element is bounded by

$$\text{(4.1)}\qquad \int\limits_{-\lambda_j/2}^{\lambda_j/2}\int\limits_{-\mu_k/2}^{\mu_k/2}\int\limits_{-\nu_l/2}^{\nu_l/2} K_n^T(\omega_{jkl}-\omega)\{f(\omega)-f(\omega_{jkl})\}\,d\omega$$

$$\leqslant \frac{C\|N\|^3}{j^6k^6l^6}\int\limits_{-\lambda_j/2}^{\lambda_j/2}\int\limits_{-\mu_k/2}^{\mu_k/2}\int\limits_{-\nu_l/2}^{\nu_l/2}\{\|\omega\|^\alpha+f(\omega_{jkl})\}\,d\omega$$

$$\leqslant \frac{C\|N\|^3}{j^6k^6l^6}\int\limits_0^{\nu_l}\int\limits_0^{2\pi}\int\limits_0^{2\pi}\{\varrho^\alpha+f(\omega_{jkl})\}\,\varrho^2 d\vartheta d\phi d\varrho$$

$$\leqslant \frac{C\|N\|^3}{j^6k^6l^6}\{\nu_l^{\alpha+3}+\|\omega_{jkl}\|^\alpha\nu_l^3\} = O\left(\frac{\|\omega_{jkl}\|^\alpha}{j^6k^6l^3}\right),$$

where the second inequality follows from moving into polar coordinates, $\lambda = \varrho\sin\varphi\cos\vartheta$, $\mu = \varrho\sin\varphi\sin\vartheta$, $\nu = \varrho\cos\varphi$. For the second term, by (2.4) we have the bound

$$Cn_1^{-5}\lambda_j^{-6}n_3^{-5}\nu_l^{-6}\int\limits_{-\lambda_j/2}^{\lambda_j/2}\int\limits_{-\pi}^{\pi}\int\limits_{-\nu_l/2}^{\nu_l/2}K_{n_2}(\mu-\mu_k)\,1_{\{|\mu|>\mu_k/2\}}\{\|\omega\|^\alpha+\|\omega_{jkl}\|^\alpha\}\,d\lambda d\mu d\nu$$

$$\leqslant C\|N\|^{-10}\lambda_j^{-6}\nu_l^{-6}\lambda_j\nu_l\{|\mu_k|^\alpha+\|\omega_{jkl}\|^\alpha\} = O\left(\frac{\|\omega_{jkl}\|^\alpha}{j^5l^2}\right).$$

Similarly, for the third and fourth term we obtain bounds of order

$$O\left(\frac{\|\omega_{jkl}\|^\alpha}{k^5l^2}\right)\quad\text{and}\quad O\left(\frac{\|\omega_{jkl}\|^\alpha}{l^2}\right),$$

respectively, and each of these terms is clearly $O(M_1^{-2}\|\omega_{jkl}\|^\alpha) = O(M_1^{-\beta}\|\omega_{jkl}\|^\alpha)$. Finally, we consider

$$\text{(4.2)}\qquad \int\limits_{-\varepsilon}^{\varepsilon}\int\limits_{-\varepsilon}^{\varepsilon}\int\limits_{\nu_l/2}^{3\nu_l/2} = \Big\{\int\limits_{-\varepsilon}^{\lambda_j/2}+\int\limits_{\lambda_j/2}^{3\lambda_j/2}+\int\limits_{3\lambda_j/2}^{\varepsilon}\Big\}\Big\{\int\limits_{-\varepsilon}^{\mu_k/2}+\int\limits_{\mu_k/2}^{3\mu_k/2}+\int\limits_{3\mu_k/2}^{\varepsilon}\Big\}\int\limits_{\nu_l/2}^{3\nu_l/2},$$

and hence, using

$$\Big(\int\limits_{-\varepsilon}^{\xi_\tau/2}+\int\limits_{3\xi_\tau/2}^{\varepsilon}\Big)K_n(\xi-\xi_\tau)\,d\xi \leqslant Cn^{-5}\int\limits_{\xi_\tau/2}^{\infty}\xi^{-6}\,d\xi = O(\tau^{-5}),\qquad \xi=\lambda,\mu,\nu,\ \tau=j,k,l,$$

we obtain

$$\int_{-\varepsilon}^{\lambda_j/2}\int_{-\varepsilon}^{\mu_k/2}\int_{\nu_l/2}^{3\nu_l/2} K_n^T(\omega_{jkl}-\omega)\{f(\omega)-f(\omega_{jkl})\}\,d\omega \leqslant \nu_l^\alpha \int_{-\varepsilon}^{\lambda_j/2}\int_{-\varepsilon}^{\mu_k/2}\int_{\nu_l/2}^{3\nu_l/2} K_n^T(\omega_{jkl}-\omega)\,d\omega$$

$$= O\left(\frac{\|\omega_{jkl}\|^\alpha}{j^5 k^5}\right) = O\left(\frac{\|\omega_{jkl}\|^\alpha}{N_1^5}\right).$$

The same argument can be applied to all but one terms in (4.2), giving bounds of order $O(j^{-5}\|\omega_{jkl}\|^\alpha)$ or smaller. A different proof is needed only for $\int_{\lambda_j/2}^{3\lambda_j/2}\int_{\mu_k/2}^{3\mu_k/2}\int_{\nu_l/2}^{3\nu_l/2}$, where the argument is more delicate. Define $f_\xi = \partial f/\partial\xi$; following Velasco [24] and by Assumption A, for $|\lambda| \leqslant \lambda_j/2$, $|\mu| \leqslant \mu_k/2$, $|\nu| \leqslant \nu_l/2$, for some $0 \leqslant \theta \leqslant 1$

$$f(\omega_{jkl}-\omega)-f(\omega_{jkl}) = -\sum_{\xi=\lambda,\mu,\nu} f_\xi(\omega_{jkl}-\theta\omega)\,\xi$$

$$= -\sum_{\xi=\lambda,\mu,\nu} f_\xi(\omega_{jkl})\,\xi + \sum_{\xi=\lambda,\mu,\nu}\{f_\xi(\omega_{jkl})-f_\xi(\omega_{jkl}-\theta\omega)\}\,\xi.$$

Now

$$|f_\lambda(\omega_{jkl})| = O(\|\omega_{jkl}\|^{\alpha-2}\lambda_j) = O(\|\omega_{jkl}\|^{\alpha-1}),$$

the same bound also holding for $\xi = \mu, \nu$. On the other hand, note that

$$f_\lambda(\omega) = L_\lambda(\omega)\,\|\omega\|^\alpha + \alpha L(\|\omega\|)\,\|\omega\|^{\alpha-2}\lambda,$$

and hence

$$(4.3)\qquad |f_\lambda(\omega_{jkl}-\theta\omega)-f_\lambda(\omega_{jkl})|$$

$$(4.4)\qquad = \left|L_\lambda(\omega_{jkl}-\theta\omega)\,\|\omega_{jkl}-\theta\omega\|^\alpha - L_\lambda(\omega_{jkl})\,\|\omega_{jkl}\|^\alpha\right|$$

$$(4.5)\qquad + \left|L(\omega_{jkl}-\theta\omega)\,\|\omega_{jkl}-\theta\omega\|^{\alpha-2}(\lambda_j-\theta\lambda) - L(\omega_{jkl})\,\|\omega_{jkl}\|^{\alpha-2}\lambda_j\right|.$$

Now for (4.4) we obtain the bound

$$\begin{aligned}
(4.4) &\leqslant |L_\lambda(\omega_{jkl}-\theta\omega)-L_\lambda(\omega_{jkl})|\,\|\omega_{jkl}-\theta\omega\|^\alpha \\
&\quad + \left|\|\omega_{jkl}-\theta\omega\|^\alpha - \|\omega_{jkl}\|^\alpha\right| L_\lambda(\omega_{jkl}) \\
&\leqslant C\left|L'(\|\omega_{jkl}-\theta\omega\|)\frac{\lambda_j-\theta\lambda}{\|\omega_{jkl}-\theta\omega\|} - L'(\|\omega_{jkl}\|)\frac{\lambda_j}{\|\omega_{jkl}\|}\right| \|\omega_{jkl}-\theta\omega\|^\alpha \\
&\quad + O(\|\omega_{jkl}\|^{\alpha-3}\lambda_j\,\|\omega\|) \\
&\leqslant C\left|\frac{L'(\|\omega_{jkl}-\theta\omega\|)}{\|\omega_{jkl}-\theta\omega\|} - \frac{L'(\|\omega_{jkl}\|)}{\|\omega_{jkl}\|}\right| \|\omega_{jkl}\|^\alpha\lambda_j \\
&\quad + C\theta\lambda\frac{L'(\|\omega_{jkl}\|)}{\|\omega_{jkl}\|}\|\omega_{jkl}\|^\alpha + O(\|\omega_{jkl}\|^{\alpha-3}\lambda_j\,\|\omega\|) \\
&= O(\|\omega_{jkl}\|^{\alpha-2}\,\|\omega\|),
\end{aligned}$$

where $L' = dL/d\|\omega\|$, and recalling that, by Assumption A, $L'(\|\omega_{jkl}\|) = O(\|\omega_{jkl}\|^{\beta-1})$,

$$\left|\frac{L'(\|\omega_{jkl}-\theta\omega\|)}{\|\omega_{jkl}-\theta\omega\|}-\frac{L'(\|\omega_{jkl}\|)}{\|\omega_{jkl}\|}\right|\|\omega_{jkl}\|^{\alpha}\lambda_j$$

$$\leqslant |L'(\|\omega_{jkl}-\theta\omega\|)-L'(\|\omega_{jkl}\|)|\,\|\omega_{jkl}\|^{\alpha-1}\lambda_j+|L'(\|\omega_{jkl}\|)|\left|\frac{1}{\|\omega_{jkl}-\theta\omega\|}-\frac{1}{\|\omega_{jkl}\|}\right|\|\omega_{jkl}\|^{\alpha}\lambda_j$$

$$\leqslant C\left\{\big|\,\|\omega_{jkl}-\theta\omega\|-\|\omega_{jkl}\|\,\big|^{\beta-1}\|\omega_{jkl}\|^{\alpha-1}\lambda_j+\|\omega_{jkl}\|^{\beta-1}\frac{\|\omega\|}{\|\omega_{jkl}\|^{2}}\|\omega_{jkl}\|^{\alpha}\lambda_j\right\}$$

$$= O(\|\omega\|^{\beta-1}\|\omega_{jkl}\|^{\alpha}+\|\omega_{jkl}\|^{\beta-2}\|\omega\|\,\|\omega_{jkl}\|^{\alpha})$$

$$= O(\|\omega\|^{\beta-1}\|\omega_{jkl}\|^{\alpha}+\|\omega_{jkl}\|^{\alpha-2}\|\omega\|),$$

by the Mean Value Theorem, the triangle inequality and $\lambda_j = O(\|\omega_{jkl}\|)$. Also,

$$\begin{aligned}(4.5) &\leqslant \big|\,\|\omega_{jkl}-\theta\omega\|^{\alpha-2}(\lambda_j-\theta\lambda)-\|\omega_{jkl}\|^{\alpha-2}\lambda_j\big|\,L(\omega_{jkl})\\ &\quad+|L(\|\omega_{jkl}-\theta\omega\|)-L(\|\omega_{jkl}\|)|\,\|\omega_{jkl}\|^{\alpha-2}\lambda_j\\ &\leqslant C\{|\theta\lambda|\,\|\omega_{jkl}\|^{\alpha-2}L(\omega_{jkl})+\big|\,\|\omega_{jkl}-\theta\omega\|^{\alpha-2}-\|\omega_{jkl}\|^{\alpha-2}\big|\,\lambda_j L(\omega_{jkl})\}\\ &\quad+O(\|\omega_{jkl}\|^{-1}\|\omega\|\,\|\omega_{jkl}\|^{\alpha-2}\lambda_j)\\ &\leqslant C\{\|\omega\|\,\|\omega_{jkl}\|^{\alpha-2}+\|\omega_{jkl}\|^{\alpha-3}\|\omega\|\lambda_j+\|\omega_{jkl}\|^{\alpha-3}\|\omega\|\lambda_j\}\\ &= O(\|\omega\|\,\|\omega_{jkl}\|^{\alpha-2}).\end{aligned}$$

An identical argument holds for $f_\mu(\cdot)$ and $f_\nu(\cdot)$, and hence

$$(4.3) = O(\max\{\|\omega_{jkl}\|^{\alpha-2}\|\omega\|,\ \|\omega\|^{\beta-1}\|\omega_{jkl}\|^{\alpha}\})$$

(note that $|\lambda_j-\lambda| \leqslant \lambda_j/2$, and likewise for μ, ν). Hence

$$\int_{\lambda_j/2}^{3\lambda_j/2}\int_{\mu_k/2}^{3\mu_k/2}\int_{\nu_l/2}^{3\nu_l/2} K_n^T(\omega_{jkl}-\omega)\{f(\omega)-f(\omega_{jkl})\}\,d\omega$$

$$= \int_{-\lambda_j/2}^{\lambda_j/2}\int_{-\mu_k/2}^{\mu_k/2}\int_{-\nu_l/2}^{\nu_l/2} K_n^T(\omega)\{f(\omega_{jkl}-\omega)-f(\omega_{jkl})\}\,d\omega$$

$$= -\int_{-\lambda_j/2}^{\lambda_j/2}\int_{-\mu_k/2}^{\mu_k/2}\int_{-\nu_l/2}^{\nu_l/2} K_n^T(\omega)\{f_\lambda(\|\omega_{jkl}\|)\lambda+f_\mu(\|\omega_{jkl}\|)\mu+f_\nu(\|\omega_{jkl}\|)\nu\}\,d\omega$$

$$+\int_{-\lambda_j/2}^{\lambda_j/2}\int_{-\mu_k/2}^{\mu_k/2}\int_{-\nu_l/2}^{\nu_l/2} K_n^T(\omega)\{O(\|\omega_{jkl}\|^{\alpha-2}\|\omega\|^{2})\}\,d\omega$$

$$+\int_{-\lambda_j/2}^{\lambda_j/2}\int_{-\mu_k/2}^{\mu_k/2}\int_{-\nu_l/2}^{\nu_l/2} K_n^T(\omega)\{O(\|\omega_{jkl}\|^{\alpha}\|\omega\|^{\beta})\}\,d\omega$$

$$= O\Big(\|\omega_{jkl}\|^{\alpha-2}\int_{0}^{\lambda_j/2}\int_{0}^{\mu_k/2}\int_{0}^{\nu_l/2} K_n^T(\omega)\|\omega\|^{2}\,d\omega\Big)+O\Big(\|\omega_{jkl}\|^{\alpha}\int_{0}^{\lambda_j/2}\int_{0}^{\mu_k/2}\int_{0}^{\nu_l/2} K_n^T(\omega)\|\omega\|^{\beta}\,d\omega\Big),$$

because the first integral vanishes by symmetry of $K_n^T(\cdot)$. Standard manipulations yield

$$\int_0^{\lambda_j/2}\int_0^{\mu_k/2}\int_0^{\nu_l/2} K_n^T(\omega)\,\|\omega\|^\beta\,d\omega \leqslant C\,\|N\|^{-\beta};$$

hence

$$\int_{\lambda_j/2}^{3\lambda_j/2}\int_{\mu_k/2}^{3\mu_k/2}\int_{\nu_l/2}^{3\nu_l/2} K_n^T(\omega_{jkl}-\omega)\,\{f(\omega)-f(\omega_{jkl})\}\,d\omega$$

$$\leqslant C\,\{\|\omega_{jkl}\|^{\alpha-2}\,\|N\|^{-2}+\|\omega_{jkl}\|^{\alpha}\,\|N\|^{-\beta}\} \leqslant C\left\{\frac{\|\omega_{jkl}\|^{\alpha-}}{M_1^2}+\frac{\|\omega_{jkl}\|^{\alpha}}{\|N\|^{\beta}}\right\}.$$

Now let us focus on $\alpha \geqslant 0$. For $\|\omega\| > \varepsilon$, the argument is the same as before, recalling that $\|N\|^{-3} = O(\|\omega_{jkl}\|^\alpha/M_1^3)$ always, for $\alpha \leqslant 3$. Consider now the region

$$\omega_\varepsilon = \{\|\omega\| < \varepsilon\backslash[(-\lambda_j/2 \leqslant \lambda \leqslant 2\lambda_j)\cap(-\mu_k/2 \leqslant \mu \leqslant 2\mu_k)\cap(-\nu_l/2 \leqslant \nu \leqslant 2\nu_l)]\},$$

where, for $|\nu| \geqslant |\lambda|,\ |\mu|$,

$$\left\{\sup_{\omega_\varepsilon\cap(|\nu|\geqslant|\lambda|,|\mu|)} \frac{f(\omega)}{\|\omega\|^{3/2-\alpha/2}}\right\}\int_{\omega_\varepsilon\cap(|\nu|\geqslant|\lambda|,|\mu|)} \|\omega\|^{3/2-\alpha/2}\,K_n^T(\omega_{jkl}-\omega)\,d\omega$$

$$\leqslant C\nu_l^{\alpha/2-3/2}\int_{(\nu>3\nu_l/2)\cup(\nu<-\nu_l/2)} |\nu|^{3/2-\alpha/2}\,K_{n_3}^T(\nu_l-\nu)\,d\nu$$

$$\leqslant C\nu_l^{\alpha/2-3/2}\,\|N\|^{-5}\int_{(\nu>3\nu_l/2)\cup(\nu<-\nu_l/2)} |\nu|^{-9/2-\alpha/2}\,d\nu = O\left(\frac{\|\omega_{jkl}\|^\alpha}{l^5}\right),$$

and likewise we obtain the bounds

$$O\left(\frac{\|\omega_{jkl}\|^\alpha}{j^5}\right),\ O\left(\frac{\|\omega_{jkl}\|^\alpha}{k^5}\right) = O\left(\frac{\|\omega_{jkl}\|^\alpha}{N_1^5}\right)$$

for $|\lambda| \geqslant |\mu|,\ |\nu|$ and $|\mu| \geqslant |\lambda|,\ |\nu|$, respectively.

The remaining region is decomposed as

$$\int_{-\lambda_j/2}^{2\lambda_j}\int_{-\mu_k/2}^{2\mu_k}\int_{-\nu_l/2}^{2\nu_l} = \left\{\int_{-\lambda_j/2}^{\lambda_j/2}+\int_{\lambda_j/2}^{2\lambda_j}\right\}\left\{\int_{-\mu_k/2}^{\mu_k/2}+\int_{\mu_k/2}^{2\mu_k}\right\}\left\{\int_{-\nu_l/2}^{\nu_l/2}+\int_{\nu_l/2}^{2\nu_l}\right\},$$

where, for instance,

$$\int_{-\lambda_j/2}^{\lambda_j/2}\int_{-\mu_k/2}^{\mu_k/2}\int_{-\nu_l/2}^{\nu_l/2} K_n^T(\omega_{jkl}-\omega)\,\{f(\omega)-f(\omega_{jkl})\}\,d\omega$$

$$\leqslant \frac{C\,\|N\|^3}{j^5\,k^5\,l^5}\int_{-\lambda_j/2}^{\lambda_j/2}\int_{-\mu_k/2}^{\mu_k/2}\int_{-\nu_l/2}^{\nu_l/2}\{\|\omega\|^\alpha+f(\omega_{jkl})\}\,d\omega = O\left(\frac{\|\omega_{jkl}\|^\alpha}{j^4\,k^4\,l^4}\right),$$

as in (4.1). Similarly,

$$\int_{\lambda_j/2}^{2\lambda_j}\int_{-\mu_k/2}^{\mu_k/2}\int_{-\nu_l/2}^{\nu_l/2} K_n^T(\omega_{jkl}-\omega)\{f(\omega)-f(\omega_{jkl})\}\,d\omega$$

$$\leqslant \lambda_j^\alpha \int_{-\mu_k/2}^{\mu_k/2}\int_{-\nu_l/2}^{\nu_l/2} K_{n_2}^T(\mu_k-\mu)K_{n_3}^T(\nu_l-\nu)\,d\mu d\nu = O\left(\frac{\|\omega_{jkl}\|^\alpha}{k^5 l^5}\right),$$

because $\lambda_j^\alpha = O(\|\omega_{jkl}\|^\alpha)$ when α is positive. The proof for all remaining terms is similar, with the exception of

$$\int_{\lambda_j/2}^{2\lambda_j}\int_{-\mu_k/2}^{2\mu_k}\int_{-\nu_l/2}^{2\nu_l} K_n^T(\omega_{jkl}-\omega)\{f(\omega)-f(\omega_{jkl})\}\,d\omega = O\left(\frac{\|\omega_{jkl}\|^\alpha}{M_1^2}\right),$$

which follows in the same way as for a negative α. Thus (2.6) is established.

To prove (2.7), we note that, under assumptions A and A′, similarly to Robinson [20], Appendix B,

$$\left|\frac{df_{gh}(\|\omega\|)}{d\|\omega\|}\right| \leqslant \left|\frac{df_{gg}^{1/2}(\|\omega\|)}{d\|\omega\|}R_{gh}(\|\omega\|)f_{hh}^{1/2}(\|\omega\|)\right| + \left|\frac{df_{hh}^{1/2}(\|\omega\|)}{d\|\omega\|}R_{gh}(\|\omega\|)f_{gg}^{1/2}(\|\omega\|)\right|$$

$$+\left|f_{gg}^{1/2}(\|\omega\|)\frac{dR_{gh}(\|\omega\|)}{d\|\omega\|}f_{hh}^{1/2}(\|\omega\|)\right|$$

$$= O(\|\omega\|^{\alpha_{gh}-1}),$$

whence

(4.6) $$f_{gh,\lambda}(\|\omega\|) = \frac{df_{gh}(\|\omega\|)}{d\|\omega\|}\frac{\partial\|\omega\|}{\partial\lambda} = O(\|\omega\|^{\alpha_{gh}-2}\lambda)$$

because $\partial\|\omega\|/\partial\lambda = \lambda/\|\omega\|$, and likewise

(4.7) $$f_{gh,\mu}(\|\omega\|) = O(\|\omega\|^{\alpha_{gh}-2}\mu), \quad f_{gh,\nu}(\|\omega\|) = O(\|\omega\|^{\alpha_{gh}-2}\nu).$$

Since $f_{gh}(\omega) \sim C\|\omega\|^{\alpha_{gh}}$, the proof of (2.7) follows exactly from the same steps with the exception of the term $\int_{\lambda_j/2}^{3\lambda_j/2}\int_{\mu_k/2}^{3\mu_k/2}\int_{\nu_l/2}^{3\nu_l/2}$. For this term, by (4.6) and (4.7) we have

$$\left|\int_{-\lambda_j/2}^{\lambda_j/2}\int_{-\mu_k/2}^{\mu_k/2}\int_{-\nu_l/2}^{\nu_l/2} K_n^T(\omega_{jkl}-\omega)\left(\sum_\xi |f_\xi(\omega_{jkl})|+|f_\xi(\omega_{jkl}-\theta\omega)|\right)\xi d\omega\right|$$

$$= \int_{-\lambda_j/2}^{\lambda_j/2}\int_{-\mu_k/2}^{\mu_k/2}\int_{-\nu_l/2}^{\nu_l/2} K_n^T(\omega_{jkl}-\omega)\|\omega\|\,\|\omega_{jkl}\|^{\alpha-1}\,d\omega$$

$$= O(\|\omega_{jkl}\|^{\alpha-1}\|N\|^{-1}) = O\left(\frac{\|\omega_{jkl}\|^\alpha}{M_1}\right). \blacksquare$$

Proof of Theorem 2. The proof is similar (indeed simpler) to the proof of Theorem 1, and hence omitted for brevity's sake. ∎

For the results to follow, let

$$D_n^T(\xi) = \sum_{u=1}^{n} h_u e^{iu\xi}, \qquad \tilde{D}_n^T(\xi, \zeta) = D_n^T(\zeta+\xi) D_n^T(\zeta-\xi).$$

It is well known from Hannan [7] (see also Percival and Walden [19]) that

$$D_n^T(\xi) \leqslant C \min\{n, n^{-2}|\xi|^{-3}\}. \tag{4.8}$$

The following lemma will be exploited in the proof of Theorem 3 to follow.

Lemma 1. (i) *Let* $\xi_\tau = 2\pi\tau/n$; *for any* τ_2, τ_1, *as* $n \to \infty$,

$$\int_{-\pi}^{\pi} |D_n^T(\xi_{\tau_1} - \xi)| \, |D_n^T(\xi - \xi_{\tau_2})| \, d\xi = O\left(\min\left(n, \frac{n}{|\tau_2 - \tau_1|^3}\right)\right).$$

(ii) *For any* τ_2, τ_1, *as* $n \to \infty$,

$$\int_{-\pi}^{\pi} |D_n^T(\xi_{\tau_1} - \xi)| \, |D_n^T(\xi - \xi_{\tau_2})| \, |\xi - \xi_{\tau_2}| \, d\xi = O(\min(1, \tau_2 - \tau_1)^{-2}).$$

Proof. We have

$$\int_{-\pi}^{\pi} |D_n^T(\xi_{\tau_1} - \xi)| \, |D_n^T(\xi - \xi_{\tau_2})| \, d\xi = \int_{-\pi}^{(\xi_{\tau_1}+\xi_{\tau_2})/2} + \int_{(\xi_{\tau_1}+\xi_{\tau_2})/2}^{\pi},$$

where

$$\int_{-\pi}^{(\xi_{\tau_1}+\xi_{\tau_2})/2} |D_n^T(\xi_{\tau_1} - \xi)| \, |D_n^T(\xi - \xi_{\tau_2})| \, d\xi \leqslant \frac{n_2}{|\tau_2 - \tau_1|^3} \int_{-\pi}^{(\xi_{\tau_1}+\xi_{\tau_2})/2} |D_n^T(\xi_{\tau_1} - \xi)| \, d\xi$$
$$= O\left(\frac{n_2}{|\tau_2 - \tau_1|^3}\right),$$

and

$$\int_{(\xi_{\tau_1}+\xi_{\tau_2})/2}^{\pi} |D_n^T(\xi_{\tau_1} - \xi)| \, |D_n^T(\xi - \xi_{\tau_2})| \, d\xi \leqslant \frac{n_2}{|\tau_2 - \tau_1|^3} \int_{(\xi_{\tau_1}+\xi_{\tau_2})/2}^{\pi} |D_n^T(\xi - \xi_{\tau_2})| \, d\xi$$
$$= O\left(\frac{n_2}{|\tau_2 - \tau_1|^3}\right).$$

For (ii), we have

$$\int_{-\pi}^{\pi} |D_n^T(\xi_{\tau_1} - \xi)| \, |D_n^T(\xi - \xi_{\tau_2})| \, |\xi - \xi_{\tau_2}| \, d\xi \leqslant Cn \int_{-\pi}^{\pi} |D_n^T(\xi)| \, |\xi| \, d\xi$$
$$\leqslant C\left\{n^2 \int_{-1/n}^{1/n} |\xi| \, d\xi + n^{-1} \int_{1/n}^{\pi} |\xi|^{-2} \, d\xi\right\} = O(1),$$

and also, considering without loss of generality $\tau_1 > \tau_2$,

$$\int_{-\pi}^{\pi} |D_n^T(\xi_{\tau_1}-\xi)|\,|D_n^T(\xi-\xi_{\tau_2})|\,|\xi-\xi_{\tau_2}|\,d\xi$$

$$\leqslant \Big\{ \int_{-\pi}^{(\xi_{\tau_1}+\xi_{\tau_2})/2} + \int_{(\xi_{\tau_1}+\xi_{\tau_2})/2}^{\pi} \Big\} |D_n^T(\xi_{\tau_1}-\xi)|\,|D_n^T(\xi-\xi_{\tau_2})|\,|\xi-\xi_{\tau_2}|\,d\xi$$

$$\leqslant C\left[\frac{n}{|\tau_2-\tau_1|^3} \int_{-\pi}^{(\xi_{\tau_1}+\xi_{\tau_2})/2} |D_n^T(\xi-\xi_{\tau_2})|\,|\xi-\xi_{\tau_2}|\,d\xi \right.$$

$$\left. +\frac{n}{|\tau_2-\tau_1|^3} \int_{-\pi}^{(\xi_{\tau_1}-\xi_{\tau_2})/2} |D_n^T(u)|\,|(\xi_{\tau_1}-\xi_{\tau_2})-u|\,d\xi \right]$$

$$= O\left(\frac{1}{(\tau_2-\tau_1)^2}\right),$$

after the change of variable $u = \xi_{\tau_1}-\xi$, and because

$$\int_{-\pi}^{\pi} |D_n^T(\xi)|\,|\xi|\,d\xi \leqslant 2\left\{\frac{1}{n}+\int_{1/n}^{\pi}\frac{1}{n^2u^2}\,d\xi\right\} = O\left(\frac{1}{n}\right),$$

and

$$\int_{-\pi}^{(\xi_{\tau_1}-\xi_{\tau_2})/2} |D_n^T(u)|\,|(\xi_{\tau_1}-\xi_{\tau_2})-u|\,d\xi \leqslant \int_{-\pi}^{\pi} |D_n^T(u)|\,\{|u|+|(\xi_{\tau_1}-\xi_{\tau_2})|\}\,du$$

$$\leqslant O\left(\frac{1}{n}\right)+2\,|(\xi_{\tau_1}-\xi_{\tau_2})|\int_0^{\pi} |D_n^T(u)|\,du$$

$$\leqslant O\left(\frac{1}{n}\right)+C\left|\frac{\tau_1-\tau_2}{n}\right|\left(1+\sum_{k=1}^{n}\int_{k/n}^{k+1/n}\frac{1}{n^2u^3}\,du\right) = O\left(\left|\frac{\tau_1-\tau_2}{n}\right|\right). \blacksquare$$

Proof of Theorem 3. We have

$$Ew_g^T(\omega_{j_1k_1l_1})\,\bar{w}_h^T(\omega_{j_2k_2l_2}) = H^{-2}\int_T E_{j_1k_1l_1j_2k_2l_2}(\omega)\,f_{gh}(\omega)\,d\omega,$$

$$E_{j_1k_1l_1j_2k_2l_2}(\lambda,\mu,\nu) = \hat{D}_{n_1}^T(\lambda,\lambda_{j_1},\lambda_{j_2})\,\hat{D}_{n_2}^T(\mu,\mu_{k_1},\mu_{k_2})\,\hat{D}_{n_3}^T(\nu,\nu_{l_1},\nu_{l_2}),$$

$$\hat{D}_{n_1}^T(\lambda,\lambda_{j_1},\lambda_{j_2}) = D_{n_1}^T(\lambda_{j_1}-\lambda)\,D_{n_1}^T(\lambda-\lambda_{j_2}),$$

where we recall that

$$\int_{-\pi}^{\pi} D_n^T(\lambda_{j_1}-\lambda)\,D_n^T(\lambda-\lambda_{j_2})\,d\lambda = 0 \quad \text{if } |j_1-j_2|\geqslant 3 \text{ and } n-|j_1-j_2|\geqslant 3.$$

Recall that only Fourier frequencies which are not closer than $3\pi/n_1 \pmod{2\pi}$ are used in the sequel. Assume wihout loss of generality that $\|\omega_{j_2k_2l_2}\| \geqslant \|\omega_{j_1k_1l_1}\|$,

$\lambda_{j_1} \leqslant \mu_{k_1}, \nu_{l_1}, \nu_{l_2} \geqslant \lambda_{j_2}, \mu_{k_2}$; we consider

$$\int_T E_{j_1k_1l_1j_2k_2l_2}(\omega)\{f_{gh}(\omega)-f_{gh}(\omega_{j_2k_2l_2})\}\,d\omega.$$

Examine first the case $\alpha \leqslant 0$. Note that, by using (4.8) for $\tau_2 \neq \tau_1$ and under the assumption $|\xi_{\tau_1}|$, $|\xi_{\tau_2}| < \varepsilon$,

$$\sup_{|\xi|>\varepsilon} H^{-2/3} D_n^T(\xi_{\tau_1}-\xi) D_n^T(\xi-\xi_{\tau_2}) = O(n^{-5}\varepsilon^{-6}),$$

$$\sup_{-\pi<\xi<\pi} H^{-2/3} D_n^T(\xi_{\tau_1}-\xi) D_n^T(\xi-\xi_{\tau_2}) = O\left(\frac{n}{|\tau_2-\tau_1|^3}\right),$$

and hence

$$\begin{aligned}
\int_{\|\omega\|\geqslant\varepsilon} &\leqslant \int_{T\cap(|\lambda|>\varepsilon/3)} + \int_{T\cap(|\mu|>\varepsilon/3)} + \int_{T\cap(|\nu|>\varepsilon/3)} \\
&= O\left(\frac{1}{\|N\|^3}\frac{1}{(j_2-j_1)^2}\frac{1}{(k_2-k_1)^2}\right)+O\left(\frac{1}{\|N\|^3}\frac{1}{(j_2-j_1)^2}\frac{1}{(l_2-l_1)^2}\right) \\
&\quad +O\left(\frac{1}{\|N\|^3}\frac{1}{(l_2-l_1)^2}\frac{1}{(k_2-k_1)^2}\right) \\
&= O\left(\|\omega_{j_1k_1l_1}\|^{\alpha_{gh}/2}\|\omega_{j_2k_2l_2}\|^{\alpha_{gh}/2}\prod_{a=j,k,l}\{\max(a_2-a_1,1)\}^{-2}\frac{1}{l_2}\right) \\
&= O\left(\|\omega_{j_1k_1l_1}\|^{\alpha_{gh}/2}\|\omega_{j_2k_2l_2}\|^{\alpha_{gh}/2}\prod_{a=j,k,l}\{\max(a_2-a_1,1)\}^{-2}\frac{1}{l_2}\frac{l_2^{2-\alpha_{gh}/2}}{j_1^{2-\alpha_{gh}/2}}\right) \\
&= O\left(\|\omega_{j_1k_1l_1}\|^{\alpha_{gh}/2}\|\omega_{j_2k_2l_2}\|^{\alpha_{gh}/2}\prod_{a=j,k,l}\{\max(a_2-a_1,1)\}^{-2}\frac{M_2^{1-\alpha_{gh}/2}}{N_2^{2-\alpha_{gh}/2}}\right),
\end{aligned}$$

because $l_2 \geqslant j_1$. For $2\nu_{l_2} \leqslant \|\omega\| \leqslant \varepsilon$,

(4.9) $$\int_{\|\omega\|\geqslant 2\nu_{l_2}} \leqslant \int_{T\cap(|\lambda|>2\nu_{l_2}/\sqrt{3})} + \int_{T\cap(|\mu|>2\nu_{l_2}/\sqrt{3})} + \int_{T\cap(|\nu|>2\nu_{l_2}/\sqrt{3})},$$

where the first term, for instance, is bounded by

$$\begin{aligned}
&C\|N\|^{-3}\|\omega_{j_2k_2l_2}\|^{\alpha_{gh}} \int_{T\cap(|\lambda|>2\nu_{l_2}/\sqrt{3})} |E_{j_1k_1l_1j_2k_2l_2}(\omega)|\,d\omega \\
&\leqslant C\|N\|^{-1}\|\omega_{j_2k_2l_2}\|^{\alpha_{gh}} \int_{|\lambda|>2\nu_{l_2}/\sqrt{3}} |D_n^T(\lambda_{j_1}-\lambda)|\,|D_n^T(\lambda-\lambda_{j_2})|\,d\lambda \\
&\leqslant C\|N\|^{-1}\|\omega_{j_2k_2l_2}\|^{\alpha_{gh}} \int_{|\lambda|>2\nu_{l_2}/\sqrt{3}} |\lambda_{j_1}-\lambda|^{-3}|\lambda-\lambda_{j_2}|^{-3}\,d\lambda
\end{aligned}$$

$$\leqslant C\|N\|^{-5}\|\omega_{j_2k_2l_2}\|^{\alpha_{gh}} \int_{|\lambda|>2\nu_{l_2}/\sqrt{3}} \{|\lambda_{j_1}-\lambda|^{-6}+|\lambda-\lambda_{j_2}|^{-6}\}\,d\lambda$$

$$\leqslant C\|N\|^{-5}\|\omega_{j_2k_2l_2}\|^{\alpha_{gh}}\left\{\left(\frac{2\nu_{l_2}}{\sqrt{3}}-\lambda_{j_1}\right)^{-5}+\left(\frac{2\nu_{l_2}}{\sqrt{3}}-\lambda_{j_2}\right)^{-5}\right\}=O\left(\frac{\|\omega_{j_2k_2l_2}\|^{\alpha_{gh}}}{l_2^5}\right),$$

the first step following by

$$\int_{-\pi}^{\pi} |\hat{D}_n^T(\xi,\xi_{\tau_1},\xi_{\tau_2})|\,d\xi \leqslant n \qquad \text{for any } \tau_1, \tau_2,$$

by the Cauchy–Schwarz inequality. The argument for the other two terms on the right-hand side of (4.9) is identical. Next, we note that Assumptions A and A′ imply (see (4.6) and (4.7))

$$f_{gh,a}(\|\omega\|)=O(\|\omega\|^{\alpha_{gh}-2}\,a) \qquad \text{for } a=\lambda,\mu,\nu.$$

Therefore, for $(\lambda_{j_1}+\nu_{l_2})/2 \leqslant \lambda \leqslant 2\nu_{l_2}$, $0\leqslant \mu, \nu \leqslant 2\nu_{l_2}$, we have

$$|f_{gh}(\omega)-f_{gh}(\omega_{j_2k_2l_2})|$$

$$\leqslant \Big[\sup_{(\|\omega\|\leqslant 2\nu_{l_2})\cap(\lambda\geqslant(\lambda_{j_1}+\nu_{l_2})/2)} \sup\{|f_{gh,\lambda}(\omega)|, |f_{gh,\mu}(\omega)|, |f_{gh,\nu}(\omega)|\}\Big]\,|\omega-\omega_{j_2k_2l_2}|$$

$$\leqslant C\|\omega_{j_2k_2l_2}\|^{\alpha_{gh}-1}\,|\omega-\omega_{j_2k_2l_2}|.$$

Recursive application of Lemma 1 gives

$$\frac{1}{\|N\|^3}\int_{\|\omega\|\leqslant 2\nu_{l_2},\lambda\geqslant(\lambda_{j_1}+\nu_{l_2})/2} E_{j_1k_1l_1j_2k_2l_2}(\omega)\{f_{gh}(\omega)-f_{gh}(\omega_{j_2k_2l_2})\}\,d\omega$$

$$\leqslant \frac{C}{(k_2-k_1)^3}\frac{\|\omega_{j_2k_2l_2}\|^{\alpha_{gh}-1}}{(l_2-l_1)^3}\|N\|^{-1}\int_{(\lambda_{j_1}+\nu_{l_2})/2}^{2\nu_{l_2}} |D_n^T(\lambda_{j_1}-\lambda)|\,|D_n^T(\lambda-\lambda_{j_2})|\,|\lambda-\lambda_{j_2}|\,d\lambda$$

$$+\frac{C}{(j_2-j_1)^3}\frac{\|\omega_{j_2k_2l_2}\|^{\alpha_{gh}-1}}{(l_2-l_1)^3}\|N\|^{-1}\int_{-2\nu_{l_2}}^{2\nu_{l_2}} |D_n^T(\mu_{k_1}-\mu)|\,|D_n^T(\mu-\mu_{k_2})|\,|\mu-\mu_{k_2}|\,d\mu$$

$$+\frac{C}{(j_2-j_1)^3}\frac{\|\omega_{j_2k_2l_2}\|^{\alpha_{gh}-1}}{(k_2-k_1)^3}\|N\|^{-1}\int_{-2\nu_{l_2}}^{2\nu_{l_2}} |D_n^T(\nu_{l_1}-\nu)|\,|D_n^T(\nu-\nu_{l_2})|\,|\nu-\nu_{l_2}|\,d\nu$$

$$\leqslant C\|\omega_{j_2k_2l_2}\|^{\alpha_{gh}}\frac{1}{l_2}\prod_{a=j,k,l}\{\max(a_2-a_1,1)\}^{-2}$$

$$\leqslant C\frac{\|\omega_{j_1k_1l_1}\|^{\alpha_{gh}/2}}{l_2^{1-\alpha_{gh}/2}}\frac{\|\omega_{j_2k_2l_2}\|^{\alpha_{gh}/2}}{j_1^{\alpha_{gh}/2}}\prod_{a=j,k,l}\{\max(a_2-a_1,1)\}^{-2}$$

$$\leqslant O\left(\frac{\|\omega_{j_1k_1l_1}\|^{\alpha_{gh}/2}}{M_2^{\alpha_{gh}/2-1}}\frac{\|\omega_{j_2k_2l_2}\|^{\alpha_{gh}/2}}{N_2^{2-\alpha_{gh}/2}}\prod_{a=j,k,l}\{\max(a_2-a_1,1)\}^{-2}\right),$$

because

$$\frac{l_2^{1-\alpha_{gh}/2}}{j_1^{2-\alpha_{gh}/2}} = O\left(\frac{l_2^{\alpha_{gh}/2-1}}{j_1^{\alpha_{gh}/2}}\right) = O\left(\frac{M_2^{\alpha_{gh}/2-1}}{N_2^{\alpha_{gh}/2}}\right).$$

Now recall we assumed without loss of generality that $j_2 > j_1$; note that

$$\sup_{\lambda_{j_1}/2 \leqslant \lambda \leqslant (\lambda_{j_1}+v_{l_2})/2, \|\omega\| \leqslant 2v_{l_2}} \sup_{a=\lambda,\mu,v} |f_{gh,a}(\omega)|$$

$$\leqslant \lambda_j^{\alpha_{gh}-2} \sup_{\lambda_{j_1}/2 \leqslant \lambda \leqslant (\lambda_{j_1}+v_{l_2})/2, \|\omega\| \leqslant 2v_{l_2}} \{\lambda+\mu+v\} = O(\lambda_j^{\alpha_{gh}-2} v_{l_2}).$$

For $\lambda_{j_1}/2 \leqslant \lambda \leqslant (\lambda_{j_1}+v_{l_2})/2$, $0 \leqslant \mu$, $v \leqslant 2v_{l_2}$, we add and subtract $f_{gh}(\omega_{j_1k_1l_1})$ to obtain, by the same argument as before,

$$\|N\|^{-3} \int_{\lambda_{j_1}/2 \leqslant \lambda \leqslant (\lambda_{j_1}+v_{l_2})/2, \|\omega\| \leqslant 2v_{l_2}} E_{j_1k_1l_1j_2k_2l_2}(\omega)\{f_{gh}(\omega)-f_{gh}(\omega_{j_1k_1l_1})\}\, d\omega$$

$$\leqslant \frac{C}{(k_2-k_1)^3} \frac{\lambda_{j1}^{\alpha_{gh}-2} v_{l_2}}{(l_2-l_1)^3} \|N\|^{-1} \int_{-\pi}^{\pi} |D^T(\lambda-\lambda_{j_2})|\, |D^T(\lambda_{j_1}-\lambda)|\, |\lambda-\lambda_{j_1}|\, d\lambda$$

$$+\frac{C}{(j_2-j_1)^3} \frac{\lambda_{j1}^{\alpha_{gh}-2} v_{l_2}}{(l_2-l_1)^3} \|N\|^{-1} \int_{-2v_{l_2}}^{2v_{l_2}} |D^T(\mu-\mu_{k_2})|\, |D^T(\mu_{k_1}-\mu)|\, |\mu-\mu_{k_1}|\, d\mu$$

$$+\frac{C}{(j_2-j_1)^3} \frac{\lambda_{j1}^{\alpha_{gh}-2} v_{l_2}}{(k_2-k_1)^3} \|N\|^{-1} \int_{-2v_{l_2}}^{2v_{l_2}} |D^T(v_{l_2}-v)|\, |D^T(v_{l_1}-v)|\, |v-v_{l_1}|\, dv$$

$$\leqslant \frac{C}{(k_2-k_1)^3} \frac{\|\omega_{j_1k_1l_1}\|^{\alpha_{gh}}}{(l_2-l_1)^3} \frac{1}{j_1} \frac{1}{(j_2-j_1)^2}$$

$$+\frac{C}{(j_2-j_1)^3} \frac{\|\omega_{j_1k_1l_1}\|^{\alpha_{gh}}}{(l_2-l_1)^3} \frac{1}{j_1} \frac{1}{(k_2-k_1)^2} + \frac{C}{(j_2-j_1)^3} \frac{\|\omega_{j_1k_1l_1}\|^{\alpha_{gh}}}{(k_2-k_1)^3} \frac{1}{j_1} \frac{1}{(l_2-l_1)^2}$$

$$= O\left(\frac{\|\omega_{j_1k_1l_1}\|^{\alpha_{gh}/2}}{(j_2-j_1)^2} \frac{\|\omega_{j_2k_2l_2}\|^{\alpha_{gh}/2}}{(l_2-l_1)^2} \frac{l_2^{1-\alpha/2}}{(k_2-k_1)^2} \frac{1}{j_1^{2-\alpha/2}}\right).$$

Also, for $j_2 = j_1$, by Lemma 1 we have the bounds

$$O\left(\frac{\|\omega_{j_1k_1l_1}\|^{\alpha_{gh}/2}}{(l_2-l_1)^2} \frac{\|\omega_{j_2k_2l_2}\|^{\alpha_{gh}/2}}{(k_2-k_1)^2} \frac{1}{l_2}\right),\ O\left(\frac{\|\omega_{j_1k_1l_1}\|^{\alpha_{gh}/2}}{j_1^{2-\alpha/2}} \frac{\|\omega_{j_2k_2l_2}\|^{\alpha_{gh}/2}}{(l_2-l_1)^2} \frac{l_2^{1-\alpha/2}}{(k_2-k_1)^2}\right),$$

and likewise for $k_2 = k_1$, $l_2 = l_1$.

We also have a term of the form

$$\|N\|^{-3} \int_{\lambda_{j_1}/2 \leqslant \lambda \leqslant (\lambda_{j_1}+v_{l_2})/2, \|\omega\| \leqslant 2v_{l_2}} E_{j_1k_1l_1j_2k_2l_2}(\omega)\{f_{gh}(\omega_{j_1k_1l_1})-f_{gh}(\omega_{j_2k_2l_2})\}\, d\omega,$$

which is bounded by

(4.10) $$C\|N\|^{-3}\|\omega_{j_1k_1l_1}\|^{\alpha_{gh}-1}(|\omega_{j_1k_1l_1}-\omega_{j_2k_2l_2}|)$$

$$\times \int_{\lambda_{j_1}/2\leqslant\lambda\leqslant(\lambda_{j_1}+\nu_{l_2})/2,\,\|\omega\|\leqslant 2\nu_{l_2}} |E_{j_1k_1l_1j_2k_2l_2}(\omega)|\,d\omega$$

$$\leqslant \frac{C}{l_2}\|\omega_{j_1k_1l_1}\|^{\alpha_{gh}}(|j_2-j_1|+|k_2-k_1|+|l_2-l_1|)\prod_{a=j,k,l}\{\max(a_2-a_1,\,1)\}^{-3}.$$

Hence the right-hand side of (4.10) is

$$O\left(\|\omega_{j_1k_1l_1}\|^{\alpha_{gh}/2}\|\omega_{j_2k_2l_2}\|^{\alpha_{gh}/2}\frac{M_2^{1-\alpha_{gh}/2}}{N_2^{2-\alpha/2}}\prod_{a=j,k,l}\{\max(a_2-a_1,\,1)^{-2}\}\right).$$

Finally, for $-2\nu_{l_2}\leqslant\lambda\leqslant\lambda_{j_1}/2$, we consider first the case $\|\omega\|\geqslant\lambda_j/2$, which entails

$$\int_{\lambda_j/2\leqslant\|\omega\|\leqslant 2\nu_{l_2},\lambda\leqslant\lambda_{j_1}/2} |E_{j_1k_1l_1j_2k_2l_2}(\omega)|\,|f_{gh}(\omega)-f_{gh}(\omega_{j_2k_2l_2})|\,d\omega$$

$$\leqslant \max_{\|\omega\|\geqslant\lambda_j/2}|f_{gh}(\omega)-f_{gh}(\omega_{j_2k_2l_2})| \int_{\lambda_j/2\leqslant\|\omega\|\leqslant 2\nu_{l_2},\lambda\leqslant\lambda_{j_1}/2} |E_{j_1k_1l_1j_2k_2l_2}(\omega)|\,d\omega$$

$$\leqslant \frac{C}{j_1^6}\frac{1}{(k_2-k_1)^3}\frac{1}{(l_2-l_1)^3}\lambda_{j_1}^{\alpha}$$

$$= O\left(\frac{\|\omega_{j_1k_1l_1}\|^{\alpha_{gh}/2}}{j_1^5}\frac{\|\omega_{j_2k_2l_2}\|^{\alpha_{gh}/2}}{(k_2-k_1)^3}\frac{1}{(l_2-l_1)^3}\frac{M_2^{-\alpha_{gh}/2}}{N_2^{1-\alpha_{gh}/2}}\right).$$

Also, for $\|\omega\|\leqslant\lambda_{j_1}/2$

$$\left|\int_{\|\omega\|\leqslant\lambda_j/2}\right| \leqslant \max_{\|\omega\|\leqslant\lambda_j/2}|E_{j_1k_1l_1j_2k_2l_2}(\omega)| \int_{\|\omega\|\leqslant\lambda_j/2}|f_{gh}(\omega)-f_{gh}(\omega_{j_2k_2l_2})|\,d\omega$$

$$\leqslant C\frac{\|N\|^{-15}}{\lambda_{j_1}^3\lambda_{j_2}^3\mu_{k_1}^3\mu_{k_2}^3\nu_{l_1}^3\nu_{l_2}^3}\lambda_{j_1}^{\alpha_{gh}} = O\left(\frac{\|\omega_{j_1k_1l_1}\|^{\alpha_{gh}/2}\|\omega_{j_2k_2l_2}\|^{\alpha_{gh}/2}}{j_2^3k_1^3k_2^3l_1^3l_2^3}\right).$$

Now consider $\alpha_{gh}>0$; again, we discuss only regions where a different treatment than for negative α is required. For $2\nu_{l_2}\leqslant\|\omega\|\leqslant\varepsilon$,

$$\int_{\|\omega\|\geqslant 2\nu_{l_2}} \leqslant \int_{\|\omega\|\geqslant 2\nu_{l_2}} [\{1(|\nu|\geqslant|\lambda|,\,|\mu|)\}+\{1(|\mu|\geqslant|\lambda|,\,|\nu|)\}+\{1(|\lambda|\geqslant|\mu|,\,|\nu|)\}],$$

where the last term, for instance, is bounded by

$$C\|N\|^{-3}\left\{\max_{\lambda\geqslant 2\nu_{l_2}}\frac{f(\|\omega\|)}{\|\omega\|^{(3+\alpha_{gh})/2}}\right\}\int_{T\cap(|\lambda|>2\nu_{l_2}/\sqrt{3})}\|\omega\|^{(3+\alpha_{gh})/2}|E_{j_1k_1l_1j_2k_2l_2}(\omega)|\,d\omega$$

$$\leqslant C\|N\|^{-1}\|\omega_{j_2k_2l_2}\|^{(\alpha_{gh}-3)/2}\int_{|\lambda|>2\nu_{l_2}/\sqrt{3}}|\lambda|^{(3+\alpha_{gh})/2}|D_n^T(\lambda_{j_1}-\lambda)D_n^T(\lambda-\lambda_{j_2})|\,d\lambda$$

$$\leqslant C\|N\|^{-5}\|\omega_{j_2k_2l_2}\|^{(\alpha_{gh}-3)/2}\int_{|\lambda|>2\nu_{l_2}/\sqrt{3}}|\lambda|^{(-9+\alpha_{gh})/2}\,d\lambda\leqslant C\frac{\|\omega_{j_2k_2l_2}\|^{\alpha_{gh}}}{l_2^5},$$

because $|\lambda_{j_1}-\lambda|^{-3}, |\lambda-\lambda_{j_2}|^{-3}\leqslant|\lambda|^{-3}\,|\lambda|>2\nu_{l_2}/\sqrt{3}$; analogous bounds hold for the other two terms. Next, for $(\lambda_{j_1}+\nu_{l_2})/2\leqslant\lambda\leqslant 2\nu_{l_2}$, $0\leqslant\mu,\nu\leqslant 2\nu_{l_2}$, we obtain as before

$$\int_{\|\omega\|\leqslant 2\nu_{l_2},\lambda\geqslant(\lambda_{j_1}+\nu_{l_2})/2}E_{j_1k_1l_1j_2k_2l_2}(\omega)\{f_{gh}(\omega)-f_{gh}(\omega_{j_2k_2l_2})\}\,d\omega$$

$$=O\left(\|\omega_{j_2k_2l_2}\|^{\alpha_{gh}}\frac{1}{l_2}\prod_{a=j,k,l}\{\max(a_2-a_1,1)^2\}^{-1}\right)$$

$$=O\left(\|\omega_{j_1k_1l_1}\|^{\alpha_{gh}/2}\|\omega_{j_2k_2l_2}\|^{\alpha_{gh}/2}\frac{M_2^{\alpha_{gh}/2-1}}{N_2^{\alpha_{gh}/2}}\prod_{a=j,k,l}\{\max(a_2-a_1,1)^2\}^{-1}\right).$$

The remaining part of the argument is similar to that in the case of a negative α_{gh}. ■

Proof of Theorem 4. The proofs is similar (indeed slightly simpler) to that of Theorem 3, and hence omitted for brevity's sake. ■

5. PROOFS FOR SECTION 3

LEMMA 2. *Under Assumptions* A, B, C *and* D *we have*

$$\{\tilde{m}_1\tilde{m}_2\tilde{m}_3\}^{-1/2}\widehat{\sum}_{jkl}\Lambda_{jkl}\left(\frac{(2\pi)^3 I_{\varepsilon jkl}^T}{\sigma_\varepsilon^2}-1\right)\xrightarrow{d}N(0,\Phi). \tag{5.1}$$

Proof. We use the Lindeberg–Feller Central Limit Theorem for triangular arrays. Note that, by Gaussianity,

$$\left(\frac{(2\pi)^3 I_{\varepsilon jkl}^T}{\sigma_\varepsilon^2}-1\right)\equiv\text{i.i.d. }(0,1),$$

where by i.i.d. (0, 1) we denote a sequence of independent and identically distributed random variables, with zero mean and unit variance. We need to show that

$$\max_{j,k,l}|\Lambda_{jkl}|=o(\tilde{m}_1\tilde{m}_2\tilde{m}_3), \tag{5.2}$$

$$\frac{1}{\tilde{m}_1\tilde{m}_2\tilde{m}_3}\widehat{\sum}_{jkl}\Lambda_{jkl}^2=O(1), \tag{5.3}$$

$$\frac{1}{\tilde{m}_1\tilde{m}_2\tilde{m}_3}\widehat{\sum}_{jkl}|\Lambda_{jkl}|^p=O(1) \tag{5.4}$$

for some $p > 2$. Now (5.2) is trivial, whereas for (5.3) we consider

$$\frac{1}{\tilde{m}_1\tilde{m}_2\tilde{m}_3}\hat{\sum}_{jkl}\Lambda_{jkl}^2 = \{\tilde{m}_1\tilde{m}_2\tilde{m}_3\}^{-1}\hat{\sum}_{jkl}\log^2\left\|\left(\frac{j}{n_1},\frac{k}{n_2},\frac{l}{n_3}\right)\right\|$$
$$-\left\{\{\tilde{m}_1\tilde{m}_2\tilde{m}_3\}^{-1}\hat{\sum}_{jkl}\log\left\|\left(\frac{j}{n_1},\frac{k}{n_2},\frac{l}{n_3}\right)\right\|\right\}^2$$
$$= \{\tilde{m}_1\tilde{m}_2\tilde{m}_3\}^{-1}\hat{\sum}_{jkl}\log^2\left\|\left(\frac{j}{m_1},\frac{m_2/n_2}{m_1/n_1}\frac{k}{m_2},\frac{m_3/n_3}{m_1/n_1}\frac{l}{m_3}\right)\right\|$$
$$-\left\{\{\tilde{m}_1\tilde{m}_2\tilde{m}_3\}^{-1}\hat{\sum}_{jkl}\log\left\|\left(\frac{j}{m_1},\frac{m_2/n_2}{m_1/n_1}\frac{k}{m_2},\frac{m_3/n_3}{m_1/n_1}\frac{l}{m_3}\right)\right\|\right\}^2,$$

subtracting $\log^2 m_1/n_1$ from both summands; by Assumption D and integral approximation we obtain

$$\frac{1}{\tilde{m}_1\tilde{m}_2\tilde{m}_3}\hat{\sum}_{jkl}\Lambda_{jkl}^2 \simeq \sum_{j=r_1,k=r_2,l=r_3}^{\tilde{m}_1,\tilde{m}_2,\tilde{m}_3}\int_{j/\tilde{m}_1}^{(j+3)/\tilde{m}_1}\int_{k/\tilde{m}_2}^{(k+3)/\tilde{m}_2}\int_{l/\tilde{m}_3}^{(l+3)/\tilde{m}_3}\log^2\|(x)\|\,dx$$
$$-\Big[\sum_{j=r_1,k=r_2,l=r_3}^{\tilde{m}_1,\tilde{m}_2,\tilde{m}_3}\int_{j/\tilde{m}_1}^{(j+3)/\tilde{m}_1}\int_{k/\tilde{m}_2}^{(k+3)/\tilde{m}_2}\int_{l/\tilde{m}_3}^{(l+3)/\tilde{m}_3}\log^2\|(x)\|\,dx\Big]^2$$
$$\to \int_{[0,1]^3}\log^2\|(x)\|\,dx - \Big[\int_{[0,1]^3}\log\|(x)\|\,dx\Big]^2 = \Phi \quad \text{as } \tilde{m}_1,\tilde{m}_2,\tilde{m}_3\to\infty.$$

The same argument holds for (5.4), for instance considering $p = 4$; thus Lemma 2 is established. ▪

LEMMA 3. *Under Assumptions* A, B, C *and* D, *we have*

$$\hat{\sum}_{jkl}\Lambda_{jkl}^{\tau}\frac{I_{jkl}^T-|\hat{a}_{jkl}|^2 I_{\varepsilon jkl}^T}{f_{jkl}} = o_p(\sqrt{\tilde{m}_1\tilde{m}_2\tilde{m}_3}) \quad \textit{for } \tau = 0, 1, 2, \tag{5.5}$$

$$(\tilde{m}_1\tilde{m}_2\tilde{m}_3)^{-1/2}\hat{\sum}_{jkl}\Lambda_{jkl}^{\tau}\left\{\frac{I_{jkl}^T}{f_{jkl}}-1\right\}\xrightarrow{d}N(0,\Phi^{\tau}) \quad \textit{for } \tau = 0, 1, \tag{5.6}$$

and

$$E\left\{\hat{\sum}_{jkl}\frac{I_{jkl}^T}{f_{jkl}}\right\}^2 = O(\tilde{m}_1\tilde{m}_2\tilde{m}_3). \tag{5.7}$$

Proof. The convergence (5.6) is an immediate consequence of (5.5), Lemma 2 and trivial manipulations; note that subtracting unity is vacuous for $\tau = 1$ because the Λ_{jkl}'s sum to zero. The expected value of the square of (5.5) is bounded by

$$\log^{2\tau}\|N\|\hat{\sum}_{j_1k_1l_1}\hat{\sum}_{j_2k_2l_2}f_{j_1k_1l_1}^{-1}f_{j_2k_2l_2}^{-1}E(I_{j_1k_1l_1}^T-|\hat{a}_{j_1k_1l_1}|^2 I_{\varepsilon j_1k_1l_1}^T)(I_{j_2k_2l_2}^T-|\hat{a}_{j_2k_2l_2}|^2 I_{\varepsilon j_2k_2l_2}^T),$$

where, in view of the Isserlis formula for the fourth moment of Gaussian variates,

$$E(I^T_{j_1k_1l_1}-|\hat a_{j_1k_1l_1}|^2 I^T_{\varepsilon j_1k_1l_1})(I^T_{j_2k_2l_2}-|\hat a_{j_2k_2l_2}|^2 I^T_{\varepsilon j_2k_2l_2})$$

$$= EI^T_{j_1k_1l_1} EI^T_{j_2k_2l_2}-|\hat a_{j_2k_2l_2}|^2 EI^T_{j_1k_1l_1}EI^T_{\varepsilon j_2k_2l_2}-|\hat a_{j_1k_1l_1}|^2 EI^T_{\varepsilon j_1k_1l_1} EI^T_{j_2k_2l_2}$$

$$+|\hat a_{j_1k_1l_1}|^2 |\hat a_{j_2k_2l_2}|^2 EI^T_{\varepsilon j_1k_1l_1} EI^T_{\varepsilon j_2k_2l_2}+Ew^T_{j_1k_1l_1} w^T_{j_2k_2l_2} E\bar w^T_{j_1k_1l_1} \bar w^T_{j_2k_2l_2}$$

$$+Ew^T_{j_1k_1l_1} \bar w^T_{j_2k_2l_2} E\bar w^T_{j_1k_1l_1} w^T_{j_2k_2l_2}-|a_{j_1k_1l_1}|^2 Ew^T_{\varepsilon j_1k_1l_1} w^T_{j_2k_2l_2} E\bar w^T_{\varepsilon j_1k_1l_1} \bar w^T_{j_2k_2l_2}$$

$$-|\hat a_{j_1k_1l_1}|^2 Ew^T_{\varepsilon j_1k_1l_1} \bar w^T_{j_2k_2l_2} E\bar w^T_{\varepsilon j_1k_1l_1} w^T_{j_2k_2l_2}$$

$$-|\hat a_{j_2k_2l_2}|^2 Ew^T_{j_1k_1l_1} w^T_{\varepsilon j_2k_2l_2} E\bar w^T_{j_1k_1l_1} \bar w^T_{\varepsilon j_2k_2l_2}$$

$$-f_{j_2k_2l_2} Ew^T_{j_1k_1l_1} \bar w^T_{\varepsilon j_2k_2l_2} E\bar w^T_{j_1k_1l_1} w^T_{\varepsilon j_2k_2l_2}$$

$$+|\hat a_{j_1k_1l_1}|^2 |\hat a_{j_2k_2l_2}|^2 Ew^T_{\varepsilon j_1k_1l_1} w^T_{\varepsilon j_2k_2l_2} E\bar w^T_{\varepsilon j_1k_1l_1} \bar w^T_{\varepsilon j_2k_2l_2}$$

$$+|\hat a_{j_1k_1l_1}|^2 |\hat a_{j_2k_2l_2}|^2 Ew^T_{\varepsilon j_1k_1l_1} \bar w^T_{\varepsilon j_2k_2l_2} E\bar w^T_{\varepsilon j_1k_1l_1} w^T_{\varepsilon j_2k_2l_2}$$

$$= (EI^T_{j_1k_1l_1}-|\hat a_{j_1k_1l_1}|^2 EI^T_{\varepsilon j_1k_1l_1})(EI^T_{j_2k_2l_2}-|\hat a_{j_2k_2l_2}|^2 EI^T_{\varepsilon j_2k_2l_2}) \tag{5.8}$$

$$+|Ew^T_{j_1k_1l_1} w^T_{j_2k_2l_2}|^2+|\hat a_{j_1k_1l_1}|^2 |\hat a_{j_2k_2l_2}|^2 |Ew^T_{\varepsilon j_1k_1l_1} w^T_{\varepsilon j_2k_2l_2}|^2 \tag{5.9}$$

$$+|Ew^T_{j_1k_1l_1} \bar w^T_{j_2k_2l_2}|^2+|\hat a_{j_1k_1l_1}|^2 |\hat a_{j_2k_2l_2}|^2 |Ew^T_{\varepsilon j_1k_1l_1} \bar w^T_{\varepsilon j_2k_2l_2}|^2 \tag{5.10}$$

$$-|\hat a_{j_1k_1l_1}|^2 |Ew^T_{\varepsilon j_1k_1l_1} w^T_{j_2k_2l_2}|^2-|\hat a_{j_1k_1l_1}|^2 |Ew^T_{\varepsilon j_1k_1l_1} \bar w^T_{j_2k_2l_2}|^2 \tag{5.11}$$

$$-|\hat a_{j_2k_2l_2}|^2 |Ew^T_{j_1k_1l_1} w^T_{\varepsilon j_2k_2l_2}|^2-|\hat a_{j_2k_2l_2}|^2 |Ew^T_{j_1k_1l_1} \bar w^T_{\varepsilon j_2k_2l_2}|^2. \tag{5.12}$$

Now for (5.8), as in Robinson [21],

$$\left|EI^T_{j_1k_1l_1}-|\hat a_{j_1k_1l_1}|^2 EI^T_{\varepsilon j_1k_1l_1}\right| \left|EI^T_{j_2k_2l_2}-|\hat a_{j_2k_2l_2}|^2 EI^T_{\varepsilon j_2k_2l_2}\right|$$
$$\leqslant \left\{|EI^T_{j_1k_1l_1}-f_{j_1k_1l_1}|+\left|f_{j_1k_1l_1}-|\hat a_{j_1k_1l_1}|^2 EI^T_{\varepsilon j_1k_1l_1}\right|\right\}$$
$$\times \left\{|EI^T_{j_2k_2l_2}-f_{j_2k_2l_2}|+\left|f_{j_2k_2l_2}-|\hat a_{j_2k_2l_2}|^2 EI^T_{\varepsilon j_2k_2l_2}\right|\right\}$$
$$\leqslant Cf_{j_1k_1l_1} f_{j_2k_2l_2}\left\{\frac{1}{\|J_1\|^2}\frac{1}{\|J_2\|^2}+\|R\|^{-10}+\frac{\|M\|^{2\beta}}{\|N\|^{2\beta}}\right\},$$

by Theorem 1 (recall that $Ew^T_{\varepsilon jkl}\bar w^T_{\varepsilon jkl} \equiv \sigma^2_\varepsilon/(2\pi)^3$); here we write $J_i=(j_i, k_i, l_i)$, $i=1, 2$.

Note that

$$\widehat{\sum}_{jkl}\frac{1}{j^2+k^2+l^2}\leqslant C\int_{[0,\|M\|]^3}\frac{1}{\|x\|^2}dx\leqslant C\int_0^{\|M\|}\int_0^{\pi}\int_0^{2\pi}\varrho^{-2}\varrho^2\, d\varphi\, d\theta\, d\varrho\leqslant C\|M\|.$$

Hence, under Assumption D,

$$\log^{2\tau}\|N\|\widehat{\sum}_{j_1k_1l_1}\widehat{\sum}_{j_2k_2l_2} f^{-1}_{j_1k_1l_1} f^{-1}_{j_2k_2l_2}(EI^T_{j_1k_1l_1}-|\hat a_{j_1k_1l_1}|^2 EI^T_{\varepsilon j_1k_1l_1})$$
$$\times(EI^T_{j_2k_2l_2}-|\hat a_{j_2k_2l_2}|^2 EI^T_{\varepsilon j_2k_2l_2})$$

$$\leqslant \log^{2\tau}\|N\| \hat{\sum}_{j_1k_1l_1} \hat{\sum}_{j_2k_2l_2} \frac{1}{j_1^2+k_1^2+l_1^2} \frac{1}{j_2^2+k_2^2+l_2^2} + \frac{\|M\|^6}{\|R\|^{10}} + \frac{\|M\|^{6+2\beta}}{\|N\|^{2\beta}}$$

$$= O\left(\|M\|^2 + \frac{\|M\|^6}{\|R\|^{10}} + \frac{\|M\|^{6+2\beta}}{\|N\|^{2\beta}}\right) = o(\|M\|^3).$$

For (5.9)–(5.12) we consider first the case where $(j_1, k_1, l_1) \neq (j_2, k_2, l_2)$. Note that, by the orthogonality properties of the taper, $Ew^T_{\varepsilon j_1k_1l_1} w^T_{\varepsilon j_2k_2l_2} \equiv 0$. By Theorem 4 we have

$$\log^{2\tau}\|N\| \hat{\sum}_{j_1k_1l_1} \hat{\sum}_{j_2k_2l_2} f^{-1}_{j_1k_1l_1} f^{-1}_{j_2k_2l_2} |Ew^T_{j_1k_1l_1} w^T_{j_2k_2l_2}|^2 \quad -$$

$$\leqslant C \log^{2\tau}\|N\| \frac{\|M\|^6}{\|R\|^{10}} = o(\|M\|^3).$$

For (5.10), we have again by the orthogonality properties of the taper $|Ew^T_{\varepsilon j_1k_1l_1} \bar{w}^T_{\varepsilon j_2k_2l_2}| = 0$; by Theorem 3, for $\alpha_0 < 0$,

$$\log^{2\tau}\|N\| \hat{\sum}_{j_1k_1l_1} \hat{\sum}_{j_2k_2l_2} f^{-1}_{j_1k_1l_1} f^{-1}_{j_2k_2l_2} |Ew^T_{j_1k_1l_1} \bar{w}^T_{j_2k_2l_2}|^2$$

$$\leqslant C \log^{2\tau}\|N\| \hat{\sum}_{j_1k_1l_1} \hat{\sum}_{j_2k_2l_2} \frac{l_2^{-\alpha_0}}{j_1^{1-\alpha_0}} \prod_{a=j,k,l} \{\max(a_2-a_1, 1)\}^{-4}$$

$$\leqslant C \frac{\|M\|^{-\alpha_0/2}}{\|R\|^{1-\alpha_0/2}} \log^{2\tau}\|N\|$$

$$\times \sum_{j_1=r_1, j_2=j_1+1}^{\tilde{m}_1} \sum_{k_1=r_2, k_2=k_1+1}^{\tilde{m}_2} \sum_{l_1=r_3, l_2=l_1+1}^{\tilde{m}_3} \frac{1}{(j_2-j_1)^4} \frac{1}{(k_2-k_1)^4} \frac{1}{(l_2-l_1)^4}$$

$$+ C \frac{\|M\|^{-\alpha_0/2}}{\|R\|^{1-\alpha_0/2}} \log^{2\tau}\|N\| \sum_{j_1=r_1}^{\tilde{m}_1} \sum_{k_1=r_2, k_2=k_1+1}^{\tilde{m}_2} \sum_{l_1=r_3, l_2=l_1+1}^{\tilde{m}_3} \frac{1}{(k_2-k_1)^4} \frac{1}{(l_2-l_1)^4}$$

$$+ \ldots + C \frac{\|M\|^{-\alpha_0/2}}{\|R\|^{1-\alpha_0/2}} \log^{2\tau}\|N\| \sum_{j_1=r_1}^{\tilde{m}_1} \sum_{k_1=r_2}^{\tilde{m}_2} \sum_{l_1=r_3, l_2=l_1+1}^{\tilde{m}_3} \frac{1}{(l_2-l_1)^4}$$

$$= O\left(\frac{\|M\|^{3-\alpha_0/2}}{\|R\|^{1-\alpha_0/2}} \log^{2\tau}\|N\|\right),$$

the summands corresponding to the cases

$$\{(j_1 \neq j_2, k_1 \neq k_2, l_1 \neq l_2)\},$$

$$\{(j_1 = j_2, k_1 \neq k_2, l_1 \neq l_2)(j_1 \neq j_2, k_1 = k_2, l_1 \neq l_2)(j_1 \neq j_2, k_1 \neq k_2, l_1 = l_2)\},$$

and

$$\{(j_1 = j_2, k_1 = k_2, l_1 \neq l_2)(j_1 \neq j_2, k_1 = k_2, l_1 = l_2)(j_1 = j_2, k_1 \neq k_2, l_1 = l_2)\},$$

respectively; the role of the indexes can clearly be permuted. The argument for $\alpha_0 \geqslant 0$ is entirely analogous, whence we have a bound of order $o(\|M\|^3)$ under

Assumption D. Analogous bounds hold for (5.11) and (5.12), on the basis of Theorems 3 and 4. If instead $(j_1, k_1, l_1) = (j_2, k_2, l_2)$, we rearrange terms in (5.9)–(5.12), to obtain

(5.13) $\quad |E(w^T_{j_1k_1l_1})^2|^2 + |\hat{a}_{j_1k_1l_1}|^4 |E(w^T_{\varepsilon j_1k_1l_1})^2|^2$

(5.14) $\quad -|\hat{a}_{j_1k_1l_1}|^2 |Ew^T_{\varepsilon j_1k_1l_1} w^T_{j_1k_1l_1}|^2 - |\hat{a}_{j_1k_1l_1}|^2 |Ew^T_{j_1k_1l_1} w^T_{\varepsilon j_1k_1l_1}|^2$

(5.15) $\quad +|Ew^T_{j_1k_1l_1} \bar{w}^T_{j_1k_1l_1}|^2 + |\hat{a}_{j_1k_1l_1}|^4 |Ew^T_{\varepsilon j_1k_1l_1} \bar{w}^T_{\varepsilon j_1k_1l_1}|^2$

(5.16) $\quad -|\hat{a}_{j_1k_1l_1}|^2 |Ew^T_{\varepsilon j_1k_1l_1} \bar{w}^T_{j_1k_1l_1}|^2 - |\hat{a}_{j_1k_1l_1}|^2 |Ew^T_{j_1k_1l_1} \bar{w}^T_{\varepsilon j_1k_1l_1}|^2.$

(5.13) and (5.14) are $o(1)$ as an immediate application on Theorem 2. From Theorem 1, each of the four terms in (5.15)–(5.16) has the asymptotically absolute value $f^2_{j_1k_1l_1}(1+o(1))$, whence (5.15)+(5.16) = $o(1)$ also; note we are using here (2.7), which requires Assumption A′: the latter is immediately seen to be satisfied when we consider the cross-spectral density between the field $U(\cdot)$, and its Wold innovations $\varepsilon(\cdot)$. Hence the sum over (j_1, k_1, l_1) is $o(\tilde{m}_1 \tilde{m}_2 \tilde{m}_3)$, and thus (5.5) is established. Finally, for (5.7) we note that

$$E\left\{\widehat{\sum}_{jkl} \frac{I^T_{jkl}}{f_{jkl}}\right\}^2 = \widehat{\sum}_{jkl} E\left\{\frac{I^T_{jkl}}{f_{jkl}}\right\}^2 + \widehat{\sum}_{jkl} \widehat{\sum}_{j'k'l' \neq jkl} E \frac{I^T_{jkl}}{f_{jkl}} \frac{I^T_{j'k'l'}}{f_{j'k'l'}}.$$

Now

$$\widehat{\sum}_{jkl} E\left\{\frac{I^T_{jkl}}{f_{jkl}}\right\}^2 = O(\tilde{m}_1 \tilde{m}_2 \tilde{m}_3)$$

follows easily from Theorems 1 and 2 and standard manipulations for fourth moments of Gaussian variates, whereas the proof that

$$\widehat{\sum}_{jkl} \widehat{\sum}_{j'k'l' \neq jkl} E \frac{I^T_{jkl}}{f_{jkl}} \frac{I^T_{j'k'l'}}{f_{j'k'l'}} = O(\tilde{m}_1 \tilde{m}_2 \tilde{m}_3)$$

is very similar to the argument given for (5.5). Thus Lemma 3 is established. ∎

LEMMA 4. *Under Assumptions* A, B, C *and* D, *for* $\tau = 0, 1, 2$ *we have*

(5.17) $$(\tilde{m}_1 \tilde{m}_2 \tilde{m}_3)^{-1} \widehat{\sum}_{jkl} \log^\tau \|\omega_{jkl}\| \frac{I^T_{jkl}}{L(0)\|\omega_{jkl}\|^{\alpha_0}} - (\tilde{m}_1 \tilde{m}_2 \tilde{m}_3)^{-1} \widehat{\sum}_{jkl} \log^\tau \|\omega_{jkl}\| \xrightarrow{P} 0.$$

Proof. Clearly, (5.17) can be rewritten as

(5.18) $$(\tilde{m}_1 \tilde{m}_2 \tilde{m}_3)^{-1} \widehat{\sum}_{jkl} \log^\tau \|\omega_{jkl}\| \left\{\frac{I^T_{jkl}}{f_{jkl}} - 1\right\}$$

(5.19) $$+(\tilde{m}_1 \tilde{m}_2 \tilde{m}_3)^{-1} \widehat{\sum}_{jkl} \log^\tau \|\omega_{jkl}\| \left(1 - \frac{L(0)\|\omega_{jkl}\|^{\alpha_0}}{f_{jkl}}\right) \frac{I^T_{jkl}}{L(0)\|\omega_{jkl}\|^{\alpha_0}}.$$

Now (5.18) is $o_p(1)$ as an immediate consequence of Lemma 3, whereas for (5.19) we have, by Theorem 1,

$$E|(5.19)| \leqslant \left\{\sup_{j,k,l}\left|\left(1-\frac{L(0)\,\|\omega_{jkl}\|^{\alpha_0}}{f_{jkl}}\right)\log^\tau\|\omega_{jkl}\|\right|\right\}$$
$$\times(\tilde m_1\tilde m_2\tilde m_3)^{-1}\hat{\sum}_{jkl}\frac{EI_{jkl}^T}{L(0)\,\|\omega_{jkl}\|^{\alpha_0}}$$
$$\leqslant C\left\{\sup_{j,k,l}\left|\left(1-\frac{L(0)\,\|\omega_{jkl}\|^{\alpha_0}}{f_{jkl}}\right)\log^\tau\|\omega_{jkl}\|\right|\right\} = o(1). \quad\blacksquare$$

LEMMA 5. *Under Assumptions* A, B, C *and* D, *as* $\|N\|\to\infty$, *we have, for* $\hat\alpha$ *defined by* (3.3) *and* (3.4),

$$\hat\alpha \xrightarrow{P} \alpha_0.$$

Proof. Again we give the proof for the case $n_1 = n_2 = n_3 = n$, $\tilde m_1 = \tilde m_2 = \tilde m_3 = m$, $r_1 = r_2 = r_3 = r$. Let $\bar N_\delta \overset{\text{def}}{=} \{\alpha\colon |\alpha-\alpha_0| > \delta\}$, $\delta > 0$. For $S(\alpha) = Z(\alpha)-Z(\alpha_0)$,

$$P(|\hat\alpha-\alpha_0| > \delta) \leqslant P\big(\inf S(\alpha) \leqslant 0,\ \alpha\in\bar N_\delta\big).$$

Now

$$P\big(\inf S(\alpha)\leqslant 0,\ \alpha\in\bar N_\delta\big) \leqslant P\big(|\sup T(\alpha)| \geqslant \inf U(\alpha)\big),$$

where $S(\alpha) = U(\alpha)-T(\alpha)$ for

$$T(\alpha) = \log\frac{\hat G(\alpha_0)}{L(0)} - \log\frac{\hat G(\alpha)}{G(\alpha)},\qquad G(\alpha) = L(0)\frac{1}{m^3}\hat{\sum}_{jkl}\|\omega_{jkl}\|^{\alpha-\alpha_0},$$

$$U(\alpha) = \log\left\{\frac{1}{m^3}\hat{\sum}_{jkl}\|\omega_{jkl}\|^{\alpha-\alpha_0}\right\} - \frac{1}{m^3}\hat{\sum}_{jkl}\|\omega_{jkl}\|^{\alpha-\alpha_0}$$
$$\sim \log\left\{\frac{1}{m^3}\hat{\sum}_{jkl}\left\|\left(\frac{j}{m},\frac{k}{m},\frac{l}{m}\right)\right\|^{\alpha-\alpha_0}\right\} - \frac{1}{m^3}\hat{\sum}_{jkl}\log\left\|\left(\frac{j}{m},\frac{k}{m},\frac{l}{m}\right)\right\|^{\alpha-\alpha_0},$$

and $\sim$ means that the ratio between the left-hand and the right-hand side tends to one in view of Assumption D. Note that $G(\alpha_0)\equiv L(0)$. Now $U(\alpha)$ is easily seen to be uniformly bounded below by some $\eta = \eta(\alpha)\geqslant 0$, the inequality being strict for $\alpha\neq\alpha_0$ by Jensen's inequality and strict concavity of the logarithm function. Also, it is immediate to see that

$$\sup_\alpha\left|\log\frac{\hat G(\alpha_0)}{L(0)}\right| = \left|\log\frac{\hat G(\alpha_0)}{L(0)}\right| \leqslant \sup_\alpha\left|\log\frac{\hat G(\alpha_0)}{G(\alpha)}\right|.$$

By the same argument as in Robinson [21], we thus only need to prove that

$$\sup_\alpha\left|\frac{\hat G(\alpha_0)-G(\alpha)}{G(\alpha)}\right| = o_p(1).$$

Now we infer easily from Assumption D that

$$\sup_\alpha \left|\frac{\hat{G}(\alpha)-G(\alpha)}{G(\alpha)}\right| \leqslant \frac{|\sup A(\alpha)|}{|\inf B(\alpha)|},$$

$$A(\alpha) = L(0)\frac{1}{m^3}\hat{\sum}_{jkl}\left\|\left(\frac{j}{m}, \frac{k}{m}, \frac{l}{m}\right)\right\|^{\alpha-\alpha_0}\left[\frac{I_{jkl}}{L(0)\,\|\omega_{jkl}\|^{\alpha_0}}-1\right],$$

$$B(\alpha) = L(0)\frac{1}{m^3}\hat{\sum}_{jkl}\left\|\left(\frac{j}{m}, \frac{k}{m}, \frac{l}{m}\right)\right\|^{\alpha-\alpha_0}.$$

Note that

$$\inf B(\alpha) \geqslant C \int_{[0,1]^3} \|x\|^{\alpha-\alpha_0}\,dx > c > 0 \qquad \text{for } \alpha-\alpha_0 > -3,$$

$$\inf B(\alpha) \geqslant L(0)\, m^{\alpha_0-\alpha-3}\hat{\sum}_{jkl}\|(j, k, l)\|^{\alpha-\alpha_0}$$

$$\geqslant C \int_{\|R\|}^{\|M\|}\int_0^{\pi}\int_0^{2\pi} \varrho^{\alpha-\alpha_0}\varrho^2\, d\varphi\, d\theta\, d\varrho \geqslant C\,(m/r)^{\alpha_0-\alpha-3} \qquad \text{for } \alpha_0-\alpha > 3.$$

Hence

$$\sup_\alpha \left|\frac{\hat{G}(\alpha)-G(\alpha)}{G(\alpha)}\right| \leqslant C\,|\sup A(\alpha)|$$

$$= O_p\left(m^{-3}\, E\left|\hat{\sum}_{jkl}\left(\frac{I^T_{jkl}}{L(0)\,\|\omega_{jkl}\|^{\alpha_0}}-1\right)\right|\right) \qquad \text{for } \alpha-\alpha_0 > 0,$$

$$\sup_\alpha \left|\frac{\hat{G}(\alpha)-G(\alpha)}{G(\alpha)}\right| \leqslant C\,|\sup A(\alpha)|$$

$$= O_p\left(\frac{m^{\alpha_0-\alpha-3}}{r^{\alpha_0-\alpha}}\, E\left|\hat{\sum}_{jkl}\left(\frac{I^T_{jkl}}{L(0)\,\|\omega_{jkl}\|^{\alpha_0}}-1\right)\right|\right)$$

$$= O_p\left(\frac{1}{r^3}\, E\left|\hat{\sum}_{jkl}\left(\frac{I^T_{jkl}}{L(0)\,\|\omega_{jkl}\|^{\alpha_0}}-1\right)\right|\right) \qquad \text{for } -3 < \alpha-\alpha_0 < 0,$$

$$\sup_\alpha \left|\frac{\hat{G}(\alpha)-G(\alpha)}{G(\alpha)}\right| \leqslant C\,\frac{r^{\alpha_0-\alpha-3}\,|\sup A(\alpha)|}{m^{\alpha_0-\alpha-3}}$$

$$= O_p\left(\frac{1}{r^3}\, E\left|\hat{\sum}_{jkl}\left(\frac{I^T_{jkl}}{L(0)\,\|\omega_{jkl}\|^{\alpha_0}}-1\right)\right|\right) \qquad \text{for } \alpha-\alpha_0 < -3.$$

Hence consistency will follow if we just show that

$$E\left|\hat{\sum}_{jkl}\left(\frac{I^T_{jkl}}{L(0)\,\|\omega_{jkl}\|^{\alpha_0}}-1\right)\right| = o(r^3).$$

Now

$$E\left|\widehat{\sum}_{jkl}\left(\frac{I^T_{jkl}}{L(0)\|\omega_{jkl}\|^{\alpha_0}}-1\right)\right| \leqslant E\left|\widehat{\sum}_{jkl}\frac{I^T_{jkl}-|\hat{a}_{jkl}|^2 I^T_{\varepsilon jkl}}{f_{jkl}}\right| + E\left|\widehat{\sum}_{jkl}\left(\frac{(2\pi)^3 I^T_{\varepsilon jkl}}{\sigma^2_\varepsilon}-1\right)\right|$$
$$+E\left|\widehat{\sum}_{jkl}\left(1-\frac{f_{jkl}}{L(0)\|\omega_{jkl}\|^{\alpha_0}}\right)\frac{I^T_{jkl}}{f_{jkl}}\right|.$$

For the last term, we have

$$E\left|\widehat{\sum}_{jkl}\left(1-\frac{f_{jkl}}{L(0)\|\omega_{jkl}\|^{\alpha_0}}\right)\frac{I^T_{jkl}}{f_{jkl}}\right| \leqslant \left[\sup_{j,k,l}\left|1-\frac{f_{jkl}}{L(0)\|\omega_{jkl}\|^{\alpha_0}}\right|^2 E\left\{\widehat{\sum}_{jkl}\frac{I^T_{jkl}}{f_{jkl}}\right\}^2\right]^{1/2}$$
$$= O(\|\omega_{jkl}\|^\beta \|M\|^{3/2}) = O\left(\frac{\|M\|^{\beta+3/2}}{\|N\|^\beta}\right),$$

the bound following from Lemma 3. For the other two terms, we have by Gaussianity and Lemmas 2 and 3, respectively,

$$E\left[\frac{1}{m^{3/2}}\widehat{\sum}_{jkl}\left(\frac{I^T_{\varepsilon jkl}}{\sigma^2_\varepsilon}-1\right)\right]^2 = O_p(1), \quad \widehat{\sum}_{jkl}\frac{I^T_{jkl}-|\hat{a}_{jkl}|^2 I^T_{\varepsilon jkl}}{f_{jkl}\sigma^2_\varepsilon} = o_p(\|M\|^{3/2}).$$

Thus consistency is established. ■

REFERENCES

[1] S. Albeverio, S. A. Molchanov and D. Surgailis, *Stratified structure of the Universe and Burgers' equation: Probabilistic approach*, Probab. Theory Related Fields 100 (1994), pp. 457–484.

[2] V. V. Anh, J. M. Angulo and M. D. Ruiz-Medina, *Possible long-range dependence in fractional random fields*, J. Statist. Plann. Inference 80 (1999), pp. 95–110.

[3] V. V. Anh and N. N. Leonenko, *Non-Gaussian scenario for the heat equation with singular initial conditions*, Stochastic Process. Appl. 84 (1999), pp. 91–114.

[4] V. V. Anh and N. N. Leonenko, *Scaling laws for fractional diffusion-wave equations with singular data*, Statist. Probab. Lett., to appear.

[5] T. Funaki, D. Surgailis and W. A. Woyczynski, *Gibbs–Cox random fields and Burgers turbulence*, Ann. Appl. Probab. 5 (1995), pp. 461–492.

[6] X. Guyon, *Random Fields on a Network: Modelling, Statistics and Applications*, Springer, Berlin 1995.

[7] E. J. Hannan, *Multiple Time Series*, Wiley, New York 1970.

[8] C. C. Heyde and R. Gay, *Smoothed periodogram asymptotics and estimation for processes and fields with possible long-range dependence*, Stochastic Process. Appl. 45 (1993), pp. 169–182.

[9] A. V. Ivanov and N. N. Leonenko, *Statistical Analysis of Random Fields*, Kluwer Academic Publishers, Dordrecht 1989.

[10] G. W. Kolb and M. S. Turner, *The Early Universe*, Addison and Wesley, Redwood City, CA, 1990.

[11] H. R. Künsch, *Statistical analysis of self-similar processes*, in: *Proceedings of the 1st World Congress of the Bernoulli Society*, VNU Science Press, 1987, pp. 67–74.

[12] N. N. Leonenko, *Random Fields with Singular Spectrum*, Kluwer Academic Publishers, Dordrecht 1999.

[13] N. N. Leonenko and W. A. Woyczynski, *Exact parabolic asymptotics for singular n-D Burgers' random fields: Gaussian approximation*, Stochastic Process. Appl. 76 (1998), pp. 141–165.

[14] N. N. Leonenko and W. A. Woyczynski, *Parameter identification for singular random fields arising in Burgers' turbulence*, J. Statist. Plann. Inference 80 (1999), pp. 1–14.

[15] N. N. Leonenko and W. A. Woyczynski, *Parameter identification for stochastic Burgers' flows via parabolic rescaling*, Probab. Math. Statist. 21, No. 1 (2001), pp. 1–55.

[16] C. Ludena and M. Lavielle, *The Whittle estimator for strongly dependent stationary Gaussian fields*, Scand. J. Statist. 26 (1999), pp. 433–450.

[17] S. A. Molchanov, D. Surgailis and W. A. Woyczynski, *The large scale structure of the Universe and quasi-Voronoi tessellation of shock fronts in forced Burgers turbulence in R^d*, Ann. Appl. Probab. 7 (1997), pp. 200–228.

[18] J. Peebles, *Principles of Physical Cosmology*, Princeton University Press, Princeton 1990.

[19] D. B. Percival and A. T. Walden, *Spectral Analysis with Physical Applications*, Cambridge University Press, Cambridge 1993.

[20] P. M. Robinson, *Log-periodogram regression of time series with long-range dependence*, Ann. Statist. 23 (1995), pp. 1048–1072.

[21] P. M. Robinson, *Gaussian semiparametric estimation of long-range dependence*, Ann. Statist. 23 (1995), pp. 1630–1661.

[22] S. F. Shandarin and Y. B. Zeldovich, *The large-scale structure of the Universe: Turbulence, intermittency, structures in a self-gravitating medium*, Rev. Modern Phys., Vol. 1, No. 2 (1989), pp. 185–220.

[23] I. Vajda, *Theory of Statistical Inference and Information*, Kluwer Academic Publishers, London 1989.

[24] C. Velasco, *Non-Gaussian log-periodogram regression*, Econom. Theory 16, No. 1 (2000), pp. 44–79.

Dipartimento di Matematica
Università di Roma "Tor Vergata"
via della Ricerca Scientifica 1
00133 Roma, Italy
E-mail: marinucc@mat.uniroma2.it

Received on 20.4.2002;
revised version on 27.1.2003