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DISTRIBUTIONS OF SUPREMA OF LÉVY PROCESSES VIA THE HEAVY TRAFFIC INVARIANCE PRINCIPLE*

BY

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Abstract. We study the relationship between the distribution of the supremum functional $M_X = \sup_{0 \leqslant t < \infty}(X(t) - \beta t)$ for a process X with stationary, but not necessarily independent increments, and the limiting distribution of an appropriately normalized stationary waiting time for $G/G/1$ queues in heavy traffic. As a by-product we obtain explicit expressions for the distribution of M_X in several special cases of Lévy processes.

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1. INTRODUCTION

The purpose of this paper is to show that the class of limiting distributions of appropriately normalized stationary waiting time for $G/G/1$ queues in heavy traffic coincides with the class of distributions of the supremum functional

$$M_X \overset{\text{df}}{=} \sup_{0 \leqslant t < \infty} \big(X(t) - \beta t\big)$$

for stochastically continuous processes X with stationary increments. This result, which we label the *Heavy Traffic Invariance Principle*, is formulated at the end of this section. In the special case of stationary *and* independent increments, that is, in the case when X is a Lévy process, explicit formulas for the distribution of M_X are established.

In a sense, our Invariance Principle clarifies the connection between the supremum functional and the stationary waiting times. Of course, separately, these objects have been studied extensively before. In particular:

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In the case when X is a Lévy process, the distribution of M_X was considered by several authors, including Baxter and Donsker [1], Zolotarev [17], Takács [14], Bingham [3], Harrison [6], Kella and Whitt [8], and others.

The case of a spectrally negative Lévy process X was studied by Zolotarev ([17], Theorem 2) and Bingham ([3], Proposition 5) who have shown that in this case M_X has an exponential distribution with parameter λ which is a solution of a certain integral equation.

The case of a spectrally positive X was considered by Zolotarev ([17], Theorem 3), Takács ([14], Theorem 5), Harrison [6], and Kella and Whitt ([8], Theorem 4.2). Using different methods they showed that the distribution of M_X is given by the Pollaczek–Khinchine formula, although not all of them use this terminology. The special situation of a spectrally positive α-stable Lévy process, where M_X turns out to have a Mittag-Leffler distribution, was, however, not analyzed by them in any detail. To the best of our knowledge that observation has been first made by Whitt [16].

On the other hand, Boxma and Cohen [4] showed that the limiting distribution of the stationary waiting times in heavy traffic for $GI/GI/1$ queues, in the case when the service times have tails heavier than the interarrival times, and both belong to the domains of attraction of (different) α-stable distributions with parameters α, $1 < \alpha < 2$, have the Mittag-Leffler distribution. However, the connection with the supremum functional has not been noticed in their paper.

The way the *Heavy Traffic Invariance Principle* is applied is as follows: we find a sequence of queueing systems for which the distributions of appropriately normalized stationary waiting times in heavy traffic converge to the distribution of M_X. Knowing the form of the stationary waiting times for those queues we can then find their limit, that is, the distribution of M_X. In Section 3 we illustrate our method in the case of spectrally positive Lévy processes, first for the α-stable process, and then for the general case. Section 3 also contains a comparison of our method with the Takács method. In Section 4 we find an explicit formula for the distribution of the supremum functional for a symmetric α-stable process. We are not aware of any other result of this type. Our method can also be applied to find the distribution of M_X when X is a spectrally negative α-stable Lévy process.

The tools developed in this paper can be used to study a broad class of queueing systems. We shall apply them for this purpose in a separate paper which, from a purely mathematical perspective, may be viewed just as an expanded set of examples illustrating our Heavy Traffic Invariance Principle. However, from the perspective of queueing theory the ability to explicitly evaluate distributions for stationary waiting times in more complicated systems is of primary importance.

Now, let, for each $n \geqslant 1$, $\{(v_{n,k}, u_{n,k}), -\infty < k < \infty\}$ be a stationary sequence of pairs of nonnegative random variables. In our context they represent the generating sequence of the n-th queue: $v_{n,k}$ is the service time of the k-th customer, $k \geqslant 1$, and $u_{n,k}$ is the interarrival time between the k-th and $(k+1)$-st customers, $k \geqslant 1$. Then the random variable

$$\omega_n \stackrel{\mathrm{df}}{=} \sup_{0 \leqslant k < \infty} S_{n,k},$$

where

$$\xi_{n,k} = v_{n,-k} - u_{n,-k} \quad \text{and} \quad S_{n,0} = 0, \quad S_{n,k} = \sum_{j=1}^{k} \xi_{n,j} \quad \text{for } k \geqslant 1,$$

can be interpreted as the stationary waiting time for the n-th queue. Perhaps, at this point it is worthwhile to clarify that queues defined for different n's do not interact with each other and do not form a queueing system.

Our standing, and obvious, assumptions are: $a_n \stackrel{\mathrm{df}}{=} E\xi_{n,1} < 0$, for $n \geqslant 1$, and $S_{n,k} \to -\infty$ a.s. as $k \to \infty$.

It is now easy to see that, for any fixed $n \geqslant 1$,

$$\omega_n = \sup_{0 \leqslant t < \infty} \left(Z_n(t) - \lfloor nt \rfloor |a_n|\right), \quad \text{where } Z_n(t) = \sum_{j=1}^{\lfloor nt \rfloor} (\xi_{n,j} - a_n).$$

Under the above notation and assumptions our main result can be formulated as follows:

THEOREM 1 (Heavy Traffic Invariance Principle). *Let, for each $n = 1, 2, \ldots$, $\xi_{n,k}$, $k = 1, 2, \ldots$, be a stationary sequence of random variables such that $a_n := E\xi_{n,1} < 0$, and*

$$S_{n,k} := \sum_{j=1}^{k} \xi_{n,j} \to -\infty \text{ a.s.} \quad \text{as } k \to \infty.$$

Moreover, let

$$\omega_n = \sup_{0 \leqslant t < \infty} \left(Z_n(t) - \lfloor nt \rfloor |a_n|\right), \quad \text{where } Z_n(t) = \sum_{j=1}^{\lfloor nt \rfloor} (\xi_{n,j} - a_n),$$

and suppose that $0 < \beta < \infty$ and $\{c_n, n \geqslant 1\}$ is a sequence of positive numbers such that

(i) *$X_n \equiv Z_n/c_n \xrightarrow{\mathscr{D}} X$, in $D[0, \infty)$, considered with the J_1 Skorokhod topology, as $n \to \infty$, and X is stochastically continuous,*

(ii) *$\beta_n \equiv n|a_n|/c_n \to \beta$ as $n \to \infty$, and*

(iii) *the sequence $\{\omega_n/c_n\}$ is tight.*

Then $\omega_n/c_n \xrightarrow{\mathscr{D}} M_X = \sup_{0 \leqslant t < \infty} \left(X(t) - \beta t\right)$ as $n \to \infty$.

The proof of this Principle is provided in Section 2. The remainder of the paper is devoted to applications of the above Principle for various types of Lévy processes X. The goal is to determine the supremum M_X by an appropriate selection of the queue generating sequences $\{(v_{n,k}, u_{n,k}), -\infty < k < \infty\}$ and constants c_n for which conditions (i)–(iii) are satisfied and the limit ω_n/c_n can be identified.

2. CONVERGENCE AND TIGHTNESS CRITERIA FOR THE SUPREMUM FUNCTIONAL

In this section we establish a general convergence result for suprema of a sequence of processes. The Heavy Traffic Invariance Principle is an immediate corollary to this result. We also find usable sufficient conditions for the tightness of the sequence $\{\omega_n/c_n\}$.

The notation $Z_n \xrightarrow{\mathscr{D}} Z$ will mean the usual convergence in distribution if Z_n and Z are random variables, and the convergence in the J_1 Skorokhod topology (see Billingsley [2] and Lindvall [10]) if Z_n and Z are processes with sample paths in the functional space $D[0, \infty)$. A function β is said to be *superadditive* if $\beta(t+s) \geqslant \beta(t)+\beta(s)$ for each $t, s \geqslant 0$. The function $\beta(t) = \lfloor ct \rfloor$ is an example of a superadditive function.

THEOREM 2. *Let X and X_n, $n \geqslant 1$, be stochastic processes with stationary increments and trajectories in $D[0, \infty)$ such that $X(0) = X_n(0) = 0$ a.s. Additionally assume that*

(A) *$X_n \xrightarrow{\mathscr{D}} X$ as $n \to \infty$, and X is stochastically continuous.*

Furthermore, let β and β_n, $n \geqslant 1$, be superadditive functions in $D[0, \infty)$, positive for $t > 0$, and equal to 0 for $t = 0$, such that $M = \sup_{0 \leqslant t < \infty}(X(t)-\beta(t))$ and $M_n = \sup_{0 \leqslant t < \infty}(X_n(t)-\beta_n(t))$ are finite random variables,

(B) *$\beta_n(t) \to \beta(t)$ for each $t \geqslant 0$ as $n \to \infty$, where the function β is continuous, and*

(C) *$X(t)-\beta(t) \to -\infty$ a.s. as $t \to \infty$.*

Then the tightness of the sequence $\{M_n\}$ implies that $M_n \xrightarrow{\mathscr{D}} M$.

Proof. Let us notice that for any function $x \in D[0, \infty)$ such that $\sup_{0 \leqslant t < \infty} x(t) < \infty$ we have

$$(1) \quad \sup_{0 \leqslant t < \infty} x(t) - \sup_{0 \leqslant t \leqslant s} x(t) = \max\big(\sup_{0 \leqslant t \leqslant s} x(t), \sup_{s \leqslant t < \infty} x(t)\big) - \sup_{0 \leqslant t \leqslant s} x(t)$$

$$= \max\big(0, \sup_{s \leqslant t < \infty} x(t) - \sup_{0 \leqslant t \leqslant s} x(t)\big)$$

$$= \max\big(0, \sup_{0 \leqslant t < \infty} \big(x(t+s)-x(s)\big) + x(s) - \sup_{0 \leqslant t \leqslant s} x(t)\big).$$

Let us write

$$\tilde{X}_n(t) = X_n(t)-\beta_n(t), \quad M_n(s) = \sup_{0 \leqslant t \leqslant s} \tilde{X}_n(t), \quad M_n = \sup_{0 \leqslant t < \infty} \tilde{X}_n(t),$$

$$\hat{M}_n = \sup_{0 \leqslant t < \infty} \big(\tilde{X}_n(t+s)-\tilde{X}_n(s)\big), \quad M = \sup_{0 \leqslant t < \infty} \big(X(t)-\beta(t)\big).$$

Since X_n, $n \geqslant 1$, have stationary increments and β_n are superadditive and positive, we obtain

$$P(\hat{M}_n \geqslant x) = P\big(\sup_{s \leqslant t < \infty} (X_n(t) - X_n(s) - \beta_n(t) + \beta_n(s)) > x\big)$$

$$= P\big(\sup_{0 \leqslant t < \infty} (X_n(t+s) - X_n(s) - (\beta_n(t+s) - \beta_n(s))) > x\big)$$

$$\leqslant P\big(\sup_{0 \leqslant t < \infty} (X_n(t) - \beta_n(t)) > x\big) = P(M_n > x)$$

for each $x > 0$ and $n \geqslant 1$. This and the tightness of $\{M_n\}$ imply the tightness of $\{\hat{M}_n\}$. Hence for any $\varepsilon > 0$ there exists a K such that $P(\hat{M}_n > K) \leqslant \varepsilon$ for all n. This and (1) yield the following inequalities:

$$P(M_n - M_n(s) > 0) \leqslant P(\hat{M}_n + \tilde{X}_n(s) - M_n(s) > 0)$$

$$\leqslant \varepsilon + P(\hat{M}_n + \tilde{X}_n(s) - M_n(s) > 0,\ \hat{M}_n \leqslant K) \leqslant \varepsilon + P(M_n(s) - \tilde{X}_n(s) \leqslant K).$$

Since condition (A) and Theorem 5.1 from [2] imply the convergence

$$M_n(s) \xrightarrow{\mathscr{D}} \sup_{0 \leqslant t < s} (X(t) - \beta(t)) \equiv M(s),$$

in view of condition (A) we get

$$\limsup_n P(M_n(s) - \tilde{X}_n(s) \leqslant K) \leqslant P(M(s) - X(s) + \beta(s) \leqslant K).$$

Now, (C) and the above imply that $\lim_{s \to \infty} \limsup_n P(M_n - M_n(s) > 0) \leqslant \varepsilon$. Since ε was arbitrary, the proof is complete. ∎

Remark 1. A weaker version of the above theorem can be found in Szczotka [12] (see Lemmas 2 and 3 therein). In a sense, Theorem 2 provides a solution to a problem posed by Whitt [15] who asked about a topology in $D[0, \infty)$ under which the convergence $X_n(t) - \beta_n(t) \xrightarrow{\mathscr{D}} \mathscr{W}(t) - \beta t$ implies the convergence $\sup_{0 \leqslant t < \infty} (X_n(t) - \beta_n(t)) \xrightarrow{\mathscr{D}} \sup_{0 \leqslant t < \infty} (\mathscr{W}_n(t) - \beta t)$, where $\mathscr{W}$ is a Wiener process, and X_n are Donsker's sums.

Of course, Theorem 2 is only as good as our ability to verify tightness of the sequence $\{M_n\}$. For processes X_n of a special form, such as Donsker's sums of random variables, sufficient conditions for tightness were provided by Szczotka [13] under the assumption that the summands have finite and bounded moments of order $2+\varepsilon$, $\varepsilon > 0$. Our next result removes that restriction.

THEOREM 3. *Let*

$$X_n(t) = \frac{1}{c_n} \sum_{j=1}^{[nt]} \zeta_{n,j} \quad \textit{and} \quad \beta_n(t) = \frac{|a_n|\,[nt]}{c_n}, \quad t \geqslant 0,\ n \geqslant 1, \tag{2}$$

where the array $\{\zeta_{n,k}, k \geqslant 1, n \geqslant 1\}$ is row-wise stationary with $E\zeta_{n,k} = 0$, and $|a_n| n/c_n \to \beta$, $0 < \beta < \infty$.

(i) *If*

$$\sum_{k=1}^{\infty} \sup_{n \geqslant n_0} P\Big(\sup_{0 \leqslant t \leqslant \tau^k} \frac{1}{c_n} \sum_{j=1}^{[nt]} \zeta_{n,j} > \kappa \tau^k\Big) < \infty \tag{3}$$

for some positive integers τ and n_0 such that $|a_n| n/c_n \geqslant \frac{1}{2}\beta$ for $n \geqslant n_0$, and $\kappa = (1/2\tau)\beta$, then the sequence $\{M_n\}$ is tight.

(ii) *If we additionally assume that $\zeta_{n,k}$ are row-wise i.i.d. with $\zeta_{n,k} = \bar{\zeta}_k$, $k, n \geqslant 1$, and $c_n = n^{1/\alpha}$, where the distribution of $\bar{\zeta}_k$ belongs to the domain of attraction of a Lévy α-stable distribution with $1 < \alpha < 2$, then* (3) *holds.*

The proof of Theorem 3 will be preceded by a couple of lemmas.

LEMMA 1. *Let X be a process with stationary increments such that $X(0) = 0$ a.s., $\beta(t)$ a positive and superadditive function, $\beta(0) = 0$, and let $0 = \tau_0 < \tau_1 < \tau_2, \ldots$ be a sequence of positive integers. Then*

$$\begin{aligned} P\big(\sup_{0 \leqslant t < \infty} (X(t) - \beta(t)) > x\big) &\leqslant \sum_{k=0}^{\infty} P\big(X(\tau_k) > \tfrac{1}{2}(x + \beta(\tau_k))\big) \\ &+ \sum_{k=0}^{\infty} P\big(\sup_{0 \leqslant t \leqslant \tau_{k+1} - \tau_k} (X(t) - \beta(t)) > \tfrac{1}{2}(x + \beta(\tau_k))\big). \end{aligned} \tag{4}$$

If $\tau_0 = 0$, $\tau_k = \tau^k$, $k \geqslant 1$, for some integer τ, $\tau \geqslant 2$, then

$$\begin{aligned} P\big(\sup_{0 \leqslant t < \infty} (X(t) - \beta(t)) > x\big) &\leqslant \sum_{k=0}^{\infty} P\big(X(\tau^k) > \tfrac{1}{2}(x + \beta(\tau^k))\big) \\ &+ \sum_{k=0}^{\infty} P\big(\sup_{0 \leqslant t \leqslant \tau^{k+1}} (X(t) - \beta(t)) > \tfrac{1}{2}(x + \beta(\tau^k))\big). \end{aligned} \tag{5}$$

Proof. Notice that

$$\begin{aligned} \{\sup_{0 \leqslant t < \infty} (X(t) - \beta(t)) > x\} &\subset \bigcup_{k=0}^{\infty} \{\sup_{\tau_k \leqslant t \leqslant \tau_{k+1}} (X(t) - \beta(t)) > x\} \\ &= \bigcup_{k=0}^{\infty} \{\sup_{\tau_k \leqslant t \leqslant \tau_{k+1}} (X(t) - X(\tau_k) - \beta(t) + \beta(\tau_k)) + X(\tau_k) > x + \beta(\tau_k)\}. \end{aligned}$$

Hence

$$P\Big(\sup_{0\leqslant t<\infty}\big(X(t)-\beta(t)\big)>x\Big)$$

$$\leqslant \sum_{k=0}^{\infty} P\Big(\sup_{\tau_k\leqslant t\leqslant \tau_{k+1}}\big(X(t)-X(\tau_k)-\beta(t)+\beta(\tau_k)\big)+X(\tau_k)>x+\beta(\tau_k)\Big)$$

$$\leqslant \sum_{k=0}^{\infty} P\Big(\sup_{\tau_k\leqslant t\leqslant \tau_{k+1}}\big(X(t)-X(\tau_k)-\beta(t)+\beta(\tau_k)\big)>\tfrac{1}{2}\big(x+\beta(\tau_k)\big)\Big)$$

$$+\sum_{k=0}^{\infty} P\big(X(\tau_k)>\tfrac{1}{2}\big(x+\beta(\tau_k)\big)\big)$$

$$=\sum_{k=0}^{\infty} P\Big(\sup_{0\leqslant t\leqslant \tau_{k+1}-\tau_k}\big(X(t)-\beta(t)\big)>\tfrac{1}{2}\big(x+\beta(\tau_k)\big)\Big)+\sum_{k=0}^{\infty} P\big(X(\tau_k)>\tfrac{1}{2}\big(x+\beta(\tau_k)\big)\big).$$

The assertion (5) is an immediate consequence of (4), where we put $\tau_k=\tau^k$ for $k\geqslant 1$. Then $\tau_{k+1}-\tau_k=\tau^k(\tau-1)<\tau^{k+1}$. This completes the proof of the lemma. ∎

LEMMA 2. *Assume that processes X_n, $n\geqslant 1$, have stationary increments, $X_n(0)=0$ a.s., and functions $\beta_n(t)$, $n\geqslant 1$, are positive, nondecreasing and superadditive. Moreover, assume that, for some integers n_0 and $\tau\geqslant 2$,*

$$\sum_{k=1}^{\infty}\sup_{n\geqslant n_0} P\Big(\sup_{0\leqslant t\leqslant \tau^{k+1}}\big(X_n(t)\big)>\tfrac{1}{2}\beta_n(\tau^k)\Big)<\infty. \tag{6}$$

Then the sequence $\{M_n\}$ is tight.

Proof. The proof is an immediate consequence of the nonnegativity of functions $\beta_n(t)$, the inequality

$$P\big(X_n(\tau_k)>x\big)\leqslant P\Big(\sup_{0\leqslant t\leqslant \tau^{k+1}} X_n(t)>x\Big)\quad \text{for } x\geqslant 0,$$

and the second assertion of Lemma 1. ∎

Now we can return to the proof of Theorem 3.

Proof of Theorem 3. Since

$$\beta_n(\tau^k)=\frac{|a_n|\,n\tau^k}{c_n}\to\beta\tau^k\quad \text{as } n\to\infty,$$

let n_0 be an integer such that, for $n\geqslant n_0$, $(|a_n|\,n)/c_n\geqslant\frac{1}{2}\beta$, and let $\kappa\overset{\mathrm{df}}{=}(1/2\tau)\beta$. Then, for $n\geqslant n_0$, we have $\beta_n(\tau^k)\geqslant\kappa\tau^{k+1}$. This and Lemma 2 with X_n given in (2) give the first assertion of the theorem.

To prove part (ii) we notice that, by Doob's inequality,

$$P\left(\sup_{0\leqslant t\leqslant \tau^k}\frac{1}{n^{1/\alpha}}\sum_{j=1}^{[nt]}\zeta_{n,j}>\kappa\tau^k\right)\leqslant\frac{1}{(\kappa\tau^k)^\delta}E\left|\frac{1}{n^{1/\alpha}}\sum_{j=1}^{n\tau^k}\zeta_{n,j}\right|^\delta=\frac{1}{(\kappa\tau^k)^\delta}E\left|\frac{1}{n^{1/\alpha}}\sum_{j=1}^{n\tau^k}\bar{\zeta}_j\right|^\delta$$

$$=\frac{(\tau^{\delta/\alpha})^k}{(\kappa\tau^k)^\delta}E\left|\frac{1}{(n\tau^k)^{1/\alpha}}\sum_{j=1}^{n\tau^k}\bar{\zeta}_j\right|^\delta\leqslant\frac{1}{\kappa^\delta\tau_1^k}\sup_n E\left|\frac{1}{n^{1/\alpha}}\sum_{j=1}^{n}\bar{\zeta}_j\right|^\delta,$$

where $\delta\geqslant 1$ and $\tau_1=\tau^{\delta-\delta/\alpha}=\tau^{\delta(1-1/\alpha)}>1$. Since the distribution of $\bar{\zeta}_1$ belongs to the domain of attraction of an α-stable distribution, by the Remark on p. 36 in Kwapień and Woyczyński [9], we get

$$\sup_n E\left|\frac{1}{n^{1/\alpha}}\sum_{j=1}^{n}\bar{\zeta}_j\right|^\delta<\infty,$$

which jointly with $\tau_1>1$ gives the finiteness of the series in (3). ∎

3. CONVERGENCE TO LÉVY PROCESSES AND A DECOMPOSITION LEMMA

Let X be a Lévy process without a Gaussian component and with sample paths in the space $D[0,\infty)$. Then its characteristic function can be written in the form

$$E\exp\big(iuX(t)\big)=\exp\big(t\psi_{b,\nu}(u)\big), \tag{7}$$

where

$$\psi_{b,\nu}(u)=iub+\int_{|x|\geqslant r}(e^{iux}-1)\,\nu(dx)+\int_{0<|x|<r}(e^{iux}-1-iux)\,\nu(dx),$$

the *drift* b is a real number, the *spectral measure* ν is a positive measure on $(-\infty,\infty)$ which integrates the function $\min(1,x^2)$, and r is a positive number such that the points $-r$ and r are the continuity points of spectral measure ν.

If the spectral measure ν is concentrated on the positive half-line $(0,\infty)$, then the process X will be called *spectrally positive* or, loosely, a *process with positive jumps*. When ν is concentrated on the negative half-line $(-\infty,0)$, the process X will be called *spectrally negative* or a *process with negative jumps*.

Let

$$b(r,\nu)\overset{\text{df}}{=}-\int_{|x|\geqslant r}x\nu(dx)$$

if it is finite. Then

$$\psi_{b,\nu}(u)=iu\big(b-b(r,\nu)\big)+\int_{0<|x|<\infty}(e^{iux}-1-iux)\,\nu(dx). \tag{8}$$

If $b=b(r,\nu)$, then $\psi_{b,\nu}(u)$ does not depend on r.

A Lévy process can be considered as the limiting process of the interpolated sum processes $Y_n(t) = \sum_{k=1}^{[nt]} \zeta_{n,k}$, $t \geq 0$, $n \geq 1$, where $\{\zeta_{n,k}, k \geq 1, n \geq 1\}$ is an array of r.v.'s. In the sequel we recall the classical Prokhorov's result (see Prokhorov [11]) providing sufficient conditions for such a convergence.

LEMMA 3. *Let $\{\zeta_{n,k}, k \geq 1, n \geq 1\}$ be an infinitesimal array of row-wise i.i.d. zero-mean random variables with distribution function F_n in the n-th row. Furthermore, let*

$$(9) \qquad nP(\zeta_{n,1} < y) \to \nu(-\infty, y), \quad nP(\zeta_{n,1} > x) \to \nu(x, \infty) \quad \text{as } n \to \infty$$

for all continuity points $y < 0$ and $x > 0$ of ν,

$$(10) \qquad \lim_{x\to\infty} \sup_n nP(|\zeta_{n,1}| > x) = 0,$$

$$(11) \qquad b_{n,r} \stackrel{df}{=} n \int_{|x|<r} x\,dF_n(x) \to b_r,$$

and

$$(12) \qquad \lim_{\varepsilon\to 0} \limsup_n n \int_{|x|\leq\varepsilon} x^2\, dF_n(x) = 0.$$

Then $Y_n \stackrel{\mathcal{D}}{\to} X$ in $D[0, \infty)$ equipped with J_1 Skorokhod topology, where X is a stochastically continuous Lévy process with the characteristic function $E\exp(iuX(t)) = \exp(it\psi_{b,\nu}(u))$, where $\psi_{b,\nu}$ is given by (7) with $b = b_r$.

Applying the Heavy Traffic Invariance Principle will be also made easier if one utilizes the following Decomposition Lemma which, intuitively speaking, asserts that if the distribution of service times has heavier tails than the distribution of the interarrival times (or vice versa), then the limiting process X and, consequently, $\sup_{0\leq t<\infty}(X(t)-\beta t)$ depend only on the distribution of random variables with heavier tails. To formulate the Decomposition Lemma let us put

$$\zeta_{n,k}^{(1)} = \frac{1}{c_{n,1}}(v_{n,-k} - \bar{v}_n), \qquad \zeta_{n,k}^{(2)} = \frac{1}{c_{n,2}}(u_{n,-k} - \bar{u}_n), \qquad k, n \geq 1,$$

$$V_n(t) = \sum_{j=1}^{[nt]} \zeta_{n,j}^{(1)}, \qquad U_n(t) = \sum_{j=1}^{[nt]} \zeta_{n,j}^{(2)}, \qquad t \geq 0,\ n \geq 1,$$

and

$$X_n(t) = \frac{1}{c_n} \sum_{j=1}^{[nt]} (v_{n,-j} - \bar{v}_n) - \frac{1}{c_n} \sum_{j=1}^{[nt]} (u_{n,-j} - \bar{u}_n),$$

where

$$\bar{v}_n = Ev_{n,k}, \qquad \bar{u}_n = Eu_{n,k}, \qquad c_n = \max(c_{n,1}, c_{n,2}).$$

LEMMA 4 (Decomposition Lemma). *Let* $(V_n, U_n) \xrightarrow{\mathscr{D}} (V, U)$, *where* V *and* U *are nondegenerate and stochastically continuous processes. Assume that conditions* (ii) *and* (iii) *of Theorem* 1 *hold with* $\xi_{n,k} = v_{n,-k} - u_{n,-k} - \bar{v}_n + \bar{u}_n$. *Then*:

(i) *If* $c_{n,1}/c_{n,2} \to 0$, *then* $X_n \xrightarrow{\mathscr{D}} X \equiv -U$, *and* $\omega_n/c_{n,1} \xrightarrow{\mathscr{D}} \sup_{0 \leqslant t < \infty}(-U(t) - \beta t)$.

(ii) *If* $c_{n,2}/c_{n,1} \to 0$, *then* $X_n \xrightarrow{\mathscr{D}} X \equiv V$, *and* $\omega_n/c_{n,1} \xrightarrow{\mathscr{D}} \sup_{0 \leqslant t < \infty}(V(t) - \beta t)$.

Proof. The assertion follows immediately from the fact that the processes X_n are of the form

$$X_n = \frac{c_{n,1}}{c_n} V_n - \frac{c_{n,2}}{c_n} U_n.$$

4. SUPREMUM OF A SPECTRALLY POSITIVE LÉVY PROCESS

In this section we find the distribution of the supremum functional M_X when X is a spectrally positive Lévy process. For the sake of clarity of exposition we proceed first with the α-stable case although it can be deduced from the general case discussed in Subsection 4.2. Subsection 4.3 explains how our approach compares with the approach developed by Takács [14].

4.1. The α-stable case. If X is spectrally positive with $\nu(x, \infty) = \gamma x^{-\alpha}$ for $x > 0$, then

$$b(r, \nu) = -\gamma\alpha \frac{1}{\alpha - 1} r^{1-\alpha}.$$

Let

$$\mu = \int_{0+}^{\infty} (e^{-x} - 1 + x) x^{-(\alpha+1)} dx \quad \text{and} \quad \theta = \left(\frac{\alpha\gamma\mu}{\beta + b(r, \nu) - b}\right)^{1/(\alpha-1)}. \tag{13}$$

THEOREM 4. *Let* $1 < \alpha < 2$, $\gamma > 0$, *and* X *be a spectrally positive* α*-stable Lévy process with characteristic function* $\exp(iuX(t)) = \exp(t\psi_{b,\nu}(u))$, *where* $\psi_{b,\nu}(u)$ *is of the form* (7) *and* $\nu(x, \infty) = \gamma x^{-\alpha}$ *for all* $x > 0$. *Furthermore, let* $\beta > \max(0, b)$ *and*

$$M_X = \sup_{0 \leqslant t < \infty} (X(t) - \beta t).$$

Then the normalized supremum functional M_X/θ *has the Mittag-Leffler distribution* $R_{\alpha-1}$, *i.e.*,

$$1 - R_{\alpha-1}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{-k(\alpha-1)}}{\Gamma(k(\alpha-1)+1)},$$

with the Laplace transform

$$\int_{0-}^{\infty} e^{-sx} dR_{\alpha-1}(x) = \frac{1}{1 + s^{\alpha-1}} \quad \textit{for } s \geqslant 0.$$

Proof. Let us consider a sequence of the $M/GI/1$ queueing systems indexed by n. The n-th system is generated by the sequence $\{(v_{n,k}, u_{n,k}), -\infty < k < \infty\}$ with the following specifications: $v_{n,1}$ have distribution function F independent of n, where $F(x) = 0$ for $x \leqslant \gamma^{1/\alpha}$ and $F(x) = 1-\gamma x^{-\alpha}$ for $x \geqslant \gamma^{1/\alpha}$. Putting $\gamma_0 = \gamma^{1/\alpha}$ we have

$$\bar{v} \equiv Ev_{n,1} = \gamma\alpha \int_{\gamma_0}^{\infty} \frac{1}{x^{\alpha}}\, dx = \gamma\alpha \frac{1}{\alpha-1}\gamma^{(1-\alpha)/\alpha} = \frac{\alpha}{\alpha-1}\gamma^{1/\alpha} = \frac{\alpha}{\alpha-1}\gamma_0.$$

Assume that $u_{n,1}$ have exponential distributions with means $\bar{u}_n$, $n \geqslant 1$, respectively, such that $a_n \equiv \bar{v}-\bar{u}_n < 0$, and $a_n \uparrow 0$ in such a way that, for $c_n = n^{1/\alpha}$, we have

$$n|a_n|/c_n = n^{1-1/\alpha}|a_n| \to \tilde{\beta} \equiv \beta - \gamma\alpha\frac{1}{\alpha-1}r^{1-\alpha} - b \equiv \beta + b_r - b \equiv \beta + b(r, v) - b,$$

where

$$b_r = b(r, v) = -\gamma\frac{\alpha}{\alpha-1}r^{1-\alpha}$$

and r is the parameter appearing in the representation of the exponent $\psi_{b,v}(u)$. Let

$$\zeta_{n,k}^{(1)} = \frac{1}{n^{1/\alpha}}(v_{n,-k}-\bar{v}), \qquad \zeta_{n,k}^{(2)} = \frac{1}{\sqrt{n\sigma_n}}(u_{n,-k}-\bar{u}_n), \qquad k \geqslant 1,\ n \geqslant 1,$$

$$V_n(t) = \sum_{k=1}^{[nt]} \zeta_{n,k}^{(1)} \quad \text{and} \quad U_n(t) = \sum_{k=1}^{[nt]} \zeta_{n,k}^{(2)}, \qquad t \geqslant 0,\ n \geqslant 1.$$

Then $U_n \xrightarrow{\mathscr{D}} U$, where U is a Wiener process.

In the next step we shall show that the sequence $\{V_n\}$ converges to a spectrally positive stable Lévy process V with the characteristic function given by (7) with characteristic exponent $\psi_{b,v}(u)$, where

$$b = -\gamma\frac{\alpha}{\alpha-1}r^{1-\alpha}, \qquad r > 0,$$

and $v(x, \infty) = \gamma x^{-\alpha}$ for $x > 0$. As a matter of fact, in this case

$$\psi_{b,v}(u) = \int_{0<|x|<\infty} (e^{iux}-1-iux)\, v(dx),$$

so that the expression does not depend on r.

To show that $V_n \xrightarrow{\mathscr{D}} V$ we will verify the conditions of Lemma 3 in which we put $\zeta_{n,k} = \zeta_{n,k}^{(1)}$ and set F_n to be the distribution function of $\zeta_{n,k}^{(1)}$.

First, notice that, for n such that $n^{1/\alpha}x+\bar{v} > \gamma^{1/\alpha}$,

$$nP(\zeta_{n,1}^{(1)} > x) = nP(v_{n,1}-\bar{v} > n^{1/\alpha}x) = \gamma n(xn^{1/\alpha}+\bar{v})^{-\alpha},$$

which, in turn, implies that

$$nP(\zeta_{n,1}^{(1)} > x) \to \gamma \frac{1}{x^\alpha} = \nu(x, \infty) \quad \text{for any } x > 0.$$

Similarly, for n such that $n^{1/\alpha} y + \bar{v} < \gamma^{1/\alpha}$, $y < 0$,

$$nP(\zeta_{n,1}^{(1)} < y) = 0 = \nu(-\infty, y),$$

which implies condition (9) of Lemma 3.

Now notice that, for n such that $n^{1/\alpha} x + \bar{v} > \gamma^{1/\alpha}$, we have

$$nP(|\zeta_{n,1}^{(1)}| > x) = nP(v_{n,1} > n^{1/\alpha} x + \bar{v}) + nP(v_{n,1} < -n^{1/\alpha} x + \bar{v})$$

$$\leqslant n\gamma \frac{1}{(xn^{1/\alpha} + \bar{v})^\alpha} \leqslant \gamma x^{-\alpha}.$$

Therefore condition (10) of Lemma 3 holds.

To check the third condition of Lemma 3 notice that

$$\int_{|x|<r} x dF_n(x) = E\zeta_{n,1}^{(1)} I(|\zeta_{n,1}^{(1)}| < r) = -E\zeta_{n,1}^{(1)} I(|\zeta_{n,1}^{(1)}| \geqslant r).$$

Hence, for n such that $n^{1/\alpha} r + \bar{v} > \gamma^{1/\alpha}$,

$$\int_{|x|<r} x dF_n(x) = -n^{-1/\alpha} E v_{n,1} I(v_{n,1} > rn^{1/\alpha} + \bar{v}) + n^{-1/\alpha} \bar{v} P(v_{n,1} > rn^{1/\alpha} + \bar{v})$$

$$= -n^{-1/\alpha} \alpha\gamma \int_{rn^{1/\alpha}+\bar{v}}^{\infty} x^{-\alpha} dx + \bar{v} n^{-1/\alpha} P(v_{n,1} > rn^{1/\alpha} + \bar{v})$$

$$= -n^{-1/\alpha} \alpha\gamma \frac{1}{\alpha - 1} (rn^{1/\alpha} + \bar{v})^{1-\alpha} + \bar{v} n^{-1/\alpha} P(v_{n,1} > rn^{1/\alpha} + \bar{v}).$$

Consequently,

$$n \int_{|x|<r} x dF_n(x) = -n^{1-1/\alpha} \alpha\gamma \frac{1}{\alpha - 1} (rn^{1/\alpha} + \bar{v})^{1-\alpha}$$

$$+ n\bar{v} n^{-1/\alpha} P(v_{n,1} > rn^{1/\alpha} + \bar{v}) \to -\gamma\alpha \frac{1}{\alpha - 1} r^{1-\alpha}.$$

This implies condition (11) of Lemma 3 with

$$b_r = -\gamma\alpha \frac{1}{\alpha - 1} r^{1-\alpha} \equiv b(r, \nu).$$

Finally, we check the last condition of Lemma 3. Notice that

$$n \int_{|x|<\varepsilon} x^2 dF_n(x) = nE(\zeta_{n,1}^{(1)})^2 I(|\zeta_{n,1}^{(1)}| < \varepsilon) = n^{1-2/\alpha} E(v_{n,1} - \bar{v})^2 I(|v_{n,1} - \bar{v}| \leqslant \varepsilon n^{1/\alpha})$$

$$= n^{1-2/\alpha} Ev_{n,1}^2 I(|v_{n,1}-\bar{v}| \leqslant \varepsilon n^{1/\alpha}) - 2\bar{v} n^{1-2/\alpha} Ev_{n,1} I(|v_{n,1}-\bar{v}| \leqslant \varepsilon n^{1/\alpha})$$
$$+(\bar{v})^2 n^{1-2/\alpha} P(|v_{n,1}-\bar{v}| \leqslant \varepsilon n^{1/\alpha}).$$

Since $1-2/\alpha = (\alpha-2)/\alpha < 0$, the third expression tends to zero as $n \to \infty$, and since

$$n^{1-1/\alpha} E(v_{n,1}-\bar{v}) I(|v_{n,1}-\bar{v}| \leqslant \varepsilon n^{1/\alpha}) \to b_r, \qquad |b_r| < \infty,$$

the second expression also tends to zero as $n \to \infty$. The first expression equals

$$\gamma\alpha n^{1-2/\alpha} \int_{\gamma_0}^{\varepsilon n^{1/\alpha}+\bar{v}} x^2 \frac{1}{x^{\alpha+1}} dx = \gamma\alpha n^{1-2/\alpha} \frac{1}{2-\alpha}\left((\varepsilon n^{1/\alpha}+\bar{v})^{2-\alpha} - \gamma_0^{2-\alpha}\right)$$
$$= \gamma\alpha \frac{1}{2-\alpha}\left((\varepsilon+\bar{v}/n^{1/\alpha})^{2-\alpha} - n^{1-2/\alpha}\gamma_0^{2-\alpha}\right) \to \gamma\alpha \frac{1}{2-\alpha}\varepsilon^{2-\alpha}.$$

Since the above limit converges to zero as $\varepsilon \to 0$, condition (12) of Lemma 3 holds.

Thus, in view of Lemma 3, it follows that $V_n \xrightarrow{\mathscr{D}} V$, where V is a spectrally positive stable Lévy process with characteristic exponent $\psi_{b_r,v}$, where

$$b_r = -\gamma \frac{\alpha}{\alpha-1} r^{1-\alpha}.$$

If $X_n(t) = n^{-1/\alpha} \sum_{j=1}^{[nt]} (\xi_{n,j} - a_n)$, $t \geqslant 0$, then $X_n = V_n - n^{(\alpha-2)/(2\alpha)} \sigma_n U_n$ and $(\alpha-2)/(2\alpha) < 0$. Now, by the Decomposition Lemma, it follows that $X_n \xrightarrow{\mathscr{D}} V$, where V is a spectrally positive Lévy process with an α-stable spectral measure v and characteristic exponent $\psi_{b_r,v}(u)$. Since the process $X(t)-\beta t$ has the characteristic exponent $\psi_{b-\beta,v}(u)$, and $b-\beta = b_r - (\beta + b_r - b) = b_r - \tilde{\beta}$,

$$\sup_{0 \leqslant t < \infty} \left(X(t)-\beta t\right) \stackrel{D}{=} \sup_{0 \leqslant t < \infty} \left(V(t) - (\beta - b_r - b)t\right) = \sup_{0 \leqslant t < \infty} \left(V(t) - \tilde{\beta} t\right).$$

To show that $\omega_n/n^{1/\alpha} \xrightarrow{\mathscr{D}} \sup_{0 \leqslant t < \infty} \left(X(t)-\beta t\right)$ it is sufficient to demonstrate that the sequence $\{\omega_n/n^{1/\alpha}\}$ is tight. We will prove even more and establish that the Laplace–Stieltjes transforms of $\omega_n/n^{1/\alpha}$, $n \geqslant 1$, converge to the Laplace transform of the Mittag-Leffler distribution. To this end notice that the form of the distribution of the stationary waiting time for $M/GI/1$ queues is known (see, for example, in Cohen [5], p. 255, formula (4.8.2)) and, with the above specifications,

$$(14) \qquad P(\omega_n \leqslant x) = (1-\varrho_n) \sum_{k=0}^{\infty} \varrho_n^k \left(\frac{1}{\bar{v}} \int_0^x (1-F(t))\, dt\right)^{*k} \qquad \text{for } x \geqslant 0,$$

and $P(\omega_n \leqslant x) = 0$ for $x < 0$, where F is the distribution function of the service time, $\varrho_n = \bar{v}/\bar{u}_n$ is the traffic intensity, while G^{*k} denotes the k-fold convolution of a distribution function G. In this situation, the Laplace–Stieltjes transform of the stationary waiting time ω_n is of the form

$$(15)\qquad E\exp(-s\omega_n) = \frac{1-\varrho_n}{1-\varrho_n\dfrac{1-\hat{F}(s)}{\bar{v}s}} = \frac{1}{1+\dfrac{\varrho_n}{1-\varrho_n}\left(1-\dfrac{1-\hat{F}(s)}{s\bar{v}}\right)}, \qquad s \geqslant 0,$$

where $\hat{F}$ is the Laplace–Stieltjes transform of the distribution function F. Consequently, the Laplace–Stieltjes transform of ω_n/c_n is of the form

$$(16)\qquad E\exp(-s\omega_n/c_n) = \frac{1}{1+\dfrac{\varrho_n}{1-\varrho_n}\left(1-\dfrac{1-\hat{F}(s/c_n)}{s\bar{v}/c_n}\right)}, \qquad s \geqslant 0.$$

Since $c_n = n^{1/\alpha}$ and $n|a_n|/c_n \to \tilde{\beta}$, we have $\beta_n \overset{\text{df}}{=} n^{1-1/\alpha}|a_n| \to \tilde{\beta}$, and $|a_n| = \beta_n n^{-(\alpha-1)/\alpha}$. But $\varrho_n = \bar{v}/\bar{u}_n$. Therefore

$$\frac{\varrho_n}{1-\varrho_n} = \frac{\bar{v}}{|a_n|} = \frac{\bar{v}}{\beta_n} n^{(\alpha-1)/\alpha}.$$

At this point notice that the Laplace–Stieltjes transform of F has the following form:

$$\hat{F}(s) = \alpha\gamma \int_{\gamma_0}^{\infty} e^{-sx} x^{-(\alpha+1)}\, dx, \qquad s \geqslant 0.$$

Hence

$$1-\frac{1-\hat{F}(s)}{\bar{v}s} = \frac{1}{\bar{v}s}\{\bar{v}s-(1-\hat{F}(s))\}$$

$$= \frac{1}{\bar{v}s}\{\bar{v}s+\alpha\gamma\int_{\gamma_0}^{\infty}(e^{-sx}-1)x^{-(\alpha+1)}\,dx\} = \frac{1}{\bar{v}s}\alpha\gamma\int_{\gamma_0}^{\infty}(e^{-sx}-1+sx)x^{-(\alpha+1)}\,dx$$

$$= \frac{1}{\bar{v}}\alpha\gamma\int_{\gamma_0}^{\infty}(e^{-sx}-1+sx)\frac{1}{xs}x^{-\alpha}\,dx.$$

This implies that

$$\frac{\varrho_n}{1-\varrho_n}\left(1-\frac{1-\hat{F}(s/c_n)}{\bar{v}s/c_n}\right)$$

$$= \frac{\bar{v}}{\beta_n} n^{(\alpha-1)/\alpha}\frac{1}{\bar{v}}\alpha\gamma\int_{\gamma_0}^{\infty}\left(\exp(-xs/n^{1/\alpha})-1+xs/n^{1/\alpha}\right)\frac{1}{xs/n^{1/\alpha}}x^{-\alpha}\,dx$$

$$= s^{\alpha-1}\frac{1}{\beta_n}\alpha\gamma\int_{s_n}^{\infty}(e^{-u}-1+u)u^{-(\alpha+1)}\,du.$$

The last equality was obtained by changing variables $u = xs/n^{1/\alpha}$ in the previous integral, with u ranging over the domain $s_n \leqslant u < \infty$, where $s_n = \gamma_0 s/n^{1/\alpha}$. Since the last displayed expression tends to $s^{\alpha-1}\alpha\gamma\mu/\tilde{\beta} = s^{\alpha-1}\theta^{\alpha-1}$, in view of (13) and (16), we get the following convergence:

$$E\exp\left(-s\omega_n/(\theta n^{1/\alpha})\right) \to \frac{1}{1+s^{\alpha-1}}, \qquad s \geqslant 0. \tag{17}$$

The above limit is the Laplace–Stieltjes transform of the Mittag-Leffler distribution $R_{\alpha-1}$, that is

$$\int_{0-}^{\infty} e^{-sx}\, dR_{\alpha-1}(x) = \frac{1}{1+s^{\alpha-1}}, \qquad s \geqslant 0,$$

which implies that

$$1 - R_{\alpha-1}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{-k(\alpha-1)}}{\Gamma\left(k(\alpha-1)+1\right)}.$$

Summarizing, as $n \to \infty$, the distributions of $\omega_n/(\theta n^{1/\alpha})$ converge to the Mittag-Leffler distribution $R_{\alpha-1}$. Hence the distribution of M/θ has the Mittag-Leffler distribution $R_{\alpha-1}$ as well. ■

4.2. The general case. In this subsection we calculate the Laplace transform of the distribution of the supremum functional for a general spectrally positive Lévy process.

THEOREM 5. *Let X be a spectrally positive Lévy process with spectral measure ν and characteristic exponent $\psi_{b,\nu}(u)$. Furthermore, let F_n, $n \geqslant 1$, be a sequence of distribution functions on $[0, \infty)$ with means $\bar{v}_n$, $n \geqslant 1$, respectively, such that, for a sequence $\{c_n\}$ of positive constants monotonically increasing to infinity, the distribution functions*

$$\tilde{F}_n(x) \stackrel{df}{=} F_n(c_n x + \bar{v}_n), \qquad n \geqslant 1,$$

satisfy conditions (9)–(12) *of Lemma* 3. *Moreover, let*

$$\nu_{n,s}(x, \infty) \stackrel{df}{=} n\left(1 - F_n(c_n x/s)\right), \qquad x > 0,\ s \geqslant 0, \tag{18}$$

and

$$\phi(s) = \lim_{n\to\infty} \int_0^{\infty} (e^{-u} - 1 + u)\, \nu_{n,s}(du), \qquad s \geqslant 0. \tag{19}$$

Then

$$\Phi(s) = \frac{1}{1 - \phi(s)/(\beta s)}$$

is the Laplace–Stieltjes transform of the distribution of the supremum functional M_X, *provided* $\Phi(0) = 1$ *and* Φ *is continuous at* $s = 0$.

Proof. The proof is similar to the proof of Theorem 4 with the following modifications: The single distribution F for the service times $v_{n,k}$, $k \geq 1$, in the n-th queueing system $M/GI/1$, is now replaced by the distributions F_n. Then

$$1-\frac{1-\hat{F}_n(s/c_n)}{\bar{v}_n s/c_n} = \frac{1}{\bar{v}_n s/c_n}\{\bar{v}_n s/c_n - (1-\hat{F}_n(s/c_n))\}$$
$$= \frac{1}{\bar{v}_n s/c_n}\int_0^\infty (\exp(-sx/c_n)-1+xs/c_n)\, dF_n(x)$$
$$= \frac{1}{s\bar{v}_n}\int_0^\infty (\exp(-sx/c_n)-1+xs/c_n)\, c_n\, dF_n(x).$$

Changing variables in the last integral, i.e. putting $u = xs/c_n$, we can write the latter expression as the integral

$$\frac{1}{s\bar{v}_n}\int_0^\infty (e^{-u}-1+u)\, c_n\, dF_n(c_n u/s).$$

Hence

$$1+\frac{\varrho_n}{1-\varrho_n}\left(1-\frac{1-\hat{F}_n(s/c_n)}{\bar{v}_n s/c_n}\right) = 1+\frac{1}{s\beta_n}\frac{n}{c_n}\int_0^\infty (e^{-u}-1+u)\, c_n\, dF_n(c_n u/s)$$
$$= 1-\frac{1}{s\beta_n}\int_0^\infty (e^{-u}-1+u)\, v_{n,s}(du),$$

where $v_{n,s}$ are defined in (18). By assumption, the above expression converges to $1-\phi(s)/(s\beta)$ for $s \geq 0$. Therefore

$$E\exp(-s\omega_n/c_n) \to \frac{1}{1-\phi(s)/s\beta} = \Phi(s), \qquad s \geq 0. \tag{20}$$

But $\Phi(0) = 1$ and, by assumption, Φ is also continuous at $s = 0$. This and the fact that Φ is a limit of the Laplace–Stieltjes transforms $E\exp(-s\omega_n/c_n)$ implies that Φ itself is a Laplace–Stieltjes transform. This also yields the weak convergence and, in particular, the tightness of the sequence $\{\omega_n/c_n\}$. Hence the Heavy Traffic Invariance Principle implies that the limiting distribution of $\{\omega_n/c_n\}$ is the same as the distribution of $M_X = \sup_{0 \leq t < \infty}(X(t)-\beta t)$ and that Φ is the Laplace–Stieltjes transform of the distribution of M_X. ■

4.3. The general case via a modified Takács' approach. In this subsection we provide an alternative approach to the problem of identifying the distribution of the supremum functional M_X for a spectrally positive Lévy process X which is based on the following result of L. Takács which was rediscovered, in a slightly different context, by Kella and Whitt [8]:

THEOREM 6 (Takács [14]). *Consider a spectrally positive Lévy process X with the Laplace transform of the form $E\exp(-sX(t)) = \exp(-t\phi(s))$ for $\Re(s) \geqslant 0$, where $\phi(s)$ is such that $\varrho := \lim_{s\to 0}\phi(s)/s$ exists and $0 < \varrho < 1$. Then:*

(i) *There exists a distribution function H such that $H(x) = 0$ for $x \leqslant 0$, whose Laplace transform is of the form*

$$\int_0^\infty e^{-sx}\, dH(x) = \frac{\phi(s)}{s\varrho}, \qquad \Re(s) \geqslant 0.$$

(ii) *The distribution function F_M of the functional $M_X = \sup_{0\leqslant t<\infty}(X(t)-t)$ (here $\beta = 1$!) is of the following form:*

$$F_M(x) = (1-\varrho)\sum_{k=0}^{\infty} \varrho^k H^{*k}(x) \text{ for } x \geqslant 0 \quad \text{and} \quad G(x) = 0 \text{ for } x \leqslant 0,$$

where H^{*k} denotes the k-fold convolution of H and its Laplace transform is

$$\frac{1-\varrho}{1-\phi(s)/s}.$$

Our next result shows how, in the case when $\lim_{s\to 0}\phi(s)/s = 0$, the above theorem can be extended to the case of arbitrary $\beta > 0$.

THEOREM 7. *Let X be a spectrally positive Lévy process with the Laplace transform $E\exp(-sX(t)) = \exp(-t\phi(s))$ for $\Re(s) \geqslant 0$, where $\phi(s)$ is such that $\lim_{s\to 0}\phi(s)/s = 0$. Then the distribution F_M of the supremum functional $M_X = \sup_{0\leqslant t<\infty}(X(t)-\beta t)$ has the Laplace transform*

$$\int_0^\infty e^{-sx}\, dF(x) = \frac{1}{1-\phi(s)/(\beta s)}.$$

Proof. Let γ be such that $0 < \gamma < \beta$, and define

$$\tilde{X}(t) = \frac{1}{\beta+\gamma}(X(t)+\gamma t), \qquad t \geqslant 0.$$

Then

$$X(t)-\beta t = X(t)+\gamma t-(\beta+\gamma)t = (\beta+\gamma)(\tilde{X}(t)-t), \qquad t \geqslant 0.$$

Furthermore, $\tilde{X}$ is a spectrally positive Lévy process with the Laplace transform of the form

$$E\exp(-s\tilde{X}(t)) = E\exp\left(-\frac{s}{\beta+\gamma}X(t)\right)\exp\left(-t\frac{s\gamma}{\beta+\gamma}\right)$$
$$= \exp\left(-t\phi\left(\frac{s}{\beta+\gamma}\right)-t\frac{s\gamma}{\beta+\gamma}\right) = \exp(-t\tilde{\phi}(s)),$$

where

$$\tilde{\phi}(s) = \phi\left(\frac{s}{\beta+\gamma}\right)+\frac{s\gamma}{\beta+\gamma}.$$

Hence

$$\lim_{s\to 0}\frac{\tilde{\phi}(s)}{s} = \frac{\gamma}{\beta+\gamma}.$$

Now by Takács' theorem, the distribution of $\tilde{M} = \sup_{0\leqslant t<\infty}\big(\tilde{X}(t)-t\big)$ is of the form

$$\tilde{F}(x) = \left(1-\frac{\gamma}{\beta+\gamma}\right)\sum_{k=0}^{\infty}\left(\frac{\gamma}{\beta+\gamma}\right)^k \tilde{H}^{*k}(x), \qquad x \geqslant 0,$$

where $\tilde{H}$ is the distribution function such that, for $x \leqslant 0$, $\tilde{H}(x) = 0$, and whose Laplace transform is of the form

$$\int_0^{\infty} e^{-sx}\, d\tilde{H}(x) = \frac{1}{s\gamma/(\beta+\gamma)}\left(\phi\left(\frac{s}{\beta+\gamma}\right)+\frac{s\gamma}{\beta+\gamma}\right) = 1+\phi\left(\frac{s}{\beta+\gamma}\right)\frac{\beta+\gamma}{s\gamma}.$$

Therefore the distribution of $(\beta+\gamma)\tilde{M}$ is equal to

$$F(x) = \left(1-\frac{\gamma}{\beta+\gamma}\right)\sum_{k=0}^{\infty}\left(\frac{\gamma}{\beta+\gamma}\right)^k H^{*k}(x), \qquad x \geqslant 0,$$

where $H(x) = \tilde{H}\big(x/(\beta+\gamma)\big)$. Finally, notice that the Laplace transform of F equals

$$\int_0^{\infty} e^{-sx}\, dF(x) = \frac{1-\gamma/(\beta+\gamma)}{1-\gamma(\beta+\gamma)^{-1}\big(1+(s\gamma)^{-1}\phi(s)\big)} = \frac{\beta}{\beta-s^{-1}\phi(s)} = \frac{1}{1-(s\beta)^{-1}\phi(s)}.$$

This completes the proof. ∎

5. SUPREMUM OF A SYMMETRIC α-STABLE PROCESS

In this section we abandon the restriction of spectral positivity imposed in the preceding section and consider an arbitrary symmetric α-stable Lévy process.

THEOREM 8. *Let X be a symmetric α-stable Lévy process with tails of the spectral measure $\nu(x, \infty) = \nu(-\infty, -x) = \gamma x^{-\alpha}$ for $x > 0$ and $1 < \alpha < 2$. Then the distribution of the supremum functional $M_X = \sup_{0\leqslant t<\infty}\big(X(t)-\beta t\big)$ has the Laplace–Stieltjes transform of the form $Ee^{-sM} = e^{-\Lambda(s)/\pi}$, $s \geqslant 0$, where*

$$\Lambda(s) = \int_0^{\infty}\frac{1}{u^{1+1/\alpha}}\int_0^{\infty}(e^{-sz}-1)\int_0^{\infty}\exp(-\lambda t^{\alpha})\cos\big(t(z+u\beta)u^{-1/\alpha}\big)\,dt\,dz\,du.$$

Proof. Assume that the generating sequence $\{(v_{n,k}, u_{n,k}), -\infty < k < \infty\}$ is such that

$$\xi_{n,k} = v_{n,-k} - u_{n,-k} = \xi_k - |a_n|, \quad k \geqslant 1, n \geqslant 1,$$

where $\xi, \xi_1, \xi_2, \xi_3, \ldots$ are i.i.d. random variables with stable and symmetric distribution with density f and characteristic function $\hat{f}(t) = \exp(-\lambda|t|^\alpha)$, $t \in R$, $1 < \alpha < 2$, $\lambda = 2\alpha\gamma \int_0^\infty (1-\cos x)^{-(\alpha+1)} dx$.

Furthermore, define $a_n \uparrow 0$ in such a way that

$$\frac{|a_n| n}{n^{1/\alpha}} = |a_n| n^{1-1/\alpha} \stackrel{\mathrm{df}}{=} \beta_n \to \beta, \quad 0 < \beta < \infty.$$

Then

$$\frac{\omega_n}{n^{1/\alpha}} = \sup_{0 \leqslant t < \infty} (X_n(t) - \beta_n(t)),$$

where

$$X_n(t) = \frac{1}{n^{1/\alpha}} \sum_{j=1}^{[nt]} \xi_j \quad \text{and} \quad \beta_n(t) = \frac{|a_n|[nt]}{n^{1/\alpha}}, \quad t \geqslant 0, n \geqslant 1.$$

This implies that $X_n \xrightarrow{\mathscr{D}} X$, where X is a Lévy process with symmetric stable spectral measure ν and $\beta_n(t) \to \beta t$. Hence the characteristic function of $X(t)$ is the limit of characteristic functions of $X_n(t)$. But

$$\lim_n E \exp\left(iu \frac{1}{n^{1/\alpha}} \sum_{j=1}^{[nt]} \xi_j\right) = \lim_n E \exp\left(iu([nt]/n)^{1/\alpha} \xi\right) \tag{21}$$
$$= E \exp(iut^{1/\alpha}\xi) = \exp(-\lambda|u|^\alpha t).$$

On the other hand, for positive u, the process X has the characteristic exponent equal to

$$\psi_{0,\nu}(u) = \gamma\alpha \int_{-\infty}^{0-} (e^{iux} - 1 - iux)|x|^{-(\alpha+1)} dx + \gamma\alpha \int_{0+}^{\infty} (e^{iux} - 1 - iux) x^{-(\alpha+1)} dx$$
$$= \gamma\alpha \int_{-\infty}^{0-} (\cos|ux| - i\sin|ux| - 1 + i|ux|)|x|^{-(\alpha+1)} dx$$
$$+ \gamma\alpha \int_{0+}^{\infty} (\cos|ux| + i\sin|ux| - 1 - i|ux|) x^{-(\alpha+1)} dx$$
$$= -2\alpha\gamma \int_{0+}^{\infty} (1 - \cos ux) x^{-(\alpha+1)} dx = -2\alpha\gamma u^\alpha \int_{0+}^{\infty} (1 - \cos x) x^{-(\alpha+1)} dx.$$

For negative u the situation is similar. Finally, we get

$$\psi_{0,\nu}(u) = -2\alpha\gamma|u|^\alpha \int_{0+}^{\infty} (1 - \cos x) x^{-(\alpha+1)} dx. \tag{22}$$

Comparing the right-hand sides of (21) and of (22) we get

$$\lambda = 2\alpha\gamma \int_{0+}^{\infty} (1-\cos x)\, x^{-(\alpha+1)}\, dx.$$

To prove the tightness of the sequence $\{\omega_n/n^{1/\alpha}\}$ we can use Theorem 3 with $\zeta_{n,k} = \xi_k$, $k \geqslant 1$, and $c_n = n^{1/\alpha}$. Indeed, notice that

$$P\Big(\sup_{0\leqslant t\leqslant \tau^k} \frac{1}{n^{1/\alpha}} \sum_{j=1}^{[nt]} \xi_j > \kappa\tau^k\Big) \leqslant \frac{1}{\kappa^\delta \tau^{\delta(1-1/\alpha)k}} E\,\Big|(n\tau^k)^{-1/\alpha} \sum_{j=1}^{n\tau^k} \xi_j\Big|^\delta = \frac{\tau^{\delta/\alpha}}{\kappa^\delta} E\,|\xi|^\delta \frac{1}{b^k},$$

where $1 < \delta < \alpha$ and $b = \tau^{\delta(1-1/\alpha)}$. Since $E\,|\xi|^\delta < \infty$ and $b > 1$, the above inequality implies that the series (3) in Theorem 3 converges uniformly with respect to n. This gives the tightness of $\{\omega_n/n^{1/\alpha}\}$.

Next, we will find the limit of $E\exp(-s\omega_n/n^{1/\alpha})$ as $n \to \infty$. To do this we will use the following representation of the Laplace–Stieltjes transform of the stationary waiting time ω_n:

$$E\exp(-s\omega_n) = \exp\Big\{-\sum_{k=1}^{\infty} \frac{1}{k}\big(E\exp(-s(S_{n,k})_+)-1\big)\Big\}, \tag{23}$$

where $S_{n,k} = \sum_{j=1}^{k} \xi_{n,j}$, $k \geqslant 1$, $n \geqslant 1$ and $(x)_+ = \max(0, x)$. Since $S_{n,k} = \sum_{j=1}^{k} \xi_j - |a_n|\,k$ and ξ_k are symmetric and α-stable, $S_{n,k} \overset{\mathscr{D}}{=} k^{1/\alpha}\xi - |a_n|\,k$. Hence

$$\begin{aligned} E\exp\big(-s(S_{n,k})_+\big)-1 &= E\exp\big(-s(k^{1/\alpha}\xi - k\,|a_n|)_+\big)-1 \\ &= P(\xi \leqslant k^{1-1/\alpha}|a_n|)-1+\int_{k^{1-1/\alpha}|a_n|}^{\infty} \exp\big(-s(k^{1/\alpha}x-k\,|a_n|)\big) f(x)\,dx \\ &= -P(\xi > k^{1-1/\alpha}|a_n|)+\exp(sk\,|a_n|)\int_{k^{1-1/\alpha}|a_n|}^{\infty} \exp(-sk^{1/\alpha}x) f(x)\,dx \\ &= \int_{k^{1-1/\alpha}|a_n|}^{\infty} \big(\exp\big(-s(k^{1/\alpha}x-k\,|a_n|)\big)-1\big) f(x)\,dx. \end{aligned} \tag{24}$$

On the other hand,

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_R e^{-itx}\hat{f}(t)\,dt = \frac{1}{2\pi}\int_R \big(\cos(tx)-i\sin(tx)\big)\exp(-\lambda\,|t|^\alpha)\,dt \\ &= \frac{1}{\pi}\int_0^{\infty} \cos(tx)\exp(-\lambda t^\alpha)\,dt. \end{aligned}$$

Hence, by (24), we get

$$\begin{aligned} &E\exp\big(-s(S_{n,k})_+\big)-1 \\ &= \frac{1}{\pi}\int_{k^{1-1/\alpha}|a_n|}^{\infty}\int_0^{\infty} \big(\exp\big(-s(k^{1/\alpha}x-k\,|a_n|)\big)-1\big)\cos(tx)\exp(-\lambda t^\alpha)\,dt\,dx. \end{aligned} \tag{25}$$

Using (25) and (23) we get

$$(26)\quad E\exp(-s\omega_n/n^{1/\alpha})$$
$$=\exp\left\{-\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\left(\int_{\beta_n(k/n)^{1-1/\alpha}}^{\infty}\int_0^{\infty}\left(\exp\left(-s\left(\left(\frac{k}{n}\right)^{1/\alpha}x-\beta_n\frac{k}{n}\right)\right)-1\right)\right.\right.$$
$$\left.\left.\times\cos(tx)\exp(-\lambda t^{\alpha})\,dt\,dx\right)\right\}$$
$$=\exp\left\{-\frac{1}{\pi}\Lambda_n(s)\right\},$$

where

$$\Lambda_n(s)=\frac{1}{n}\sum_{k=1}^{\infty}\frac{n}{k}\int_{\beta_n(k/n)^{1-1/\alpha}}^{\infty}\left(\exp\left(-s\left(\frac{k}{n}\right)^{1/\alpha}x+\beta_n s\frac{k}{n}\right)-1\right)\int_0^{\infty}\exp(-\lambda t^{\alpha})\cos(tx)\,dt\,dx.$$

Notice that, for each $s\geqslant 0$, $\Lambda_n(s)\to\Lambda(s)$, where

$$\Lambda(s)=\int_0^{\infty}\frac{1}{u}\int_{\beta u^{1-1/\alpha}}^{\infty}(\exp(-su^{1/\alpha}x+s\beta u)-1)\int_0^{\infty}\exp(-\lambda t^{\alpha})\cos(tx)\,dt\,dx\,du.$$

Substituting $z=u^{1/\alpha}x-\beta u$, we get $x=zu^{-1/\alpha}+\beta u^{1-1/\alpha}$ and

$$\Lambda(s)=\int_0^{\infty}\frac{1}{u^{1+1/\alpha}}\int_0^{\infty}(e^{-sz}-1)\int_0^{\infty}\exp(-\lambda t^{\alpha})\cos\left(t(z+u\beta)u^{-1/\alpha}\right)dt\,dz\,du.$$

Since $\{\omega_n n^{-1/\alpha}\}$ is tight, the function $\exp(-\pi^{-1}\Lambda(s))$ must be the Laplace–-Stieltjes transform of a distribution function on $[0,\infty)$. Therefore, by the Heavy Traffic Invariance Principle, $\exp(-\pi^{-1}\Lambda(s))$ must be the Laplace–Stieltjes transform of $M_X=\sup_{0\leqslant t<\infty}(X(t)-\beta t)$, where X is a Lévy process with symmetric α-stable spectral measure ν. ∎

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