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TSALLIS' ENTROPY BOUNDS FOR GENERALIZED ORDER STATISTICS

BY

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Abstract. We present sharp bounds for expectations of generalized order statistics with random indices expressed in terms of Tsallis' entropy. The bounds are attainable and provide new characterizations of some nontrivial distributions. No restrictions are imposed on the parameters of the generalized order statistics model.

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1. INTRODUCTION

Universal bounds for moments of order statistics from an i.i.d. sample were first derived by Hartley and David (1954), Gumbel (1954), Moriguti (1953) and Ludwig (1960). The bounds are expressed in terms of the mean and standard derivation of the underlying distribution and they provide characterizations of some distributions. Analogous evaluations for records and kth records were established by Nagaraja (1978), Grudzień and Szynal (1983) and Raqab (1997). Extensions of these results to progressive type II censored order statistic and generalized order statistic are given in Balakrishnan et al. (2001) and Kamps (1995). In Gajek and Gather (1991) p-norm bounds for order and record statistics were determined. The bounds for generalized order statistics based on inequalities of Diaz and Metcalf, and Pólya and Szegö were derived by Kamps (1995). In Gajek and Okolewski (2000a) some evaluations for generalized order statistics were obtained by the approach which is equivalent to the combination of the Moriguti inequality and the Steffensen inequality. The results proved by the Steffensen inequality alone are given in Gajek and Okolewski (2000b). In the case of restricted families of distributions there are known improvements of Moriguti-type bounds for order and record statistics determined by applying projections of elements of functional Hilbert spaces onto convex cones (see Rychlik (2001)). A summary of known bounds for generalized order statistics is presented in Kamps (1995). The results for order and record statistics are presented e.g. in David and Nagaraja (2003), Arnold and Balakrishnan (1989), Arnold et al. (1998) and Rychlik (1998), (2001).

Boltzmann's entropy bounds for expectations of generalized order statistics were established for the first time by Kałuszka and Okolewski (2003). The bounds characterize e.g. shifted Pareto distribution. They can also provide some rate of convergence evaluations. In 1988 Tsallis proposed a new definition of the entropy of the random variable X:

$$T_p(X) = E\left(\frac{X^p - X}{p-1}\right)$$

with the entropy index $p > 0$ and $p \neq 1$, which returns the classical Boltzmann entropy for $p \to 1$. Since then researchers have used it in many physical applications, such as developing the statistical mechanics of large scale astrophysical systems (Nakamichi et al. (2002)), investigating thermodynamic properties of stellar self-gravitating system (Taruya and Sakagami (2003)), describing fully developed turbulence (Armitsu and Armitsu (2002)). Furthermore, Tsallis' entropy was employed in solving inverse (Shiguemori et al. (2002)) and optimization problems (Andricioaei and Straub (1996), Serra et al. (1997), Franz and Hoffmann (2003)), constructing nonparametric tests of independence between stochastic processes (Fernandes (2000)), establishing a fragment size distribution function which undergoes a transition to scaling (Sotolongo-Costa et al. (2000)).

In this paper we propose sharp bounds for expectations of generalized order statistics $X(r, n, \tilde{m}, k)$ (see Definition 1 below) with random parameters r and n, expressed in terms of the Tsallis entropy of the underlying distribution. The bounds are attainable and provide characterizations of e.g. shifted Weibull and Pareto distributions. Some relations between these bounds and the Boltzmann entropy bounds are presented in Remarks 1 and 2. In the particular case of order statistics, random r and n appear naturally, e.g. in the context of stochastic scenario of relaxation (see Jurlewicz and Weron (2002)).

2. THE RESULT

Let $X, X_1, X_2, \ldots$ be i.i.d. random variables with a common distribution function F. Let $X_{r:n}$ denote the rth order statistics from the sample $X_1, \ldots, X_n$, and let $Y_r^{(k)}$, $r = 1, 2, \ldots$, be the kth record statistics, i.e.

$$Y_r^{(k)} = X_{L_k(r):L_k(r)+k-1}, \qquad r = 1, 2, \ldots, \ k = 1, 2, \ldots,$$

where $L_k(1) = 1$, $L_k(r+1) = \min\{j\colon X_{L_k(r):L_k(r)+k-1} < X_{j:j+k-1}\}$ for $r = 1, 2, \ldots$ (cf. Dziubdziela and Kopociński (1976)). Define the quantile function $F^{-1}(t) =$

$\inf\{s\in \boldsymbol{R}\colon F(s)\geqslant t\}$, $t\in(0,1)$. The generalized order statistics are defined by Kamps (1995) as follows:

DEFINITION 1. Let $n\in \boldsymbol{N}$, $k>0$, $\tilde{m}=(m_1,\ldots,m_{n-1})\in \boldsymbol{R}^{n-1}$ be parameters such that $\eta_r=k+n-r+\sum_{j=r}^{n-1} m_j>0$ for all $r\in\{1,\ldots,n\}$. If the random variables $U(r,n,\tilde{m},k)$, $r=1,\ldots,n$, have a joint density function of the form

$$f^{U(1,n,\tilde{m},k),\ldots,U(n,n,\tilde{m},k)}(u_1,\ldots,u_n)=k\Big(\prod_{j=1}^{n-1}\eta_j\Big)\Big(\prod_{i=1}^{n-1}(1-u_i)^{m_i}\Big)(1-u_n)^{k-1}$$

on the cone $0\leqslant u_1\leqslant\ldots\leqslant u_n<1$ of $\boldsymbol{R}^n$, then they are called *uniform generalized order statistics*. The random variables

$$X(r,n,\tilde{m},k)=F^{-1}\big(U(r,n,\tilde{m},k)\big),\qquad r=1,\ldots,n,$$

are called *generalized order statistics* based on the distribution function F. If $m_1=\ldots=m_{n-1}=m$, say, the random variables $X(r,n,\tilde{m},k)$ are denoted by $X(r,n,m,k)$.

In the case of $m=0$ and $k=1$ the $X(r,n,m,k)$ reduces to the $X_{r:n}$ from the sample $X_1,\ldots,X_n$, for continuous F, $m=-1$ and $k\in \boldsymbol{N}$ we obtain $Y_r^{(k)}$ based on the sequence $X_1,X_2,\ldots$, while for absolutely continuous F, $m_i=R_i$, where $R_i\in\{0,1,2,\ldots\}$, $i=1,2,\ldots,n$, are such that $R_1+\ldots+R_n+n=M$, and $k=M-\sum_{i=1}^{n-1} m_i-n+1$, the $X(r,n,\tilde{m},k)$ recovers the progressive censored type II order statistic $X_{r:n}^{\tilde{R},M}$ with the censored scheme $\tilde{R}=(R_1,\ldots,R_n)$ (see e.g. Balakrishnan et al. (2001), Balakrishnan and Aggarwala (2000)).

Let us put $\Gamma=\{(r,n)\in \boldsymbol{N}\times \boldsymbol{N}\colon 1\leqslant r\leqslant n\}$ and $\tilde{m}=(m_{1,n},\ldots,m_{n-1,n})\in \boldsymbol{R}^{n-1}$, $n\geqslant 2$. Let (R,N) be a random vector with values in Γ, independent of $X(r,n,\tilde{m},k)$, $(r,n)\in\Gamma$. Random parameters R and N are the natural ones for both order and record statistics. Define

$$\Phi(t)=\sum_{(r,n)\in\Gamma}\Phi_{r,n}(t)\,p_{r,n},\tag{1}$$

where $p_{r,n}=\boldsymbol{P}(R=r,N=n)$, $(r,n)\in\Gamma$ and

$$\Phi_{r,n}(t)=\int_0^t\Big(\prod_{j=1}^{r}\eta_j\Big)G_{r,r}^{r,0}\left(1-s\,\middle|\,\begin{matrix}\eta_1,\ \ldots,\ \eta_r\\ \eta_1-1,\ldots,\eta_r-1\end{matrix}\right)ds,\qquad t\in[0,1];\tag{2}$$

$G_{r,r}^{r,0}$ denotes a particular Meijers G-function. Let $\underline{\Phi}$ and $\bar{\Phi}$ be the greatest convex minorant and the smallest concave majorant of Φ, and let $\underline{\varphi}$ and $\bar{\varphi}$ be the right-hand side derivatives of $\underline{\Phi}$ and $\bar{\Phi}$, respectively. Recall that X stands for a random variable with the distribution function F, and $T_p(X)=\boldsymbol{E}\big((X^p-X)/(p-1)\big)$ means the Tsallis entropy of X.

THEOREM 1. *Suppose that* $\boldsymbol{P}(X\geqslant 0)=1$, $\boldsymbol{E}X^{p\vee 1}<\infty$ *and* $c>0$. *If* $p>1$, *then*

$$\boldsymbol{E}X(R,N,\tilde{m},k)\leqslant\frac{1}{p}\Big(c^{p-1}\,T_p(X)+\frac{c^{p-1}-p}{p-1}\boldsymbol{E}X+\frac{1}{c}u(p)\Big),\tag{3}$$

where

$$u(p) = \int_0^1 \left(1+(p-1)\underline{\varphi}(t)\right)^{p/(p-1)} dt. \tag{4}$$

If $0 < p < 1$, *then*

$$EX(R, N, \tilde{m}, k) \geqslant -\frac{1}{p}\left(c^{p-1} T_p(X) + \frac{c^{p-1}-p}{p-1} EX + \frac{1}{c} l(p)\right), \tag{5}$$

where

$$l(p) = -\int_0^1 \left(1-(p-1)\bar{\varphi}(t)\right)^{p/(p-1)} dt. \tag{6}$$

Equalities in (3) *and* (5) *are attained if and only if*

$$F^{-1}(t) = \frac{1}{c}\left(1+(p-1)\underline{\varphi}(t)\right)^{1/(p-1)}$$

and

$$F^{-1}(t) = \frac{1}{c}\left(1-(p-1)\bar{\varphi}(t)\right)^{1/(p-1)}, \quad t \in (0, 1),$$

respectively.

Proof. It was shown in Kamps (1995) and Cramer et al. (2002) that the expected value of $X(r, n, \tilde{m}, k)$ can be represented as follows:

$$EX(r, n, \tilde{m}, k) = \int_0^1 F^{-1}(t)\, d\Phi_{r,n}(t),$$

where $\Phi_{r,n}$ is given by (2). By Fubini's theorem we have

$$EX(R, N, \tilde{m}, k) = \sum_{(r,n)\in\Gamma} \int_0^1 F^{-1}(t)\, d\Phi_{r,n}(t)\, p_{r,n} = \int_0^1 F^{-1}(t)\, d\Phi(t)$$

with Φ defined by (1). We shall now apply Moriguti's lemma. Let us recall it for completeness of the presentation.

Lemma (Moriguti (1953)). *Let* Φ, $\underline{\Phi}$ *and* $\bar{\Phi}$: $[a, b] \to \mathbf{R}$ *be continuous, nondecreasing functions such that* $\Phi(a) = \underline{\Phi}(a) = \bar{\Phi}(a)$, $\Phi(b) = \underline{\Phi}(b) = \bar{\Phi}(b)$ *and* $\underline{\Phi}(t) \leqslant \Phi(t) \leqslant \bar{\Phi}(t)$ *for every* $t \in [a, b]$. *Then the following inequalities hold*:

(i) $\int_a^b x(t)\, d\Phi(t) \leqslant \int_a^b x(t)\, d\underline{\Phi}(t)$,

(ii) $\int_a^b x(t)\, d\Phi(t) \geqslant \int_a^b x(t)\, d\bar{\Phi}(t)$

for any nondecreasing function x: $(a, b) \to \mathbf{R}$ *for which the corresponding integrals exist. The equality in* (i) *holds if and only if* x *is constant on each connected interval from the set* $\{t \in (a, b)$: $\underline{\Phi}(t) < \Phi(t)\}$. *The equality in* (ii) *holds if and only if* x *is constant on each connected interval from the set* $\{t \in (a, b)$: $\bar{\Phi}(t) > \Phi(t)\}$.

By the lemma we get

$$\int_0^1 F^{-1}(t)\bar{\varphi}(t)\,dt \leqslant EX(R, N, \tilde{m}, k) \leqslant \int_0^1 F^{-1}(t)\underline{\varphi}(t)\,dt. \tag{7}$$

One can easily check that for all $x \geqslant 0$ and $1 \neq p > 0$

$$-\frac{x^p - x}{p-1} \leqslant 1-x \tag{8}$$

with equality iff $x = 1$. Putting $x = a/(1+|p-1|\,b)^{1/(p-1)}$ with $a, b \geqslant 0$ in (8), after some algebra, yields

$$p\,\mathrm{sgn}(p-1)\,ab - \frac{a^p - a}{p-1} \leqslant (1+|p-1|\,b)^{p/(p-1)} - a, \tag{9}$$

where $\mathrm{sgn}\,x = x/|x|$. The equality is attainable iff $a = (1+|p-1|\,b)^{1/(p-1)}$. Taking $a = cF^{-1}(t)$, $c > 0$, $b = \underline{\varphi}(t)$ for $p > 1$ and $b = \bar{\varphi}(t)$ for $p \in (0, 1)$, integrating both sides of (9) and dividing by c we obtain

$$p\int_0^1 F^{-1}(t)\underline{\varphi}(t)\,dt \leqslant \int_0^1 \frac{c^{p-1}(F^{-1}(t))^p - F^{-1}(t)}{p-1}\,dt + \frac{1}{c}\int_0^1 (1+(p-1)\underline{\varphi}(t))^{p/(p-1)}\,dt - \int_0^1 F^{-1}(t)\,dt, \tag{10}$$

and

$$p\int_0^1 F^{-1}(t)\bar{\varphi}(t)\,dt \geqslant -\int_0^1 \frac{c^{p-1}(F^{-1}(t))^p - F^{-1}(t)}{p-1}\,dt - \frac{1}{c}\int_0^1 (1-(p-1)\bar{\varphi}(t))^{p/(p-1)}\,dt + \int_0^1 F^{-1}(t)\,dt. \tag{11}$$

Combining (10) and (11) with (7) completes the proof. ∎

Remark 1. If $\boldsymbol{P}(R = r, N = n) = 1$ for some $(r, n) \in \Gamma$ and $\boldsymbol{P}(X > 0) = 1$, then deriving the limits as $p \to 1^+$ and $p \to 1^-$ of the right-hand sides of (3) and (5) and minimizing with respect to c we recover, for the proper class of distributions of X, Boltzmann's entropy bounds given in Kałuszka and Okolewski ((2003), Theorem 2 with $\lambda = 1$). Of course, this approach needs additional assumptions which enable taking limits under the integral signs. Above all, the approach requires finiteness of $EX^{1+\varepsilon}$ for some $\varepsilon > 0$, and so it does not cover the case of the distributions such that $EX\log X < \infty$ and $EX^{1+\varepsilon} = \infty$ for all $\varepsilon > 0$. As an example of such a situation, consider the density function

$$f(x) = \begin{cases} \dfrac{c}{x^2(\log x)^b}, & x \geqslant a, \\ 0, & x < a, \end{cases} \tag{12}$$

where $a > 1$, $b > 2$ and c is the normalizing constant. Consequently, Theorem 1 is not a generalization of the bounds given in Kałuszka and Okolewski (2003).

Remark 2. Optimizing (3) with respect to $c > 0$ yields

$$(13)\quad EX(R, N, \tilde{m}, k) \leqslant ((p-1)\,T_p(X)+EX)^{1/p}\Big(\int_0^1 (1+(p-1)\,\underline{\varphi}(t))^q\,dt\Big)^{1/q} - EX$$

with equality iff

$$c(p)\,F^{-1}(t) = \big(1+(p-1)\,\underline{\varphi}(t)\big)^{q/p},$$

in which $1/p+1/q = 1$ and the optimal value

$$c(p) = \left(\frac{\int_0^1 (1+(p-1)\,\underline{\varphi}(t))^q\,dt}{EX^p}\right)^{1/p}.$$

Observe that (13) can be obtained directly by Hölder's inequality. Of course, the attainability condition remains unchanged. However, it seems difficult to derive the limit of the left-hand side of (13) as $p \to 1^+$. Even if one managed to show that for deterministic R and N the resulting limit inequality takes the same form as the upper Boltzmann's entropy bound of Kałuszka and Okolewski (2003), it would hold as long as $EX^p < \infty$ for some $p > 1$.

3. EXAMPLES

In this section we shall not rewrite particular bounds for order statistics, records and progressive censored type II order statistics. Instead, we shall present the distributions characterized by the corresponding attainability conditions as well as some distribution-free coefficients $u(p)$ and $l(p)$ defined by (4) and (6). Since all the distribution functions are continuous, we only provide the analytical formulae.

3.1. Order statistics. Choose $m = 0$ and $k = 1$. If $P(R = 1, N = n) = 1$, then $\bar{\varphi}(t) = n(1-t)^{n-1}$, and the lower bound (5) for $EX_{1:n}$ is attained iff

$$F(t) = 1-\left(\frac{1-(ct)^{p-1}}{n(p-1)}\right)^{1/(n-1)}$$

for $n \geqslant 2$, $c > 0$ and $p \in (0, 1)$.

Let $P(R = N) = 1$ and assume that N has the Poisson distribution with mean λ. Conventionally, we set $X_{0:0} = 0$. Then $\underline{\varphi}(t) = \lambda e^{-\lambda(1-t)}$. Equality in (3) for $EX_{N:N}$ holds iff

$$F(t) = 1+\frac{1}{\lambda}\log\frac{(ct)^{p-1}-1}{\lambda(p-1)},$$

where $c > 0$ and $p > 1$.

Now we consider bounds for $EX_{1:N}$. The first order statistic from a sample of random length is used in a stochastic model of relaxation proposed by Jurlewicz and Weron (2002). Namely, the distribution of the relaxation time $\bar{\Theta}$ of the entire system is determined by the first passage of the system from its initial state, so

$$P(\bar{\Theta} \geqslant t) = P\big(\min(\bar{\Theta}_{1,N_0}, \ldots, \bar{\Theta}_{N,N_0}) \geqslant t\big),$$

where N_0 denotes the system size, N is a random number of dipoles taking essentially part in the relaxation process, and variables $\bar{\Theta}_{i,N_0}$, $i = 1, \ldots, N$, represent the random waiting times of the particular responding dipoles for the initial state transition. There are several theories concerning the limit distribution of $\bar{\Theta}$ (see Jurlewicz and Weron (2002)). Theorem 1 provides some evaluations on expected relaxation time expressed in terms of Tsallis' entropy.

Suppose that $P(R = 1) = 1$. Assume that N has the binomial distribution with $a \in (0, 1)$, $M \geqslant 2$. Then $\bar{\varphi}(t) = aM(1-at)^{M-1}$, and the bound (5) is attainable for

$$F(t) = \frac{1}{a}\left\{1 - \left(\frac{1-(ct)^{p-1}}{aM(p-1)}\right)^{1/(M-1)}\right\},$$

where $c > 0$ and $p \in (0, 1)$. If N has the Poisson distribution with mean λ, then $\bar{\varphi}(t) = \lambda e^{-\lambda t}$. The lower bound is attained iff

$$F(t) = -\frac{1}{\lambda}\log\frac{1-(ct)^{p-1}}{\lambda(p-1)},$$

where $c > 0$ and $p \in (0, 1)$. If N has the geometrical distribution with mean $1/a$, i.e. $P(N = n) = a(1-a)^{n-1}$, $n = 1, 2, \ldots$, $a \in (0, 1)$, then $\bar{\varphi}(t) = a/(1-(1-a)(1-t))^2$ and equality holds in (5) iff

$$F(t) = 1 - \frac{1}{1-a}\left(1 - \frac{a(p-1)}{1-(ct)^{p-1}}\right)^{1/2}, \quad p \in (0, 1).$$

In particular, for $p = 0.5$, the distribution-free coefficient

$$l(p) = \frac{\sqrt{a/2}}{1-a}\arctan\frac{\sqrt{2/a}-\sqrt{2a}}{3} - 1.$$

3.2. Record statistics. Set $m = -1$ and $k = 1$. If $R = r$ with probability one, then

$$\underline{\varphi}(t) = \frac{1}{(r-1)!}\left(-\log(1-t)\right)^{r-1}.$$

The upper bound (3) for $EY_r^{(1)}$, $r \geqslant 2$, is attained iff

$$F(t) = 1 - \exp\left\{-\left(\frac{(r-1)!}{p-1}\left((ct)^{p-1} - 1\right)\right)^{1/(r-1)}\right\}, \quad t \geqslant 1/c,\ p > 1.$$

Observe that for $p=2$ we obtain the shifted Weibull distribution. Moreover, the coefficient $u(2)=3+(2r-2)!/[(r-1)!]^2$.

Assume that R has the geometrical distribution with mean $1/a$. Then

$$\varphi(t)=a/(1-t)^{1-a}.$$

The bound (3) is meaningful if $u(p)<\infty$, i.e. for $a>1/p$, $p>1$. Equality holds iff

$$F(t)=1-\left(\frac{a(p-1)}{(ct)^{p-1}-1}\right)^{1/(1-a)}$$

In case $p=2$ we get the shifted Pareto distribution. The coefficient $u(2)=3+a^2/(2a-1)$, $a>1/2$.

3.3. Progressive censored type II order statistics. Set $m_i=R_i$, where $R_i \in \{0, 1, 2, \ldots\}$, $i=1, 2, \ldots, n$, are such that $R_1+\ldots+R_n+n=M$, and $k=M-\sum_{i=1}^{n-1} m_i-n+1$. Let $P(R=N=n)=1$ for fixed $n \leqslant M$. Suppose that $\eta_n=1$. Then (see Balakrishnan et al. (2001))

$$\varphi(t)=c_{n-1}\sum_{i=1}^{n} a_{in}(1-t)^{\eta_i-1},$$

where

$$c_{n-1}=\prod_{j=1}^{n}\eta_j \quad \text{and} \quad a_{ir}=\prod_{j=1, j\neq i}^{n}\frac{1}{\eta_j-\eta_i}.$$

The upper bound for $EX_{n:n}^{\tilde{R},M}$ is attained iff

$$F^{-1}(t)=\frac{1}{c}\left(1+(p-1)c_{n-1}\sum_{i=1}^{n} a_{in}(1-t)^{\eta_i-1}\right)^{1/(p-1)}, \quad t\in(0, 1),\ p>1.$$

Note that describing the distribution via the quantile function simplifies generating pseudorandom samples from this distribution. For $p=2$,

$$u(p)=c_{n-1}^2\sum_{i,j=1}^{n}\frac{a_{in}a_{jn}}{\eta_i+\eta_j-1}+3.$$

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