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BESSEL POTENTIALS, GREEN FUNCTIONS AND EXPONENTIAL FUNCTIONALS ON HALF-SPACES*

BY

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Abstract. The purpose of the paper is to provide precise estimates for the Green function corresponding to the operator $(I-\Delta)^{\alpha/2}$, $0 < \alpha < 2$. The potential theory of this operator is based on Bessel potentials $J_\alpha = (I-\Delta)^{-\alpha/2}$. In probabilistic terms it corresponds to a subprobabilistic process obtained from the so-called *relativistic α-stable process.* We are interested in the theory of the *killed process* when exiting a fixed half-space. The crucial rôle in our research is played by (recently found) an explicit form of the *Green function* of a half-space. We also examine properties of some exponential functionals corresponding to the operator $(I-\Delta)^{\alpha/2}$.

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1. INTRODUCTION

As E. M. Stein remarked in his monograph [10], *fractional Sobolev spaces* and *potential spaces* are among the most important Banach spaces of functions to analyze various problems from analysis and potential theory.

While fractional Sobolev spaces are defined in terms of *Riesz potentials* $I_\alpha = (-\Delta)^{-\alpha/2}$, potential spaces employ *Bessel potentials* $J_\alpha = (I-\Delta)^{-\alpha/2}$.

It is remarkable that both these objects are closely related to the potential theory of specific Lévy processes: in the first case it is the *d-dimensional symmetric* (rotation invariant) *α-stable* Lévy process; in the latter case we have to deal with the so-called *α-stable relativistic* Lévy process. More specifically, the operator $-(-\Delta)^{\alpha/2}$ is the infinitesimal generator of the symmetric *α-stable* Lévy process and I_α is its (formal) inverse. For the Bessel potential J_α the situation is

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more complicated: the operator $-(I-\Delta)^{\alpha/2}$ is the generator for a subprobabilistic process with the potential J_α. More precisely, the operator $I-(I-\Delta)^{\alpha/2}$ is the generator of the relativistic (α-stable) process and, in probabilistic terms, J_α is its 1-*potential*.

Potential theory based on Riesz kernels (or, equivalently: potential theory for α-stable rotation invariant Lévy process) is well developed and rich in explicit formulas, much like in the classical case of Brownian motion process. The homogeneity of Riesz kernels yields many elegant and transparent formulas for harmonic measure and Green function for such basic sets as balls and half-spaces in $\boldsymbol{R}^d$ (see e.g. [1]). These formulas played an important rôle in setting up the so-called *boundary potential theory* of the operator $-(-\Delta)^{\alpha/2}$ and the Schrödinger operator based on it (see e.g. [2] and [3]).

In contrast to this situation, up to now there were no explicit formulas known either for harmonic measure or for Green function for the relativistic process for sets such as half-planes or balls. Nevertheless, an adequate boundary potential theory for bounded smooth sets was set up by Ryznar [9].

In the recent paper [5] explicit formulas for harmonic measure and Green function for half-spaces for the operator $-(I-\Delta)^{\alpha/2}$ are given. The purpose of the present paper is to extend results obtained in [4] for the operator $-(-\Delta)^{\alpha/2}$ to the case of the operator $-(I-\Delta)^{\alpha/2}$. The basic tool employed in this paper consists of the formula for the Green function. Section 3 contains very precise estimates for this function. In the next section we examine an exponential functional $u_q^1(x, b)$. The decisive rôle is played again by the Green function. The paper ends with some examples where the *critical value* b_0 for $u_q^1(x, b)$ is evaluated for various potentials q.

2. PRELIMINARIES

We present here some basic material regarding the α-stable relativistic processes. For more detailed informations the reader is referred to [9] and [6].

We first introduce an appropriate class of subordinating processes. By $T_\beta(t)$ we denote the strictly β-stable positive standard subordinator with the Laplace transform

$$E^0 \exp\big(-\lambda T_\beta(t)\big) = \exp(-t\lambda^\beta), \qquad 0 < \beta < 1. \tag{1}$$

Let $\theta_\beta(t, u)$, $u > 0$, denote the density function of $T_\beta(t)$. Next, if B_t is the symmetric Brownian motion in $\boldsymbol{R}^d$ with characteristic function of the form

$$E^0 \exp(i\xi \cdot B_t) = \exp(-t|\xi|^2), \tag{2}$$

then the process $B_{T_\beta(t)}$ is the standard symmetric α-stable process, under the usual assumption that the processes $T_\beta(t)$ and $B(t)$ are stochastically independent.

Now, for $m > 0$ and $t > 0$ define a probability density function

$$\theta_\beta(t, u, m) = e^{mt}\theta_\beta(t, u)\exp(-m^{1/\beta}u), \quad u > 0.$$

Applying (1) we derive the Laplace transform of $\theta_\beta(t, u, m)$:

$$E^0\exp(-\lambda T_\beta(t, u, m)) = e^{mt}\exp(-t(\lambda+m^{1/\beta})^\beta). \tag{3}$$

We define the *α-stable relativistic density* (with parameter m) by the following formula:

$$p_t^m(x) = \int_0^\infty \theta_\beta(t, u, m)g_u(x)\,du, \tag{4}$$

where $g_u(x)$ is the Brownian semigroup, defined by (2).

Let K_ν, $\nu \in \mathbf{R}$, be the Macdonald function with index ν, called also the *modified Bessel function of the second kind*, which is given by the following formula:

$$K_\nu(r) = 2^{-1-\nu}r^\nu\int_0^\infty e^{-v}\exp\left(-\frac{r^2}{4v}\right)v^{-1-\nu}\,dv, \quad r > 0.$$

For properties of K_ν we refer the reader to [8]. In the sequel we will use the asymptotic behaviour of K_ν:

$$K_\nu(r) \cong \frac{\Gamma(\nu)}{2}\left(\frac{r}{2}\right)^{-\nu}, \quad r \to 0^+,\ \nu > 0, \tag{5}$$

$$K_0(r) \cong -\log r, \quad r \to 0^+, \tag{6}$$

$$K_\nu(r) \cong \frac{\sqrt{\pi}}{\sqrt{2r}}e^{-r}, \quad r \to \infty, \tag{7}$$

where $g(r) \cong f(r)$ means that the ratio of g and f tends to 1. For $\nu < 0$ we have $K_\nu(r) = K_{-\nu}(r)$, which determines the asymptotic behaviour for negative indices.

A particular case of an α-stable relativistic density when $\alpha = 1$ is called the *relativistic Cauchy semigroup* on $\mathbf{R}^d$ with parameter m. The following formula exhibits the explicit form of this density:

LEMMA 2.1 (*relativistic Cauchy semigroup*). *The density $\tilde{p}_t^m$ of the relativistic Cauchy process is of the form*:

$$\tilde{p}_t^m(x) = 2(m/2\pi)^{(d+1)/2}te^{mt}\frac{K_{(d+1)/2}\left(m(|x|^2+t^2)^{1/2}\right)}{(|x|^2+t^2)^{(d+1)/4}}.$$

The Fourier transform of the transition density (4) is of the following form:

LEMMA 2.2 (*Fourier transform of p_t^m*). *The Fourier transform of α-stable relativistic density p_t^m is of the form*:

$$\widehat{p_t^m}(z) = e^{mt}\exp\left(-t\left(|z|^2+m^{2/\alpha}\right)^{\alpha/2}\right).$$

Using the Fourier transform we obtain the following scaling property:

$$p_t^m(x) = m^{d/\alpha}\, p_{mt}^1(m^{1/\alpha}x).$$

In terms of one-dimensional distributions of the relativistic process (starting from the point 0) we obtain

$$X_t^m \stackrel{d}{=} m^{-1/\alpha} X_{mt}^1, \tag{8}$$

where X_t^m denotes the relativistic α-stable process with parameter m, and "$\stackrel{d}{=}$" means equality of distributions.

In what follows we put $p_t^1(x) = p_t(x)$ (i.e. for $m = 1$). By $U_\lambda^m(x)$ we denote the λ-potential of $p_t^m(x)$, that is,

$$U_\lambda^m(x) = \int_0^\infty e^{-\lambda t} p_t^m(x)\,dt.$$

Again we denote by $U_\lambda(x)$ the λ-potential in the case $m = 1$.

LEMMA 2.3 (*m-potential for relativistic process with parameter m*). *We have*

$$U_m^m(x) = C(\alpha, d)\, m^{(d-\alpha)/2\alpha}\frac{K_{(d-\alpha)/2}(m^{1/\alpha}|x|)}{|x|^{(d-\alpha)/2}},$$

where

$$C(\alpha, d) = \frac{2^{1-(d+\alpha)/2}}{\Gamma(\alpha/2)\,\pi^{d/2}}.$$

We also recall the form of the density function $\nu(x)$ of the Lévy measure and the infinitesimal generator of the relativistic α-stable process (see, e.g., [9]):

LEMMA 2.4 (*Lévy measure and generator of relativistic process*). *The density ν of the Lévy measure of the relativistic process with parameter m is of the form*:

$$\nu(x) = \frac{\alpha 2^{(\alpha-d)/2}}{\pi^{d/2}\,\Gamma(1-\alpha/2)}\left(\frac{m^{1/\alpha}}{|x|}\right)^{(d+\alpha)/2} K_{(d+\alpha)/2}(m^{1/\alpha}|x|),$$

while the generator is given by the formula

$$H = mI - (m^{2/\alpha}I - \Delta)^{\alpha/2}.$$

We now state two results from the paper [5]. In what follows we put $\boldsymbol{H}_b = \{x \in \boldsymbol{R}^d;\ x_d < b\}$, and $\delta(x)$ denotes the distance from the point $x \in \boldsymbol{H}_b$ to the boundary of the set $\boldsymbol{H}_b$. The first result provides the formula for the density function of m-harmonic measure for the set $\boldsymbol{H}_b$ for the α-stable relativistic process with parameter m:

$$P_m(x, u) = 2C_\alpha^1 \left(\frac{m^{1/\alpha}}{2\pi}\right)^{d/2} \left(\frac{b-x_d}{u_d-b}\right)^{\alpha/2} \frac{K_{d/2}(m^{1/\alpha}|x-u|)}{|x-u|^{d/2}}, \quad x_d < b < u_d.$$

The second formula is of a prime importance here; it provides the m-Green function for the set H_b for the relativistic process with parameter m. The m-Green function can be defined as the density function of the m-potential (or m-resolvent) for the process killed when leaving the set H_b. We have

(9) $\quad G^m_{H_b}(x, y)$

$$= \frac{2^{1-\alpha} m^{d/2\alpha}|x-y|^{\alpha-d/2}}{(2\pi)^{d/2}\Gamma(\alpha/2)^2} \int\limits_0^{4\delta(x)\delta(y)/|x-y|^2} \frac{t^{\alpha/2-1}}{(t+1)^{d/4}} K_{d/2}\left(m^{1/\alpha}|x-y|(t+1)^{1/2}\right) dt.$$

In the sequel we always take $m = 1$; results for the general case can be easily obtained taking into account the appropriate form of the scaling property (8). Hence in the notation we drop the parameter $m = 1$: for example, $G_{H_b}(x, y)$ denotes $G^1_{H_b}(x, y)$.

Throughout the paper, by c, C we denote nonnegative constants which may depend on other constant parameters only. The value of c or C may change from line to line in a chain of estimates.

The notion $p(u) \approx q(u)$, $u \in A$, means that the ratio $p(u)/q(u)$, $u \in A$, is bounded from below and above by positive constants which may depend on other constant parameters only.

3. GREEN FUNCTION OF $-(I-\Delta)^{\alpha/2}$ FOR H_b

In the one-dimensional case we have

$$H_b = (-\infty, b), \qquad K_{1/2}(r) = \frac{\sqrt{\pi}}{\sqrt{2r}} e^{-r}$$

and the formula (9) becomes

$$\text{(10)} \qquad G_{(-\infty,b)}(x, y) = \frac{|x-y|^{\alpha-1}}{2^\alpha \Gamma(\alpha/2)^2} \int\limits_0^{4\delta(x)\delta(y)/(x-y)^2} \frac{\exp\left(-|x-y|(t+1)^{1/2}\right)}{t^{1-\alpha/2}(t+1)^{1/2}} dt.$$

An equivalent and very useful version of the above formula reads as follows:

$$\text{(11)} \qquad G_{(-\infty,b)}(x, y) = \frac{e^{-|x-y|}}{\Gamma(\alpha/2)^2} \int\limits_0^{\delta(x)\wedge\delta(y)} \frac{e^{-2v} v^{\alpha/2-1}}{(v+|x-y|)^{1-\alpha/2}} dv.$$

In the general d-dimensional case apart from (9) we have the following equivalent formula for the Green function of the set $\boldsymbol{H}_b$:

$$G_{\boldsymbol{H}_b}(x, y) = \frac{2^{1-\alpha}}{(2\pi)^{d/2}\,\Gamma(\alpha/2)^2} \int\limits_0^{4\delta(x)\delta(y)} s^{\alpha/2-1} \frac{K_{d/2}\big((|x-y|^2+s)^{1/2}\big)}{(|x-y|^2+s)^{d/4}}\,ds.$$

If we substitute $(|x-y|^2+s)^{1/2} = v+|x-y|$, then

$$G_{\boldsymbol{H}_b}(x, y) = \frac{2^{1-\alpha}}{(2\pi)^{d/2}\,\Gamma(\alpha/2)^2} \int\limits_0^{z^*|x-y|} \frac{v^{\alpha/2-1}\,(2\,|x-y|+v)^{\alpha/2-1}}{(|x-y|+v)^{d/2-1}} K_{d/2}(|x-y|+v)\,dv,$$

where

$$z^* = \sqrt{\frac{4\delta(x)\,\delta(y)}{|x-y|^2}+1}-1.$$

In order to provide estimates for the Green function in the case $\alpha < d$ we consider the following integral for $a, b > 0$:

$$I(a, b) = \int\limits_0^b \frac{v^{\alpha/2-1}\,(2a+v)^{\alpha/2-1}}{(a+v)^{d/2-1}} K_{d/2}(a+v)\,dv.$$

To get estimates in the case $\alpha \geqslant 1 = d$ it is convenient to use another integral depending on $a, b > 0$:

$$L(a, b) = \int\limits_0^b \frac{e^{-2v}\,v^{\alpha/2-1}}{(v+a)^{1-\alpha/2}}\,dv.$$

Note that

$$G_{\boldsymbol{H}_b}(x, y) = \frac{2^{1-\alpha}}{(2\pi)^{d/2}\,\Gamma(\alpha/2)^2} I(|x-y|,\, z^*\,|x-y|)$$

and

$$G_{(-\infty,b)}(x, y) = \frac{e^{-|x-y|}}{\Gamma(\alpha/2)^2} L\big(|x-y|,\, \delta(x)\wedge\delta(y)\big)$$

in the one-dimensional case. Thus to obtain estimates for the Green function it is enough to find estimates for the integrals I and L, which is carried out in the following lemma.

LEMMA 3.1. *If* $\alpha < d$, *then*

(12) $$I(a, b) \approx \frac{K_{(d-\alpha)/2}(a)}{a^{(d-\alpha)/2}} \left[\left(\frac{b\wedge 1}{a\wedge 1}\right)^{\alpha/2} \wedge 1\right].$$

If $\alpha = 1 = d$, *then*

(13) $$L(a, b) \approx a^{\alpha/2-1}\,(b\wedge 1)^{\alpha/2}, \qquad b \leqslant a \ \textit{or}\ 1 \leqslant a \leqslant b,$$

(14) $$L(a, b) \approx \log\left(2\frac{b\wedge 1}{a}\right), \qquad a \leqslant 1 \leqslant b \ \textit{or}\ a \leqslant b \leqslant 1.$$

If $\alpha > d = 1$, *then*

(15) $$L(a, b) \approx a^{\alpha/2-1}(b \wedge 1)^{\alpha/2}, \qquad b \leqslant a \text{ or } 1 \leqslant a \leqslant b,$$

(16) $$L(a, b) \approx (a \vee 1)^{\alpha/2}(b \wedge 1)^{\alpha-1}, \qquad a \leqslant 1 \leqslant b \text{ or } a \leqslant b \leqslant 1.$$

Proof. We begin with estimating I:

$$\begin{aligned} I(a, b) &= \int_0^b \frac{v^{\alpha/2-1}(2a+v)^{\alpha/2-1}}{(a+v)^{d/2-1}} K_{d/2}(a+v)\,dv \\ &= \int_0^{b\wedge 1} \frac{v^{\alpha/2-1}(2a+v)^{\alpha/2-1}}{(a+v)^{d/2-1}} K_{d/2}(a+v)\,dv \\ &\quad + \int_{b\wedge 1}^{b} \frac{v^{\alpha/2-1}(2a+v)^{\alpha/2-1}}{(a+v)^{d/2-1}} K_{d/2}(a+v)\,dv \\ &= I(a, b\wedge 1)+R(a, b), \end{aligned}$$

where

$$R(a, b) = \int_{b\wedge 1}^{b} \frac{v^{\alpha/2-1}(2a+v)^{\alpha/2-1}}{(a+v)^{d/2-1}} K_{d/2}(a+v)\,dv.$$

We will later show that $R(a, b)$ is at most of the same order as $I(a, b\wedge 1)$. Namely, there is a constant C such that

(17) $$R(a, \infty) \leqslant CI(a, 1).$$

Hence $I(a, b) \approx I(a, b\wedge 1)$ and it is enough to consider the case $b \leqslant 1$. Note that

(18) $$\begin{aligned} I(a, b) &= \int_0^b \frac{v^{\alpha/2-1}(2a+v)^{\alpha/2-1}}{(a+v)^{d/2-1}} K_{d/2}(a+v)\,dv \\ &= a^{\alpha-d/2} \int_0^{b/a} \frac{v^{\alpha/2-1}(2+v)^{\alpha/2-1}}{(1+v)^{d/2-1}} K_{d/2}\big(a(1+v)\big)\,dv \\ &\approx a^{\alpha-d/2} \int_0^{b/a} v^{\alpha/2-1}(1+v)^{(\alpha-d)/2} K_{d/2}\big(a(1+v)\big)\,dv. \end{aligned}$$

First assume that $a \leqslant 1$ and $b \leqslant 1$. From (5) we have $K_{d/2}(v) \approx v^{-d/2}$, $0 < v < 1$. Thus

$$I(a, b) \approx a^{\alpha-d} \int_0^{b/a} v^{\alpha/2-1}(1+v)^{\alpha/2-d}\,dv \approx a^{\alpha-d}\{(b/a)^{\alpha/2} \wedge 1\} \qquad \text{if } \alpha < d.$$

Next we consider $b \leqslant 1 \leqslant a$. Then by (7) we have

$$K_{d/2}(a+v) \approx \frac{1}{(a+v)^{1/2}} e^{-a-v},$$

and hence

$$I(a, b) = \int_0^b \frac{v^{\alpha/2-1}(2a+v)^{\alpha/2-1}}{(a+v)^{d/2-1}} K_{d/2}(a+v)\, dv$$
$$\approx e^{-a} \int_0^b \frac{v^{\alpha/2-1}(2a+v)^{\alpha/2-1}}{(a+v)^{(d-1)/2}} e^{-v}\, dv \approx a^{-(d+1)/2+\alpha/2} b^{\alpha/2} e^{-a}.$$

Now, if $\alpha < d$, taking into account that $I(a, b) \approx I(a, b \wedge 1)$ and applying the estimates for the Macdonald function K_ν (see (5) and (7)) we can write a unified estimate

$$I(a, b) \approx \frac{K_{(d-\alpha)/2}(a)}{a^{(d-\alpha)/2}} \left[\left(\frac{b \wedge 1}{a \wedge 1}\right)^{\alpha/2} \wedge 1\right],$$

which is (12). To end the case $\alpha < d$ we need to establish (17). We have

$$R(a, \infty) = \int_1^\infty \frac{v^{\alpha/2-1}(2a+v)^{\alpha/2-1}}{(a+v)^{d/2-1}} K_{d/2}(a+v)\, dv$$
$$\approx e^{-a} \int_1^\infty v^{\alpha/2-1}(a+v)^{(\alpha-d-1)/2-1} e^{-v}\, dv$$
$$\approx e^{-a}\left(\frac{1}{a^{(d+1-\alpha)/2}} \wedge 1\right) \leqslant C \frac{K_{(d-\alpha)/2}(a)}{a^{(d-\alpha)/2}},$$

which proves (17).

Now we deal with the situation $\alpha \geqslant d = 1$. Consider first the case when $b \leqslant a$:

$$\frac{1}{(2a)^{1-\alpha/2}} \int_0^b \frac{e^{-2v}}{v^{1-\alpha/2}}\, dv \leqslant \int_0^b \frac{e^{-2v} v^{\alpha/2-1}}{(v+a)^{1-\alpha/2}}\, dv \leqslant \frac{1}{a^{1-\alpha/2}} \int_0^b \frac{e^{-2v}}{v^{1-\alpha/2}}\, dv.$$

This shows that

$$L(a, b) \approx a^{\alpha/2-1}(b \wedge 1)^{\alpha/2},$$

which proves (13) and (15) for $b \leqslant a$.

Next, we consider the case when $b \geqslant a \geqslant 1$:

$$\frac{1}{(2a)^{1-\alpha/2}} \int_0^1 \frac{e^{-2v}}{v^{1-\alpha/2}}\, dv \leqslant \int_0^b \frac{e^{-2v} v^{\alpha/2-1}}{(v+a)^{1-\alpha/2}}\, dv \leqslant \frac{1}{a^{1-\alpha/2}} \int_0^\infty \frac{e^{-2v}}{v^{1-\alpha/2}}\, dv.$$

This obviously shows that

$$L(a, b) \approx \frac{1}{a^{1-\alpha/2}},$$

which proves (13) and (15) for $1 \leqslant a \leqslant b$.

Now, what remains is the case when $b \geqslant a$ but $a \leqslant 1$. Suppose that $\alpha > 1$ and $b \geqslant 1$. Then, taking into account that $a \leqslant 1$, we have

$$\frac{1}{(1+a)^{1-\alpha/2}} \int_0^1 \frac{e^{-2v}}{v^{1-\alpha/2}} dv \leqslant \int_0^b \frac{e^{-2v} v^{\alpha/2-1}}{(v+a)^{1-\alpha/2}} dv \leqslant \int_0^\infty \frac{e^{-2v}}{v^{2-\alpha}} dv.$$

This shows that in this case

$$L(a, b) \approx 1. \tag{19}$$

Assuming next that $\alpha = 1$ and $a \leqslant 1 \leqslant b$ we obtain

$$\begin{aligned} \int_0^1 \frac{e^{-2v}}{v^{1/2}(v+a)^{1/2}} dv &\leqslant \int_0^b \frac{e^{-2v}}{v^{1/2}(v+a)^{1/2}} dv \\ &\leqslant \int_0^1 \frac{e^{-2v}}{v^{1/2}(v+a)^{1/2}} dv + \int_1^\infty \frac{e^{-2v}}{v} dv \leqslant \int_0^1 \frac{e^{-2v}}{v^{1/2}(v+a)^{1/2}} dv + 1. \end{aligned}$$

Next we have

$$\int_0^1 \frac{e^{-2v}}{v^{1/2}(v+a)^{1/2}} dv = \int_0^{1/a} \frac{e^{-2ua}}{u^{1/2}(u+1)^{1/2}} du \approx \log\frac{2}{a}. \tag{20}$$

Thus (19) and (20) give (16) and (14), respectively, for $a \leqslant 1 \leqslant b$.

The last case to examine is $a \leqslant b \leqslant 1$. Then

$$e^{-2} \int_0^b \frac{u^{\alpha/2-1}}{(u+a)^{1-\alpha/2}} du \leqslant \int_0^b \frac{e^{-2u} u^{\alpha/2-1}}{(u+a)^{1-\alpha/2}} du \leqslant \int_0^b \frac{u^{\alpha/2-1}}{(u+a)^{1-\alpha/2}} du.$$

Hence in this case

$$\begin{aligned} L(a, b) \approx \int_0^b \frac{u^{\alpha/2-1}}{(u+a)^{1-\alpha/2}} du &= a^{\alpha-1} \int_0^{b/a} \frac{u^{\alpha/2-1}}{(u+1)^{1-\alpha/2}} du \\ &\approx \begin{cases} \log(2b/a) & \text{if } \alpha = 1, \\ b^{\alpha-1} & \text{if } \alpha > 1. \end{cases} \end{aligned}$$

This completes the proof of the last case and ends the proof of the lemma. ∎

Lemma 3.1 together with the observation that

$$z^* |x-y| \approx \frac{\delta(x) \wedge \delta(y)}{|x-y|} \quad \text{for } \frac{\delta(x)\delta(y)}{|x-y|^2} \leqslant 1$$

gives immediately the following estimates for the Green functions.

THEOREM 3.2. *Assume that $d = 1$ and $\alpha \geqslant 1$. When $|x-y| \geqslant 1 \wedge \delta(x) \wedge \delta(y)$ we obtain*

$$G_{(-\infty,b)}(x, y) \approx \frac{e^{-|x-y|}}{|x-y|^{1-\alpha/2}} (1 \wedge \delta(x) \wedge \delta(y))^{\alpha/2},$$

while for $|x-y| < 1 \wedge \delta(x) \wedge \delta(y)$ *we get*

$$G_{(-\infty,b)}(x, y) \approx \begin{cases} \log\left[2\dfrac{1 \wedge \delta(x) \wedge \delta(y)}{|x-y|}\right] & \text{if } \alpha = 1, \\ \left(1 \wedge \delta(x) \wedge \delta(y)\right)^{\alpha-1} & \text{if } \alpha > 1. \end{cases}$$

In the remaining case, $\alpha < d$, *we have*

$$G_{H_b}(x, y) \approx \frac{K_{(d-\alpha)/2}(|x-y|)}{|x-y|^{(d-\alpha)/2}}\left[\left(\frac{\delta(x) \wedge \delta(y) \wedge 1}{|x-y| \wedge 1}\right)^{\alpha/2} \wedge 1\right] \tag{21}$$

$$\approx U_1(x, y)\left[\left(\frac{\delta(x) \wedge \delta(y) \wedge 1}{|x-y| \wedge 1}\right)^{\alpha/2} \wedge 1\right]. \tag{22}$$

For $0 < \alpha < 1 = d$ *we may write a more explicit form of* (21):

$$G_{H_b}(x, y) \approx \frac{e^{-|x-y|}}{|x-y|^{1-\alpha/2}}\left(|x-y| \wedge \delta(x) \wedge \delta(y)\right)^{\alpha/2}.$$

Another very useful estimate for the Green function is obtained from the "sweeping out" principle. As a result, we obtain

$$G_{H_b}(x, y) = U_1(x-y) - \int_{H_b^c} U_1(u, y) P_1(x, u)\, du$$

$$\leqslant U_1(x, y) = \frac{2^{1-(d+\alpha)/2}}{\pi^{d/2}\Gamma(\alpha/2)} \frac{K_{(d-\alpha)/2}(|x-y|)}{|x-y|^{(d-\alpha)/2}}.$$

Applying the estimates for the Macdonald function K_ν (see (5)–(7)) we obtain

COROLLARY 3.3. *We have the following estimates*:

$$G_{H_b}(x, y) \leqslant \frac{2^{1-(d+\alpha)/2}}{\pi^{d/2}\Gamma(\alpha/2)} \frac{K_{(d-\alpha)/2}(|x-y|)}{|x-y|^{(d-\alpha)/2}}$$

$$\approx \frac{e^{-|x-y|}}{|x-y|^{(d-\alpha)/2}}\left(\frac{1}{|x-y|^{(d-\alpha)/2}} \wedge \frac{1}{|x-y|^{1/2}}\right) \quad \text{for } \alpha < d \leqslant \alpha+1,$$

$$G_{H_b}(x, y) \leqslant \frac{2^{1-(d+\alpha)/2}}{\pi^{d/2}\Gamma(\alpha/2)} \frac{K_{(d-\alpha)/2}(|x-y|)}{|x-y|^{(d-\alpha)/2}}$$

$$\approx \frac{e^{-|x-y|}}{|x-y|^{(d-\alpha)/2}}\left(\frac{1}{|x-y|^{(d-\alpha)/2}} \vee \frac{1}{|x-y|^{1/2}}\right) \quad \text{for } \alpha+1 < d,$$

$$G_{H_b}(x, y) \leqslant \frac{1}{\pi} K_0(|x-y|) \approx e^{-|x-y|}\left(\log\frac{2}{|x-y|} \wedge \frac{1}{|x-y|^{1/2}}\right) \quad \text{for } \alpha = 1 = d,$$

$$G_{H_b}(x, y) \leqslant \frac{2^{(1-\alpha)/2}}{\sqrt{\pi}\,\Gamma(\alpha/2)} \frac{K_{(1-\alpha)/2}(|x-y|)}{|x-y|^{(1-\alpha)/2}}$$

$$\approx e^{-|x-y|}\left(1 \wedge \frac{1}{|x-y|^{1-\alpha/2}}\right) \quad \textit{for } \alpha > 1 = d.$$

4. EXPONENTIAL FUNCTIONALS CORRESPONDING TO $-(I-\Delta)^{\alpha/2}$

We now consider some exponential functionals for the operator $-(I-\Delta)^{\alpha/2}$.

The typical assumption on potentials q defining the exponential factor in the Feynman–Kac theory is that it belongs to the *Kato class* $\mathcal{J}_1^\alpha$, determined by the operator $-(I-\Delta)^{\alpha/2}$. It is, in fact, defined in terms of the potential U_1 as follows:

DEFINITION 4.1. We say that a Borel function q on $\boldsymbol{R}^d$ belongs to the *Kato class* $\mathcal{J}_1^\alpha$ if

$$\limsup_{\gamma \downarrow 0} \sup_{x \in \boldsymbol{R}^d} \int_{|x-y| \leqslant \gamma} U_1(|x-y|)\,|q(y)|\,dy = 0;$$

we write $q \in \mathcal{J}_{1\mathrm{loc}}^\alpha$ if for every bounded Borel set B we have $\mathbf{1}_B q \in \mathcal{J}_1^\alpha$.

LEMMA 4.2 (*properties of the class* $\mathcal{J}_1^\alpha$). (i) *We have* $L^\infty(\boldsymbol{R}^d) \subseteq \mathcal{J}_1^\alpha$. *If* $f \in L^\infty(\boldsymbol{R}^d)$ *and* $q \in \mathcal{J}_1^\alpha$, *then* $fq \in \mathcal{J}_1^\alpha$.

(ii) *If* $q \in \mathcal{J}_1^\alpha$, *then*

$$\sup_{x \in \boldsymbol{R}^d} \int_{|x-y| \leqslant 1} |q(y)|\,dy < \infty.$$

Hence, if $q \in \mathcal{J}_{1\mathrm{loc}}^\alpha$, *then* $q \in L_{\mathrm{loc}}^1(\boldsymbol{R}^d)$.

Remark 1. Since the local behaviour of the potential U_1 is identical with that of potentials K_α (or compensated potentials, if $d = 1 \leqslant \alpha$) in the case of the standard symmetric (rotation invariant) stable processes (see, e.g., [4]), the Kato class defined in terms of the potential U_1 coincides with the corresponding one for the symmetric (i.e. rotation invariant) stable process.

THEOREM 4.3. *Assume that* $q \in \mathcal{J}_1^\alpha$ *and*

$$\int_{|y|>1} \frac{e^{-|y|}\,|q(y)|\,dy}{(1+|y|)^{(d+1-\alpha)/2}} < \infty. \tag{23}$$

Then $G_{H_b}|q|(x) < \infty$. *If* $d = 1$ *and* $q \geqslant 0$, *then the condition* (23) *is also necessary. If* $q \in \mathcal{J}_1^\alpha$ *and* $q \in L^1(\boldsymbol{R}^d)$, *then* $G_{H_b}q(x)$ *is a continuous function of* x *and*

$$\lim_{b \to -\infty} \sup_{x_d \leqslant b} G_{H_b}|q|(x) = 0.$$

Proof. We prove the first part of the theorem. For a fixed γ, $0<\gamma<1$, we obtain

$$G_{H_b}|q|(x) = \int\limits_{y_d \leqslant b} G_{H_b}(x, y)|q(y)|\,dy$$

$$\leqslant \int\limits_{y_d \leqslant b, |x-y| \leqslant \gamma} U_1(|x-y|)|q(y)|\,dy + \int\limits_{y_d \leqslant b, |x-y| > \gamma} G_{H_b}(x, y)|q(y)|\,dy.$$

We have here $G_{H_b}(x, y) \leqslant U_1(|x-y|)$. This and the assumption $q \in \mathscr{J}_1^\alpha$ yield that for a given $\varepsilon > 0$ the supremum over $x \in \boldsymbol{R}^1$ of the first term on the right-hand side of the above equality is less than ε, whenever γ is small enough. On the other hand, if $|x-y| \geqslant \gamma$, then

$$\frac{|y|+1}{|x-y|} \leqslant \frac{|x-y|+|x|+1}{|x-y|} \leqslant 1 + \frac{|x|+1}{|x-y|} \leqslant 1+\gamma^{-1}(|x|+1).$$

Therefore, the integrand in the second term is continuous in x, vanishes at b and is bounded by

$$U_1(|x-y|)|q(y)| \leqslant Ce^{|x|}[1+\gamma^{-1}(|x|+1)]^{(d+1-\alpha)/2} \frac{e^{-|y|}|q(y)|}{(1+|y|)^{(d+1-\alpha)/2}} \in L^1(\boldsymbol{H}_b).$$

We now prove that if $d=1$, $q \geqslant 0$ and the condition (23) fails, then $G_{(-\infty,b)}q(x) = \infty$. Assume that $y < 2x-b = x-(b-x) < x$. Then we have $(b-x)/(x-y) < 1$. Thus, we obtain

$$G_{(-\infty,b)}q(x) \geqslant \frac{1}{\Gamma(\alpha/2)^2} \int\limits_{-\infty}^{2x-b} \frac{e^{-(x-y)}}{(x-y)^{1-\alpha}} \int\limits_0^{\delta(x)/(x-y)} \frac{e^{-2u(x-y)}u^{\alpha/2-1}\,du}{(u+1)^{1-\alpha/2}}\, q(y)\,dy$$

$$\geqslant \frac{2^{\alpha/2}e^{x-2b}}{\alpha\Gamma(\alpha/2)^2} \int\limits_{-\infty}^{2x-b} \frac{e^y}{(x-y)^{1-\alpha/2}} (b-x)^{\alpha/2} q(y)\,dy.$$

Since obviously $x-y \leqslant b-y$, the last expression is not less than

$$\frac{2^{\alpha/2}e^{x-2b}}{\alpha\Gamma(\alpha/2)^2}(b-x)^{\alpha/2} \int\limits_{-\infty}^{2x-b} \frac{e^y q(y)}{(b-y)^{1-\alpha/2}}\,dy$$

$$\geqslant \frac{2^{\alpha/2}e^{x-2b}}{\alpha\Gamma(\alpha/2)^2}(b-x)^{\alpha/2} \int\limits_{-\infty}^{2x-b} \frac{e^{-|y|}q(y)}{(|b|+|y|)^{1-\alpha/2}}\,dy.$$

This proves that for $d=1$ and $q \geqslant 0$ the condition (23) is also necessary.

If we assume that $q \in L^1(\boldsymbol{R}^d)$, then we obtain

$$\sup_{x_d \leqslant b} G_{H_b}|q|(x) \leqslant \varepsilon + Ce^{-\gamma}\gamma^{\alpha/2-1} \int\limits_{y_d \leqslant b} |q(y)|\,dy,$$

where ε and γ are as before. This completes the proof. ∎

As explained in the Preliminaries we consider the operator $-(I-\Delta)^{\alpha/2}$ as the infinitesimal generator of the Feynman–Kac semigroup $(T_t)_{t>0}$ based on

$I-(I-\Delta)^{\alpha/2}$ with the potential $q = -1$. This semigroup acts according to the following formula:

$$T_t f(x) = E^x e^{-t} f(X_t), \qquad f \in L^\infty.$$

We thus define the exponential functional with respect to this semigroup as follows:

$$e_q^1(t) = \exp\Big(\int_0^t e^{-u} q(X_u)\, du\Big).$$

We define the exit time of H_b as $\tau_{H_b} = \inf\{t\colon X_t \notin H_b\}$. We further define the *fundamental expectation* related to the operator $-(I-\Delta)^{\alpha/2}$ as $u_q^1(x, b) = E^x e_q^1(\tau_{H_b})$. In the remaining part of the paper we attempt to establish criteria under which the functional $u_q^1(x, b)$ is finite.

Write

$$A(t) = \int_0^t e^{-u} q(X_u)\, du.$$

We obtain

$$A(t+s) = A(s) + e^{-s} A(t) \circ \theta_s.$$

Observe that for nonnegative potentials q this implies subadditivity of the functional $A(t)$, so a version of Khasminskii's lemma follows (see, e.g., [7]).

LEMMA 4.4 (*Khasminskii's lemma*). *Suppose that* $q \geqslant 0$. *Then for all* n

$$\sup_x E^x [A(\tau_{H_b})^n] \leqslant n! \sup_x E^x [A(\tau_{H_b})]^n.$$

If

$$\sup_x E^x [A(\tau_{H_b})] = r < 1,$$

then

$$\sup_x E^x \big[\exp\big(A(\tau_{H_b})\big)\big] \leqslant 1/(1-r).$$

Our purpose is to evaluate for which x and b such that $x_d < b$ the functional $u_q^1(x, b)$ is finite. Under the usual convention that the supremum over empty set equals $-\infty$ we have the following

LEMMA 4.5. *Define for* $q \in \mathscr{J}_1^\alpha$

$$b_0 = \sup\{b \in \boldsymbol{R}^1;\ \sup_{x_d \leqslant b} G_{H_b} |q|(x) < 1\}.$$

If $q \in L^1(\boldsymbol{R}^d)$, *then* $b_0 > -\infty$ *and for* $x_d < b < b_0$ *we obtain* $0 < u_q^1(x, b) < \infty$.

Proof. The proof follows easily from Khasminskii's lemma and Theorem 4.3. ∎

We call b_0 the *critical value* for the functional $u_q^1(x, b)$.

Below we present examples of potentials q and evaluate the critical value b_0. In all examples potentials depend only on the last variable: $q(y) := q(y_d)$. This, in fact, reduces the computations to the one-dimensional case:

$$G_{H_b}|q|(x) = E^x \int_0^{\tau_{H_b}} e^{-u} q(X_d(u))\,du \tag{24}$$

$$= E^{x_d} \int_0^{\tau_{(-\infty,b)}} e^{-u} q(X_d(u))\,du = G_{(-\infty,b)}|q|(x_d). \tag{25}$$

Therefore, we consider first the one-dimensional case and transform the formula for the Green operator into a form more suitable for computation. We always assume that $q \in \mathcal{J}_1^\alpha \cap L^1(\mathbf{R}^1)$.

LEMMA 4.6. *Assume that $q \in \mathcal{J}_1^\alpha \cap L^1(\mathbf{R}^1)$. Then for $x \leqslant b$ we have*

$$G_{(-\infty,b)} q(x) = \frac{1}{\Gamma(\alpha/2)^2} \int_x^b \frac{e^{x-u}}{(u-x)^{1-\alpha/2}} \left\{ \int_{-\infty}^u \frac{e^{y-u} q(y)\,dy}{(u-y)^{1-\alpha/2}} \right\} du. \tag{26}$$

Proof. We follow the calculations provided in [4] for the stable case. After changing the order of integration we obtain for $x \leqslant b$:

$$\begin{aligned}
G_{(-\infty,b)} q(x) &= \int_{-\infty}^b \frac{e^{-|x-y|}}{\Gamma(\alpha/2)^2} \left\{ \int_0^{(b-x)\wedge(b-y)} \frac{e^{-2u} u^{\alpha/2-1}\,du}{(u+|x-y|)^{1-\alpha/2}} \right\} q(y)\,dy \\
&= \int_{-\infty}^x \frac{e^{-(x-y)}}{\Gamma(\alpha/2)^2} \left\{ \int_0^{b-x} \frac{e^{-2u}\,du}{\big(u(u+x-y)\big)^{1-\alpha/2}} \right\} q(y)\,dy \\
&\quad + \int_x^b \frac{e^{-(y-x)}}{\Gamma(\alpha/2)^2} \left\{ \int_0^{b-y} \frac{e^{-2u}\,du}{\big(u(u+y-x)\big)^{1-\alpha/2}} \right\} q(y)\,dy \\
&= \int_{-\infty}^x \frac{e^{y+x}}{\Gamma(\alpha/2)^2} \left\{ \int_x^b \frac{e^{-2u}\,du}{\big((u-x)(u-y)\big)^{1-\alpha/2}} \right\} q(y)\,dy \\
&\quad + \int_x^b \frac{e^{x+y}}{\Gamma(\alpha/2)^2} \left\{ \int_y^b \frac{e^{-2u}\,du}{\big((u-x)(u-y)\big)^{1-\alpha/2}} \right\} q(y)\,dy \\
&= \int_x^b \frac{e^{x-u}}{\Gamma(\alpha/2)^2} \left\{ \int_{-\infty}^x \frac{e^{y-u} q(y)\,dy}{\big((u-x)(u-y)\big)^{1-\alpha/2}} \right\} du \\
&\quad + \int_x^b \frac{e^{x-u}}{\Gamma(\alpha/2)^2} \left\{ \int_x^u \frac{e^{y-u} q(y)\,dy}{\big((u-x)(u-y)\big)^{1-\alpha/2}} \right\} du \\
&= \frac{1}{\Gamma(\alpha/2)^2} \int_x^b \frac{e^{x-u}}{(u-x)^{1-\alpha/2}} \left\{ \int_{-\infty}^u \frac{e^{y-u} q(y)\,dy}{(u-y)^{1-\alpha/2}} \right\} du
\end{aligned}$$

$$= \frac{1}{\Gamma(\alpha/2)^2} \int_x^b \frac{e^{x-u}}{(u-x)^{1-\alpha/2}} \left\{ \int_0^\infty \frac{e^{-v} q(u-v)\, dv}{v^{1-\alpha/2}} \right\} du$$

$$= \frac{1}{\Gamma(\alpha/2)^2} \int_0^{b-x} \frac{e^{-u}}{u^{1-\alpha/2}} \left\{ \int_0^\infty \frac{e^{-v} q(u-v+x)\, dv}{v^{1-\alpha/2}} \right\} du. \ \blacksquare$$

As a corollary, we provide a simple proof of the formula for $E^x \exp(-\tau_{H_b})$.

COROLLARY 4.7. *We have*

$$1 - E^x \exp(-\tau_{H_b}) = 1 - E^{x_d} \exp(-\tau_{(-\infty,b)}) = \frac{\gamma(\alpha/2, \delta(x))}{\Gamma(\alpha/2)},$$

where by γ we denote the incomplete gamma function.

Proof. Applying the formula (24) we easily see that

$$E^x \exp(-\tau_{H_b}) = E^{x_d} \exp(-\tau_{(-\infty,b)}).$$

Then, by the last formula from the proof of Lemma 4.6 we obtain

$$1 - E^{x_d} \exp(-\tau_{(-\infty,b)}) = G_{(-\infty,b)} 1(x_d)$$

$$= \frac{1}{\Gamma(\alpha/2)^2} \int_0^{b-x_d} \frac{e^{-u}}{u^{1-\alpha/2}} \left\{ \int_0^\infty \frac{e^{-v}\, dv}{v^{1-\alpha/2}} \right\} du$$

$$= \frac{1}{\Gamma(\alpha/2)} \int_0^{\delta(x)} e^{-u} u^{\alpha/2-1}\, du = \frac{\gamma(\alpha/2, \delta(x))}{\Gamma(\alpha/2)}. \ \blacksquare$$

We now consider specific potentials q and evaluate $\sup_{x_d \leqslant b} G_{H_b} q(x)$. We begin with a particularly simple situation which occurs when $q(y) = \exp(y_d)$. We then obtain

EXAMPLE 4.8. *Let $q(y) = \exp(y_d)$. Then we have*

$$G_{H_b} q(x) = \frac{2^{1-\alpha/2} \exp(x_d)(b-x_d)^{\alpha/2}}{\alpha\Gamma(\alpha/2)}.$$

Thus, if

$$e^b < \frac{2^{\alpha-1} e^{\alpha/2} \Gamma(\alpha/2)}{\alpha^{\alpha/2}},$$

then $u_q^1(x, b) < \infty$ for $x_d < b$. For $\alpha = 1$ the critical value is $b_0 = \ln(\pi e)/2$.

Proof. Justifications of the above formulas follow easily from (26) and elementary calculus, and are omitted. $\blacksquare$

We now examine the case of the potential $q(y) = \exp(-y_d)$. Note that this function is unbounded over the set $(-\infty, b)$. Applying again the formula (26) we easily infer that for all x such that $x_d < b$ we have $G_{H_b} q(x) = \infty$. By Jensen's inequality and Theorem 4.3 we obtain $u_q^1(x, b) = \infty$ as well, for all

$x_d < b$. Therefore, we consider the potential $q(y) = \exp(-y_d)\mathbf{1}_{(-c,c)}(y_d)$ with $c > 0$.

We evaluate the critical value b_0 for this potential.

EXAMPLE 4.9. *Let* $q(y) = \exp(-y_d)\mathbf{1}_{(-c,c)}(y_d)$, $c > 0$. *We obtain for* $b > -c$

$$\sup_{x_d \leqslant b} G_{H_b} q(x) = \max_{-c < x_d < c} G_{H_b} q(x) \leqslant \frac{2^{2-\alpha} e^{b+2c}}{\alpha \Gamma(\alpha/2)^2} \gamma\big(\alpha, 2(b+c)\big).$$

Thus, if b *is such that*

$$e^{b+2c} \gamma\big(\alpha, 2(b+c)\big) < \frac{\alpha \Gamma(\alpha/2)^2}{2^{2-\alpha}},$$

then $u_q^1(x, b) < \infty$ *for* $x_d < b$.

Proof. Observe that it is enough to restrict our attention to the one-dimensional case.

We then notice that since $G_{(-\infty,b)}(x, y) = 0$ for $y \geqslant b$, we obtain

$$G_{(-\infty,b)} q(x) = \int_{-\infty}^{b} G_{(-\infty,b)}(x, y) e^{-y} \mathbf{1}_{(-c,c)}(y)\, dy = 0$$

whenever $b \leqslant -c$. We therefore assume throughout the remainder that $b > -c$. We always assume that $x \leqslant b$.

We consider first the case $x \leqslant -c$. By the form of the Green operator we obtain

$$\begin{aligned}
G_{(-\infty,b)} q(x) &= \frac{1}{\Gamma(\alpha/2)^2} \int_x^b \frac{e^{-(u-x)}}{(u-x)^{1-\alpha/2}} \left\{ \int_{-\infty}^{u} \frac{e^{y-u} q(y)\, dy}{(u-y)^{1-\alpha/2}} \right\} du \\
&= \frac{1}{\Gamma(\alpha/2)^2} \int_{x \vee (-c)}^{b} \frac{e^{-(u-x)}}{(u-x)^{1-\alpha/2}} \left\{ \int_{-c}^{u \wedge c} \frac{e^{-(u-y)} e^{-y}\, dy}{(u-y)^{1-\alpha/2}} \right\} du \\
&= \frac{e^x}{\Gamma(\alpha/2)^2} \int_{-c}^{b} \frac{e^{-2u}}{(u-x)^{1-\alpha/2}} \left\{ \int_{-c}^{u \wedge c} \frac{dy}{(u-y)^{1-\alpha/2}} \right\} du.
\end{aligned}$$

We further obtain

$$\begin{aligned}
G_{(-\infty,b)} q(x) &= \frac{e^x}{\Gamma(\alpha/2)^2} \int_{-c}^{b} \frac{e^{-2u}}{(u-x)^{1-\alpha/2}} \left\{ \int_{-c}^{u \wedge c} \frac{dy}{(u-y)^{1-\alpha/2}} \right\} du \\
&= \frac{2e^x}{\alpha \Gamma(\alpha/2)^2} \int_{-c}^{b} \frac{e^{-2u}}{(u-x)^{1-\alpha/2}} [-(u-y)^{\alpha/2}]_{-c}^{u \wedge c}\, du \\
&\leqslant \frac{2e^{-c}}{\alpha \Gamma(\alpha/2)^2} \int_{-c}^{b} \frac{e^{-2u}}{(u+c)^{1-\alpha/2}} [-(u-y)^{\alpha/2}]_{-c}^{u \wedge c}\, du,
\end{aligned}$$

where the last inequality follows from the fact that the previous expression is a nondecreasing function of x. Thus, we obtain

$$G_{(-\infty,b)}q(x) \leqslant G_{(-\infty,b)}q(-c), \qquad x \leqslant -c.$$

Direct calculations provide the value of the last quantity for $|b| \leqslant c$ as follows:

$$2^{1-\alpha} e^c \alpha^{-1} \Gamma(\alpha/2)^{-2} \gamma(\alpha, 2(b+c)),$$

where γ is the incomplete gamma function, or

$$\frac{2^{1-\alpha} e^c}{\alpha\Gamma(\alpha/2)^2}\gamma(\alpha, 2(b+c)) - \frac{(2c)^\alpha e^{-3c}}{\alpha 2^\alpha \Gamma(\alpha/2)^2}\int_0^{(b-c)/c} e^{-4c}(u+1)^{\alpha/2-1} u^{\alpha/2}\,du$$

$$\leqslant \frac{2^{1-\alpha} e^c}{\alpha\Gamma(\alpha/2)^2}\gamma(\alpha, 2(b+c))$$

whenever $b \geqslant c$. We now consider the case when $-c < x < b \leqslant c$. Then, integrating by parts, we obtain

$$\begin{aligned}
G_{(-\infty,b)}q(x) &= \frac{e^x}{\Gamma(\alpha/2)^2}\int_x^b \frac{e^{-2u}}{(u-x)^{1-\alpha/2}}\left\{\int_{-c}^{u\wedge c}\frac{dy}{(u-y)^{1-\alpha/2}}\right\}du \\
&= \frac{2e^x}{\alpha\Gamma(\alpha/2)^2}\int_x^b \frac{e^{-2u}}{(u-x)^{1-\alpha/2}}(u+c)^{\alpha/2}\,du \\
&= \frac{2^2 e^x}{\alpha^2\Gamma(\alpha/2)^2} e^{-2b}(b-x)^{\alpha/2}(b+c)^{\alpha/2} \\
&\quad + \frac{2^2 e^x}{\alpha^2\Gamma(\alpha/2)^2}\int_x^b e^{-2u}(u+c)^{\alpha/2}(u-x)^{\alpha/2}\left[2 - \frac{\alpha}{2(u+c)}\right]du \\
&\leqslant \frac{2^2 e^x}{\alpha^2\Gamma(\alpha/2)^2} e^{-2b}(b-x)^{\alpha/2}(b+c)^{\alpha/2} \\
&\quad + \frac{2^3 e^x}{\alpha^2\Gamma(\alpha/2)^2}\int_x^b e^{-2u}(u+c)^{\alpha/2}(u-x)^{\alpha/2}\,du \\
&\leqslant \frac{2^2 e^{-b}}{\alpha^2\Gamma(\alpha/2)^2}(b+c)^\alpha + \frac{2^3 e^b}{\alpha^2\Gamma(\alpha/2)^2}\int_{-c}^b e^{-2u}(u+c)^\alpha\,du \\
&= \frac{4e^b}{\alpha\Gamma(\alpha/2)^2}\int_{-c}^b e^{-2u}(u+c)^{\alpha-1}\,du = \frac{2^{2-\alpha}e^{b+2c}}{\alpha\Gamma(\alpha/2)^2}\gamma(\alpha, 2(b+c)).
\end{aligned}$$

Next, we consider the case when $-c < x < c < b$. We obtain

$$G_{(-\infty,b)}q(x) = \frac{e^x}{\Gamma(\alpha/2)^2}\int_x^b \frac{e^{-2u}}{(u-x)^{1-\alpha/2}}\left\{\int_{-c}^{u\wedge c}\frac{dy}{(u-y)^{1-\alpha/2}}\right\}du$$

$$= \frac{2e^x}{\alpha\Gamma(\alpha/2)^2} \int_x^c \frac{e^{-2u}}{(u-x)^{1-\alpha/2}} (u+c)^{\alpha/2}\, du$$

$$+ \frac{2e^x}{\alpha\Gamma(\alpha/2)^2} \int_c^b \frac{e^{-2u}}{(u-x)^{1-\alpha/2}} [(u+c)^{\alpha/2} - (u-c)^{\alpha/2}]\, du$$

$$= \frac{2e^x}{\alpha\Gamma(\alpha/2)^2} \left[\int_x^b \frac{e^{-2u}}{(u-x)^{1-\alpha/2}} (u+c)^{\alpha/2}\, du - \int_c^b \frac{e^{-2u}}{(u-x)^{1-\alpha/2}} (u-c)^{\alpha/2}\, du \right]$$

$$\leqslant \frac{2e^x}{\alpha\Gamma(\alpha/2)^2} \int_x^b \frac{e^{-2u}}{(u-x)^{1-\alpha/2}} (u+c)^{\alpha/2}\, du.$$

The last integral is the same as in the previous case and is evaluated in the same way. Thus, in this case we obtain

$$G_{(-\infty,b)}\, q(x) \leqslant \frac{2^{2-\alpha} e^{b+2c}}{\alpha\Gamma(\alpha/2)^2} \gamma(\alpha, 2(b+c)).$$

The remaining case is when $c < x < b$ and in this case we have

$$G_{(-\infty,b)}\, q(x) = \frac{e^x}{\Gamma(\alpha/2)^2} \int_x^b \frac{e^{-2u}}{(u-x)^{1-\alpha/2}} \left\{ \int_{-c}^{c} \frac{dy}{(u-y)^{1-\alpha/2}} \right\} du$$

$$= \frac{2e^x}{\alpha\Gamma(\alpha/2)^2} \int_x^b \frac{e^{-2u}}{(u-x)^{1-\alpha/2}} [(u+c)^{\alpha/2} - (u-c)^{\alpha/2}]\, du$$

$$\leqslant \frac{2e^x}{\alpha\Gamma(\alpha/2)^2} \int_x^b \frac{e^{-2u}}{(u-x)^{1-\alpha/2}} (u+c)^{\alpha/2}\, du.$$

Again, the last integral was already evaluated. This observation justifies our claim. ∎

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