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MINIMAL INTEGRAL REPRESENTATIONS OF STABLE PROCESSES

BY

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Dedicated to the memory of Kazimierz Urbanik

Abstract. Minimal integral representations are defined for general stochastic processes and completely characterized for stable processes (symmetric and asymmetric). In the stable case, minimal representations are described by rigid subsets of the L^p-spaces which are investigated here in detail. Exploiting this relationship, various tests for the minimality of representations of stable processes are obtained and used to verify this property for many representations of processes of interest.

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1. INTRODUCTION

The notion of minimal representations of symmetric α-stable (SαS) processes, introduced by Hardin [3], plays an important role in the study of stationary SαS processes (see, e.g., [3], [5], [7], [8], [11]–[13]). This is due to the fact that the usual harmonizable representations are available for only a small subclass of SαS processes but minimal representations always exist, and other representations of SαS processes can be obtained from the minimal ones. In addition, the notion of minimality is not limited to stationary processes. Minimal representations seem to be natural for the "spectral" analysis of stable processes. However, with an exception of a few simple cases, the minimality of representations for many SαS processes of interest has not been established because of the lack of workable tests for this property. A different problem is to place Hardin's notion in a general framework that can be used for the study of other infinitely divisible processes. The present paper deals with these problems.

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In Section 2 we define minimal integral representations for general stochastic processes. It turns out that minimal representations of strictly stable or symmetric stable processes are characterized by the so-called rigid subsets of L^p-spaces, which are introduced and investigated in this paper. In Theorem 3.8 of Section 3 we characterize rigid sets by a series of more or less easily verifiable conditions. We use this characterization in Section 4 to establish the minimality of representations of many stable processes of interest. Section 5 contains an auxiliary material needed for the proof of Theorem 3.8. Finally, we would like to mention that our definition of minimality is slightly different from the primary definition of [3] (see also [4], p. 118). Due to this little change, we can get a clear functional analytic interpretation. Besides of treating the stable case in detail, this paper gives foundations for a study of minimal representations of other processes, such as tempered stable ones, that will be considered in a separate work.

2. MINIMAL INTEGRAL REPRESENTATIONS OF STOCHASTIC PROCESSES

We begin with the following definitions. Let (Ω, P) and (Ω', P') be probability spaces, and let $\mathbf{X} \subset L^0(\Omega, P)$. A transformation $V\colon \mathbf{X} \mapsto L^0(\Omega', P')$ is said to be *distribution preserving* (*d.p.*) if, for every $n \geqslant 1$ and $X_1, \ldots, X_n \in \mathbf{X}$,

$$(X_1, \ldots, X_n) \stackrel{d}{=} (VX_1, \ldots, VX_n),$$

where "$\stackrel{d}{=}$" means "equal in distribution". It is clear that any d.p. transformation V has the following properties:

- V is one-to-one.
- V has a unique extension to a linear d.p. transformation

$$V\colon \mathrm{lin}(\mathbf{X}) \mapsto L^0(\Omega', P').$$

- If $\mathbf{X} \subset L^p(\Omega, P)$ $(p \geqslant 0)$, then $V(\mathbf{X}) \subset L^p(\Omega', P')$ and V has a unique extension to a linear isomorphism between the L^p-closures of $\mathrm{lin}(\mathbf{X})$ and $\mathrm{lin}(V(\mathbf{X}))$.

Moreover, the composition of d.p. transformations is d.p.

Let $M\colon \mathcal{S} \mapsto L^0(\Omega', P')$ be a random measure, i.e., a countably additive set-function defined on a σ-ring $\mathcal{S}$ of subsets of some set S. Let $L(M)$ be an appropriately defined linear space of stochastic integrals of the form $I_M(f) = \int_S f\, dM$, where f is a deterministic function on S. We say that a stochastic process $X = \{X_t\}_{t\in T}$ defined on (Ω, P) has a *representation* in $L(M)$ if there exists a d.p. transformation

$$V\colon \mathbf{X} \mapsto L(M). \tag{2.1}$$

Thus we have the correspondence

$$X_t \overset{V}{\mapsto} I_M(f_t), \qquad t \in T. \tag{2.2}$$

Such a d.p. transformation V is called an *(integral) representation of* X *in* $L(M)$. We now define a minimal representation. Let $\mathcal{M}$ be a class of random measures (for example, independently scattered, infinitely divisible, symmetric stable, strictly stable, etc).

DEFINITION 2.1. Let M belong to some class of random measures $\mathcal{M}$. A representation V of X in $L(M)$ is said to be *minimal* in the class $\mathcal{M}$ if for any other representation V' of X in $L(M')$, where $M' \in \mathcal{M}$, there exists a unique d.p. transformation $W: L(M) \mapsto L(M')$ such that $V' = W \circ V$.

In other words, by definition a *minimal representation* is a factor of any other representation in the class $\mathcal{M}$, as illustrated below, with W being unique:

$$\begin{array}{ccc} X & \overset{V'}{\to} & L(M') \\ & {\scriptstyle V}\searrow & \uparrow {\scriptstyle W} \\ & & L(M) \end{array} \tag{2.3}$$

PROPOSITION 2.2. *Let* $V: X \mapsto L(M)$ *be a minimal representation of* X *in* $L(M)$, *where* $M \in \mathcal{M}$.

(i) *If* V' *is another minimal representation of* X *in* $L(M')$, *then*

$$W: L(M) \mapsto L(M')$$

in (2.3) *is a d.p. bijection.*

(ii) *Let* $U: X \mapsto X$ *be a d.p. bijection. Then* $V \circ U$ *is also a minimal representation of* X *in* $L(M)$ *and there exists a unique d.p. bijection* $W_U: L(M) \mapsto L(M)$ *such that*

$$V \circ U = W_U \circ V.$$

Proof. (i) The existence and uniqueness of W comes from Definition 2.1. We need to prove that W is a bijection. Since V' is also minimal, there exists a d.p. map $W': L(M') \mapsto L(M)$ such that $V = W' \circ V'$. Hence

$$V = W' \circ V' = W' \circ W \circ V.$$

Putting $V' = V$ and $M' = M$ in the diagram (2.3), we see that it commutes when W equals $W' \circ W$ or the identity $\mathrm{id}_{L(M)}$. By the uniqueness, $W' \circ W = \mathrm{id}_{L(M)}$. Reversing the roles of V and V' we get $W \circ W' = \mathrm{id}_{L(M')}$. Hence W is a bijection and $W' = W^{-1}$.

(ii) Let V' be a representation of X in $L(M')$, $M \in \mathcal{M}$. Consider another representation of X in $L(M')$ given by $V' \circ U^{-1}: X \mapsto L(M')$. Since V is minimal, there exists unique $W: L(M) \mapsto L(M')$ such that $V' \circ U^{-1} = W \circ V$ on X.

Thus, $V' = W \circ (V \circ U)$, and $V \circ U$ is minimal. Applying part (i) to V and $V \circ U$ we conclude the proof. ∎

Because of the essential uniqueness of $L(M)$ given by Proposition 2.2 (i), $L(M)$ can be called the *second linear extension of* X (the first one is $L(X) := \operatorname{lin}(X)$).

Intuitively, a minimal representation is the best fit of a noise M to the process X. When X is a zero-mean Gaussian process, then

$$\overline{L(X)}_{L^2} = L(M),$$

where M is an independently scattered Gaussian random measure. This fact does not extend to symmetric α-stable processes and the best one can have in general is

$$\overline{L(X)}_{L^p} \subset L(M),$$

where $p < \alpha$ and M is an independently scattered symmetric α-stable random measure (see Examples 4.4 and 4.5). The minimal representation connects both sides of this inclusion in such a way that spectral analysis of X can be carried on $L(M)$. The latter space is isomorphic to an L^α-space of deterministic functions, which greatly facilitates analysis of stable processes.

Definition 2.1 applies to any classes of processes and random measures, real, complex or vector valued. Therefore, the pattern of analysis of stable processes can be carried over to other processes once the existence of minimal representations is established.

3. RIGID SUBSETS OF L^p-SPACES

In this section we summarize and develop certain aspects of the theory of isometries on subspaces of L^p-spaces that are pertinent to the study of representations of stable processes. We begin by recalling the Banach–Lamperti theorem (see [6], Theorem 3.1, and [1], p. 178) in the point-transformation form. This form is available in view of Sikorski's theorem ([15], Theorem 32.5) and under the assumption that the considered measure spaces are Borel. Throughout this section μ and ν will denote σ-finite measures on Borel spaces $(S, \mathscr{B}_S)$ and $(T, \mathscr{B}_T)$, respectively.

THEOREM 3.1 (Banach–Lamperti). *Let $U: L^p(S, \mu) \mapsto L^p(T, \nu)$ be a linear isometry, where $p \neq 2$. Then there exist maps $\phi: T \mapsto S$ and $h: T \mapsto \boldsymbol{R}$ (or $\boldsymbol{C}$ in the case of complex L^p-spaces) such that*

$$Uf = h \cdot f \circ \phi \tag{3.1}$$

and

$$d\mu = (|h|^p\, d\nu) \circ \phi^{-1}. \tag{3.2}$$

Conversely, (3.1) *defines a linear isometry whenever* ϕ *and* h *satisfy* (3.2). *Furthermore, if* U *is also onto, then* ϕ *can be chosen as a null-preserving isomorphism such that*

$$\frac{d(\mu\circ\phi)}{d\nu} = |h|^p > 0 \quad \nu\text{-a.e.} \tag{3.3}$$

If ϕ *is a null-preserving isomorphism satisfying* (3.3), *then* (3.1) *defines an isometric isomorphism between* $L^p(S, \mu)$ *and* $L^p(T, \nu)$.

Remarks 3.2. (a) If U is an isometry into $L^p(T, \nu)$ and

$$T_0 := \operatorname{supp}\{Uf : f \in L^p(S, \mu)\} = T \quad \text{mod-}\nu,$$

then $\nu\circ\phi^{-1} \sim \mu$. Indeed, if $T_0 = T$ mod-ν, then $h \neq 0$ ν-a.e. In view of (3.2),

$$\mu(A) = \int \mathbf{1}_A(\phi)|h|^p\, d\nu = 0 \text{ if and only if } \nu(\phi^{-1}A) = 0.$$

(b) In general, U determines $\phi_{|T_0}$ and ϕ can be arbitrary on T_0^c; $h = \sum U\mathbf{1}_{E_n}$, where $\{E_n\}$ is a countable partition of S into sets of finite measure. Thus U determines both h and $\phi_{|T_0}$ uniquely mod-ν.

(c) Following Lamperti's condition (3.6) of Theorem 3.1 in [6], an erroneous statement claiming that $|h|^p = V' = d(\mu V^{-1})/d\nu$ has been reappearing in the literature. Here V is a set σ-homomorphism from $\mathscr{B}_S$ into $\mathscr{B}_T$ (modulo null sets) given by

$$VA = \operatorname{supp}\{U\mathbf{1}_E\colon\ E \subset A,\ \mu(E) < \infty\}.$$

The incorrectness of this claim becomes clear in the following simple example. Take $S = T = N$, $\mu = \nu = \sum 2^{-n}\delta_n$, and

$$Uf(n) = \begin{cases} af(1), & n = 1,\\ bf(1), & n = 2,\\ 2f(n-1), & n \geqslant 3.\end{cases}$$

Then $U\colon L^1(S, \mu) \mapsto L^1(S, \mu)$ is a linear isometry provided $|a| + |b|/2 = 1$. Here $V\{1\} = \{1, 2\}$, $V\{n\} = \{n+1\}$ if $n \geqslant 2$; $h(1) = a$, $h(2) = b$, and $h(n) = 2$ for $n \geqslant 3$. Since one can choose infinitely many pairs (a, b) for which U is an isometry, V does not determine $|h|$. Notice that, in general, the relation between V and ϕ in the above theorem is given by $VA = \phi^{-1}A \cap TS$.

We will now consider linear isometries defined on subsets of L^p-spaces,

$$U_0\colon F \to L^p(T, \nu), \tag{3.4}$$

where $F \subset L^p(S, \mu)$. Such isometries for $p \notin 2Z$ were described in Hardin [2] in terms of set-transformations. We should notice that more informative point-transformation form, as (3.1) for (3.4), may not exist, even in trivial cases of isometries of subspaces. This is shown in the following example.

EXAMPLE 3.3. Let S be the unit interval with Lebesgue measure, and let $F = \{\mathbf{1}_S\}$. Let $T = \{y_0\}$ be a one-point space with the probability measure. Consider an isometry $U: F \mapsto L^1(T)$ given by $U\mathbf{1}_S = \mathbf{1}_T$. Then there is no function $\phi: T \mapsto S$ such that (3.1) holds *for every* f in the equivalence class of $\mathbf{1}_S$.

The existence of point-transformation forms for (3.4) was studied in [9]. The first step in obtaining such forms is to consider linear isometries defined on collections of functions, not on the equivalence classes. To make this distinction explicit, we will denote by $\boldsymbol{F}$ a collection of p-integrable functions and by F the corresponding equivalence class. Let $\overline{\operatorname{lin}}(\boldsymbol{F})_p$ be the L^p-closure of $\operatorname{lin}(\boldsymbol{F})$, the linear space generated by $\boldsymbol{F}$. A function $f^* \in \overline{\operatorname{lin}}(\boldsymbol{F})_p$ is said to have *full support* in $\boldsymbol{F}$ if

$$\operatorname{supp}\{f^*\} = \operatorname{supp}(\boldsymbol{F}) \ \text{mod-}\mu. \tag{3.5}$$

Such a function f^* always exists ([2], Lemma 3.4). Consider the following condition: there exists a countably generated σ-field $\mathscr{A}$ such that

$$\sigma\{f/f^*: f \in \boldsymbol{F}\} \subset \mathscr{A} \subset \bar{\sigma}\{f/f^*: f \in \boldsymbol{F}\}, \tag{CG}$$

where the bar on the right-hand side denotes the operation of completion in $\mathscr{B}_S$ and $f^* \in \overline{\operatorname{lin}}(\boldsymbol{F})_p$ is a function of full support in $\boldsymbol{F}$ (arbitrary but fixed). (If $f^*(x) = 0$, then $(f/f^*)(x) := \partial$, the infinity point of the one-point compactification of $\boldsymbol{R}$ (or $\boldsymbol{C}$).) In applications, condition (CD) is not restrictive (see Lemma 4.10 below), but permits to avoid pathological cases as in Example 3.3.

The following theorem combines Theorems 4.1 and 4.3 in [9].

THEOREM 3.4. *Let* $U_0: \boldsymbol{F} \mapsto L^p(T, \nu)$ *be a linear isometry, where* $\boldsymbol{F} \subset L^p(S, \mu)$ *satisfies* (CG) *and* $p \notin 2\boldsymbol{Z}$. *Then there exist maps* $\phi: T \mapsto S$ *and* $h: T \mapsto \boldsymbol{R}$ *(or* $\boldsymbol{C}$, *respectively) such that*

$$U_0 f = h \cdot f \circ \phi \qquad \textit{for every } f \in \boldsymbol{F} \tag{3.6}$$

and

$$|f^*|^p \, d\mu = |f^*|^p (|h|^p \, d\nu_h) \circ \phi^{-1} \qquad \textit{on } \mathscr{A}. \tag{3.7}$$

Conversely, if (3.7) *holds, then* (3.6) *defines an isometry on* $\boldsymbol{F}$.

The following property will play a crucial role in the study of minimal representations of stable processes.

DEFINITION 3.5. Let $p \in (0, \infty)$. A set $F \subset L^p(S, \mu)$ is said to be *rigid* if, for every space $L^p(T, \nu)$ and for every linear isometry $U_0: F \mapsto L^p(T, \nu)$, there exists a unique linear isometry $U: L^p(S, \mu) \mapsto L^p(T, \nu)$ such that $U = U_0$ on F.

We immediately notice that if F is rigid in $L^p(S, \mu)$, then

$$\operatorname{supp}(F) = S \ \ \mu\text{-a.e.} \tag{3.8}$$

Indeed, if $\mu(S \backslash \operatorname{supp}(F)) > 0$, then the identity operator $U_0 f = f$, $f \in F$, has two different extensions: one of them is the identity on $L^p(S, \mu)$ and the other is given by

$$Ug = (2\mathbf{1}_{\mathrm{supp}(F)} - 1)g, \qquad g \in L^p(S, \mu).$$

Hence (3.8) holds when F is rigid.

Our goal is to characterize rigid collections of functions F. First we observe from Theorem 3.4 (or from the set-transformation form for U_0 given in [2]) the following.

COROLLARY 3.6. *Suppose that* $\mathrm{supp}(\boldsymbol{F}) = S$ *and* $\sigma\{f/f^*: f \in \boldsymbol{F}\} = \mathscr{B}_S$ mod-μ. *Then* $\boldsymbol{F}$ *is rigid.*

Proof. Indeed, under these assumptions (CG) holds with $\mathscr{A} = \mathscr{B}_S$ and $|f^*|^p$ can be canceled on both sides of (3.7). Consequently, (3.7) becomes (3.2), so that, by Theorem 3.1, (3.6) gives a formula for an isometry on the whole $L^p(S, \mu)$. ∎

We will now define another property of $\boldsymbol{F}$ which is often easier to verify than the rigidity.

DEFINITION 3.7. Let $\boldsymbol{F} \subset L^0(S, \mu)$, and let $\psi: S \mapsto S$ be a measurable map. We say that $\boldsymbol{F}$ is *ψ-quasi-invariant* if there exists a measurable function $k: S \mapsto \boldsymbol{R}\backslash\{0\}$ ($k: S \mapsto \boldsymbol{C}\backslash\{0\}$ in the complex case) such that

$$f \circ \psi = k \cdot f \qquad \mu\text{-a.e. for every } f \in \boldsymbol{F}. \tag{3.9}$$

The quasi-invariance becomes the usual ψ-invariance when $k = 1$. The latter holds, in particular, when $\mathbf{1}_S \in \boldsymbol{F}$ or, more generally, when $\boldsymbol{F}$ contains a sequence of indicators of sets ascending to S. Every $\boldsymbol{F}$ is (quasi-) invariant with respect to id_S, the identity map on S. Recall that ψ is said to be *nonsingular* if $\mu \circ \psi^{-1}$ is absolutely continuous with respect to μ.

Now we will give the main result of this section.

THEOREM 3.8. *Let* $\boldsymbol{F} \subset L^p(S, \mu)$, *where* $p \notin 2\boldsymbol{Z}$, *and let* $\mathrm{supp}(\boldsymbol{F}) = S$ mod-μ. *Then the following are equivalent*:

(i) $\boldsymbol{F}$ *is rigid in* $L^p(S, \mu)$.

(ii) *If* $\boldsymbol{F}$ *is* ψ*-quasi-invariant with respect to some null-preserving Borel isomorphism* $\psi: S \mapsto S$ *with* $\psi \circ \psi = \mathrm{id}_S$, *then* $\psi = \mathrm{id}_S$ mod-μ.

(iii) *If* $\boldsymbol{F}$ *is* ψ*-quasi-invariant with respect to some nonsingular* $\psi: S \mapsto S$, *then* $\psi = \mathrm{id}_S$ mod-μ.

(iv) $$\sigma\{f/g: f, g \in \boldsymbol{F}\} = \mathscr{B}_S \text{ mod-}\mu.$$

(v) *There exists an* $f^* \in \overline{\mathrm{lin}(\boldsymbol{F})}_{L^p}$ *with* $\mathrm{supp}(f^*) = S$ mod-μ *such that*

$$\sigma\{f/f^*: f \in \boldsymbol{F}\} = \mathscr{B}_S \text{ mod-}\mu.$$

(vi) $$\sigma\{f/g: f, g \in \overline{\mathrm{lin}(\boldsymbol{F})}_{L^p}\} = \mathscr{B}_S.$$

(vii) *There exist a sequence* $\{f_n\} \subset F$ *and an* $f^* \in \overline{\operatorname{lin}(F)}_{L^p}$ *with* $\operatorname{supp}(f^*) = S$ mod-μ *such that the map*

$$s \mapsto \left(\frac{f_1(s)}{f^*(s)}, \frac{f_2(s)}{f^*(s)}, \frac{f_3(s)}{f^*(s)}, \ldots\right)$$

is one-to-one μ*-a.e.*

(viii) *There exist sequences* $\{f_n, g_n\} \subset \overline{\operatorname{lin}(F)}_{L^p}$ *such that the map*

$$s \mapsto \left(\frac{f_1(s)}{g_1(s)}, \frac{f_2(s)}{g_2(s)}, \frac{f_3(s)}{g_3(s)}, \ldots\right)$$

is one-to-one μ*-a.e.*

Proof. We will first show (i) $\Rightarrow$ (ii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (i). Then we will establish (v) $\Rightarrow$ (vii) $\Rightarrow$ (viii) $\Rightarrow$ (vi) $\Rightarrow$ (iii) $\Rightarrow$ (v).

(i) $\Rightarrow$ (ii). Let f^* be a function of full support in F (see (3.5)); by (3.8) we may assume that

$$\operatorname{supp}(f^*) = S. \tag{3.10}$$

Suppose that F is ψ-quasi-invariant, where ψ is specified in (ii). Then, for every $f \in F$,

$$f/f^* = (f/f^*) \circ \psi \quad \mu\text{-a.e.} \tag{3.11}$$

Let $d\mu_* := |f^*|^p \, d\mu$, and let

$$\nu := \tfrac{1}{2}\mu_* + \tfrac{1}{2}\mu_* \circ \psi^{-1}.$$

Consider the map $U_0 \colon F \mapsto L^p(S, \nu)$ given by

$$U_0 f := f/f^*.$$

U_0 is a linear isometry because, for every $f_1, \ldots, f_n \in F$ and $f := \sum a_i f_i$, we have

$$\|U_0 f\|_{L^p(\nu)}^p = \frac{1}{2}\int_S \left|\frac{f}{f^*}\right|^p d\mu_* + \frac{1}{2}\int_S \left|\frac{f}{f^*} \circ \psi\right|^p d\mu_* = \|f\|_{L^p(\mu)}^p$$

by (3.11). Since F is rigid, there is an isometry $U \colon L^p(S, \mu) \mapsto L^p(S, \nu)$ extending U_0. We have $Ug = h \cdot g \circ \phi$ and $d\mu = (|h|^p \, d\nu) \circ \phi^{-1}$, by the Banach–Lamperti theorem (Theorem 3.1). Since $Uf^* = 1$,

$$h = [f^*(\phi)]^{-1}.$$

Now notice that ν is ψ-invariant because $\psi^2 = \mathrm{id}_S$. Hence

$$U_1 g := h(\psi) \, g(\phi\psi)$$

is also an isometry, $U_1 \colon L^p(S, \mu) \mapsto L^p(S, \nu)$. Since, for every $f \in F$,

$$U_1 f = (Uf)(\psi) = (U_0 f)(\psi) = \frac{f}{f^*}(\psi) = U_0 f$$

by (3.11), and F is rigid, we get $U = U_1$, which yields

$$\phi \circ \psi = \phi \quad \mu\text{-a.e.} \tag{3.12}$$

Then, for every $A \in \mathscr{B}_S$,

$$\mu(A) = \int_S 1_A(\phi)|h|^p\,dv = \frac{1}{2}\int_S 1_A(\phi)\,|f^*(\phi)|^{-p}\,d\mu_* + \frac{1}{2}\int_S 1_A(\phi\psi)\,|f^*(\phi\psi)|^{-p}\,d\mu_*,$$

which implies

$$d\mu = |f^*|^{-p}\,d(\mu_* \circ \phi^{-1}).$$

This ensures that $V_1: L^p(S, \mu) \mapsto L^p(S, \mu_*)$, defined by $V_1 g = (g/f^*)(\phi)$, is an isometry. Since also $V_2: L^p(S, \mu) \mapsto L^p(S, \mu_*)$, given by $V_2 g = g/f^*$, is an isometry and V_2 coincides with V_1 on F, we get $\phi = \mathrm{id}_S$. This in conjunction with (3.12) yields $\psi = \mathrm{id}_S$, which proves (ii).

(ii) $\Rightarrow$ (iv). Suppose that (iv) does not hold. Then, by Proposition 5.1, applied to the case $X = S$, $Y = \boldsymbol{R} \cup \{\partial\}$ ($Y = \boldsymbol{C} \cup \{\partial\}$ in the complex case), and $\Gamma = \{f/g\colon f, g \in F\}$, there exists a null-preserving isomorphism $\psi: S \mapsto S$ such that $\psi \neq \mathrm{id}_S$ (mod-μ), $\psi \circ \psi = \mathrm{id}_S$ and

$$(f/g) \circ \psi = f/g \quad \mu\text{-a.e.} \tag{3.13}$$

for every $f, g \in F$. This gives

$$g \cdot f \circ \psi = f \cdot g \circ \psi. \tag{3.14}$$

By the nonsingularity of ψ, (3.14) holds for $g = f^*$ (given by (3.10)) and all $f \in F$. Thus (3.9) holds with $k = f^*(\psi)/f^*$ and ψ as above, which contradicts (ii).

(iv) $\Rightarrow$ (v). Obvious.

(v) $\Rightarrow$ (i). This is given in Corollary 3.6.

(v) $\Rightarrow$ (vii). It follows from Proposition 5.2 below.

(vii) $\Rightarrow$ (viii). Obvious.

(viii) $\Rightarrow$ (vi). It follows from Proposition 5.2 below.

(vi) $\Rightarrow$ (iii). Suppose that (3.9) holds for some nonsingular ψ. Hence (3.14) holds for every $f, g \in \overline{\mathrm{lin}(F)}$. Thus (3.13) holds for such f and g, which gives $\psi = \mathrm{id}_S$, by Proposition 5.1.

(iii) $\Rightarrow$ (v). Use the same arguments as in (ii) $\Rightarrow$ (iv).

The proof of theorem is complete. ∎

Following the proof of Theorem 1.1 in [3] one can show that every subset of an L^p-space is isometric to a set satisfying condition (vi) of Theorem 3.8 (see also [9], Section 4). In view of Theorem 3.8 we have the following interpretation of that result.

PROPOSITION 3.9. *Let $F \subset L^p(S, \mu)$, $0 < p < \infty$. Then there exist a rigid subset $\tilde{F}$ of some space $L^p(\tilde{S}, \tilde{\mu})$ and a linear isometry $U: L^p(\tilde{S}, \tilde{\mu}) \mapsto L^p(S, \mu)$ such that $U\tilde{F} = F$.*

4. MINIMAL REPRESENTATIONS OF STABLE PROCESSES

4.1. Symmetric stable processes. Let $\mathscr{M} = \mathscr{M}_\alpha^s$ denote the class of independently scattered symmetric α-stable (SαS) random measures. If $M \in \mathscr{M}_\alpha^s$, then the space $L(M)$, which is defined as the closure of stochastic integrals $I_M(f)$ of simple functions $f\colon S \mapsto \boldsymbol{R}$ in some (any) $L^p(\Omega, P)$ $(0 \leqslant p < \alpha)$, is isomorphic to the $L^\alpha(S, \mu)$-space (μ is the control measure of M). Specifically, we have

$$E\exp\big(iI_M(f)\big) = \exp\big(-\int_S |f|^\alpha \, d\mu\big) \tag{4.1}$$

for every $f \in L^\alpha(S, \mu)$. For these facts, further details and proofs, we refer the reader to [14]. We will also consider, as a separate case, complex-valued SαS random measures and stochastic processes; in this case the symmetry means the rotational invariance. It is well known that every SαS process $X = \{X_t\}_{t \in \boldsymbol{T}}$ separable in probability has a representation

$$X_t \mapsto I_M(f_t) \tag{4.2}$$

with $M \in \mathscr{M}_\alpha^s$ and $f_t \in L^\alpha(S, \mu)$. Put $\boldsymbol{F} = \{f_t\}_{t \in \boldsymbol{T}}$. Let $X_t \mapsto I_{M'}(f_t')$ be any other representation of X with $M' \in \mathscr{M}_\alpha^s$ and $f_t' \in L^\alpha(S', \mu')$. Then (4.2) implies that

$$U_0 f_t := f_t' \tag{4.3}$$

is a linear isometry, $U_0\colon \boldsymbol{F} \mapsto L^\alpha(S', \mu')$. Conversely, if (4.3) is a linear isometry, then $X_t \mapsto I_{M'}(f_t')$ is a representation of X. Using these facts and Definition 2.1 it is easy to verify the following.

PROPOSITION 4.1. *Let X be an SαS process. Then the following are equivalent*:

(i) *The representation* (4.2) *is minimal in the class* $\mathscr{M}_\alpha^s$.

(ii) $\{f_t\}_{t \in \boldsymbol{T}}$ *is rigid in* $L^\alpha(S, \mu)$.

Combining this proposition with Remarks 3.2 (a) we obtain the following corollary which makes precise the statement that other representations of stable processes share the properties of the minimal ones.

COROLLARY 4.2. *Under the above notation, assume that* (4.2) *is minimal. Suppose that another representation $X_t \mapsto I_{M'}(f_t')$ has the property* $\operatorname{supp}\{f_t'\colon t \in \boldsymbol{T}\} = S'$ mod-μ'. *Then, for every $t \in \boldsymbol{T}$,*

$$f_t' = h \cdot f_t \circ \phi \quad \mu'\text{-a.e.},$$

where $\phi\colon S' \to S$, $h\colon S' \to \boldsymbol{R}\backslash\{0\}$ ($\boldsymbol{C}\backslash\{0\}$, respectively) are measurable, and $\mu \sim \mu' \circ \phi^{-1}$.

We will now apply Theorem 3.8 to describe minimal representations of some SαS processes together with their first and second linear extensions (see Section 2). We will start with the simplest case of the SαS Lévy motion.

EXAMPLE 4.3. *SαS Lévy motion.*

$$X_t = \int_0^\infty \mathbf{1}_{(0,t]}(s)\, M(ds), \qquad t \geqslant 0,$$

where M has control Lebesgue measure. Clearly, this is a minimal representation by (iv) of Theorem 3.8. Here $L(X) = L(M)$.

EXAMPLE 4.4. Let $f_0, f_1 \in L^\alpha(S, \mu)$, $f_0 \neq 0$, and let $f_1/f_0 \colon S \mapsto \boldsymbol{R}$ be one-to-one a.e. Then $X_i := I_M(f_i)$, $i = 1, 2$, is given by its minimal representation. The first linear extension of X,

$$L(X) = \{I_M(af_0 + bf_1) \colon a, b \in \boldsymbol{R}\},$$

is a two-dimensional space. The second linear extension is the whole space $L(M)$, which can be infinite dimensional. Notice that $L(X)$ does not contain any pair of independent non-zero random variables, unless S is consisting of two atoms.

EXAMPLE 4.5. *Stationary sequence.* Let $S = [0, 1]^2$ and for $s = (s_1, s_2) \in S$

$$f_n(s) := \begin{cases} \operatorname{sign}\big(\sin(2^n \pi s_1)\big), & n \geqslant 1, \\ 1, & n = 0, \\ \operatorname{sign}\big(\sin(2^{|n|} \pi s_2)\big), & n \leqslant -1. \end{cases}$$

$\{f_n\}_{n \neq 0}$ is a bilateral Rademacher sequence. Using binary expansions of s_1 and s_2 we verify that the map $s \mapsto (\ldots, f_{-1}(s)/f_0(s), 1, f_{-1}(s)/f_0(s), \ldots)$ is one-to-one μ-a.e., where μ is the Lebesgue measure on S. Hence

$$X_n := I_M(f_n) = \iint_{[0,1]^2} f_n(s)\, M(ds), \qquad n \in \boldsymbol{Z},$$

is a stationary SαS sequence and this representation is minimal. We have

$$L(X) = \{I_M(\textstyle\sum a_n f_n) \colon \sum_{-\infty}^{\infty} a_n^2 < \infty\},$$

which is a much smaller space than $L(M)$.

EXAMPLE 4.6. *Moving averages.* We will show that the representation

$$X_t := \int_{\boldsymbol{R}^d} f(t+s)\, M(ds), \qquad t \in \boldsymbol{R}^d,$$

is minimal. Here M has control Lebesgue measure. We will verify condition (iii) of Theorem 3.8 with $F = \{f_t\}_{t \in \boldsymbol{R}^d}$, $f_t(s) = f(t+s)$. Suppose that there exists a nonsingular $\psi \colon \boldsymbol{R}^d \mapsto \boldsymbol{R}^d$ such that for each $t \in \boldsymbol{R}^d$

$$f\big(t + \psi(s)\big) = k(s)\, f(t+s) \qquad \text{for a.a. } s \in \boldsymbol{R}^d.$$

By Fubini's theorem and invariance of the Lebesgue measure, for a.a. $s \in \boldsymbol{R}^d$

$$f\big(u + \psi(s) - s\big) = k(s)\, f(u) \qquad \text{for a.a. } u \in \boldsymbol{R}^d.$$

This in conjunction with $f \in L^\alpha(\boldsymbol{R}^d)$ yields $\psi(s) - s = 0$, which was to be shown.

EXAMPLE 4.7. *Takenaka random field.* An (α, H)-Takenaka random field X is defined by

$$X_t := M(V_t) = \int_{\boldsymbol{R}^d \times \boldsymbol{R}_+} \mathbf{1}_{V_t}(x, r)\, M(dx, dr), \qquad t \in \boldsymbol{R}^d, \tag{4.4}$$

where

$$V_t := \{(x, r) \colon \|x\| \leqslant r\} \triangle \{(x, r) \colon \|x - t\| \leqslant r\},$$

and M is an SαS random measure on $\boldsymbol{R}^d \times \boldsymbol{R}_+$ with control measure

$$\mu(dx, dr) = r^{\alpha H - d - 1}\, dx dr,$$

$H \in (0, 1/\alpha)$ (see [14], Chapter 8.4). We will show that (4.4) is a minimal representation. To this end we will check condition (iii) of Theorem 3.8. Suppose that there exists a nonsingular $\psi \colon \boldsymbol{R}^d \times \boldsymbol{R}_+ \mapsto \boldsymbol{R}^d \times \boldsymbol{R}_+$ such that, for each $t \in \boldsymbol{R}^d$,

$$\mathbf{1}_{V_t}(x', r') = k(x, r)\mathbf{1}_{V_t}(x, r) \qquad \text{for } \mu\text{-a.a. } (x, r), \tag{4.5}$$

where $(x', r') = \psi(x, r)$. By Fubini's theorem, for μ-a.a. (x, r), (4.5) holds for a.a. $t \in \boldsymbol{R}^d$. However, for fixed (x, r), the function

$$t \mapsto \mathbf{1}_{V_t}(x, r)$$

has only two possible forms: either it is the indicator of a closed ball with center x and radius r (if $\|x\| > r$) or it is the indicator of the complement of such a ball (if $\|x\| \leqslant r$). This yields $k(x, r) = 1$ in (4.5) and identifies $x' = x$ and $r' = r$. Thus $\psi(x, r) = (x, r)$ μ-a.e., proving the minimality of (4.4).

EXAMPLE 4.8. *Harmonizable process.* Here we consider

$$X_t = \int_{\hat{\boldsymbol{T}}^d} e^{it \cdot s}\, M(ds), \tag{4.6}$$

where $\boldsymbol{T} = \boldsymbol{Z}$ or $\boldsymbol{R}$ and $\hat{\boldsymbol{T}} = [0, 2\pi)$ or $\boldsymbol{R}$, respectively. M is a rotationally invariant complex-valued random measure with a finite control measure μ. Since trigonometric polynomials are dense in $L^\alpha(\hat{\boldsymbol{T}}^d, \mu)$, the minimality of (4.6) is obvious. In this case we also have $L(X) = L(M)$.

EXAMPLE 4.9. *Real part of a harmonizable process.* If X is given by (4.6), then its real part $\Re X$ can be represented as

$$\Re X_t \mapsto k_\alpha^{-1} \int_{[0, 2\pi) \times \hat{\boldsymbol{T}}^d} \cos(s + t \cdot w)\, Z(ds, dw), \tag{4.7}$$

where Z is a real-valued SαS random measure on $[0, 2\pi) \times \hat{\boldsymbol{T}}^d$ with control measure Leb$\otimes\mu$, and

$$k_\alpha = \Big(\int_0^{2\pi} |\cos s|^\alpha\, ds\Big)^{1/\alpha}$$

(see [12]). This, however, is not a minimal representation. Indeed, we have here

$$f_t(s, w) = k_\alpha^{-1}\cos(s+t\cdot w), \qquad (s, w) \in [0, 2\pi) \times \boldsymbol{T}^d.$$

Define

$$\psi(s, w) = \begin{cases} (s+\pi, w) & \text{if } s \in [0, \pi), \\ (s-\pi, w) & \text{if } s \in [\pi, 2\pi). \end{cases}$$

Then $f_t(\psi(s, w)) = -f_t(s, w)$, contradicting (ii) of Theorem 3.8.

We will find a minimal representation for (4.7) in the case $\boldsymbol{T} = \boldsymbol{R}$. To this end we need to assume that $\mu(\{0\}) = 0$. Then there exists a hyperplane $H = H_\alpha = \{x \in \boldsymbol{R}^d : x \cdot a = 0\}$, $\|a\| = 1$, such that $\mu(H) = 0$. Put $H_+ := \{x : x \cdot a > 0\}$ and "fold" the measure μ. Formally, let $\nu := \mu \circ \phi^{-1}$, where ϕ: $\boldsymbol{R}^d \backslash H \mapsto H_+$ is given by $\phi(x) := \operatorname{sign}(x \cdot a)\, x$. Let N be an SαS random measure on $[0, \pi) \times \boldsymbol{R}^d$ with control measure Leb$\otimes\nu$. We have

$$\begin{aligned} &\int_{\boldsymbol{R}^d}\int_0^\pi \Big|\sum a_j 2^{1/\alpha} k_\alpha^{-1}\cos(s+t_j\cdot w)\Big|^\alpha ds\,\nu(dw) \\ &\quad = \int_{\boldsymbol{R}^d} \Big|\big[\sum a_j \cos(t_j\cdot w)\big]^2 + \big[\sum a_j \sin(t_j \cdot w)\big]^2\Big|^{\alpha/2} \nu(dw) \\ &\quad = \int_{\boldsymbol{R}^d\backslash H} \Big|\big[\sum a_j \cos\big((t_j\cdot \phi(w)\big)\big]^2 + \big[\sum a_j \sin\big(t_j \cdot \phi(w)\big)\big]^2\Big|^{\alpha/2} \mu(dw) \\ &\quad = \int_{\boldsymbol{R}^d} \Big|\big[\sum a_j \cos(t_j\cdot w)\big]^2 + \big[\sum a_j \sin(t_j \cdot w)\big]^2\Big|^{\alpha/2} \mu(dw), \end{aligned}$$

which implies that

$$\Re X_t \mapsto 2^{1/\alpha} k_\alpha^{-1} \int_{[0,\pi)\times \boldsymbol{R}^d} \cos(s+t\cdot w)\, N(ds, dw) \tag{4.8}$$

is also a representation of $\Re X$. We will show that (4.8) is minimal. To this end choose a dense sequence $\{t_j\} \in \boldsymbol{R}^d$ and consider the map

$$\{(0, \pi/2) \cup (\pi/2, \pi)\} \times H_+ \ni (s, w) \mapsto \left(\frac{\cos(s+t_1\cdot w)}{\cos s}, \frac{\cos(s+t_2\cdot w)}{\cos s}, \ldots\right).$$

We will verify condition (vii) of Theorem 3.8. Suppose that, for some (s_1, w_1), $(s_2, w_2) \in \{(0, \pi/2) \cup (\pi/2, \pi)\} \times H_+$, we have

$$\frac{\cos(s_1+t_j\cdot w_1)}{\cos s_1} = \frac{\cos(s_2+t_j\cdot w_2)}{\cos s_2} \quad \text{for all } j.$$

By continuity we get

$$\cos(s_1+t\cdot w_1)\cos s_2 = \cos(s_2+t\cdot w_2)\cos s_1 \quad \text{for all } t \in \boldsymbol{R}^d. \tag{4.9}$$

Taking partial derivatives in (4.9) with respect to t and then setting $t = 0$ we obtain

$$\begin{cases} w_1^k \sin s_1 \cos s_2 = w_2^k \sin s_2 \cos s_1, \\ (w_1^k)^2 \cos s_1 \cos s_2 = (w_2^k)^2 \cos s_2 \cos s_1, \qquad k = 1, \ldots, d, \end{cases} \tag{4.10}$$

where $w_i = (w_i^1, \ldots, w_i^d)$, $i = 1, 2$. From the second equation in (4.10) we get $w_1^k = \pm w_2^k$ for each k. Since $w_i \neq 0$, we infer from (4.10) that $\tan s_1 = \pm \tan s_2$. Hence $s_1 = s_2$ or $s_1 = \pi - s_2$. In the first case, (4.10) yields $w_1 = w_2$, which gives the required conclusion, in the second case, (4.10) gives $w_1 = -w_2$. But the latter is impossible since both $w_1, w_2 \in H_+$, and this completes the proof of minimality in (4.8).

Our last example contains sub-Gaussian and sub-stable processes. The fact that the natural representation of sub-stable processes is not minimal has been mentioned in [4], p. 122. We give here a simpler and more general argument to that fact.

EXAMPLE 4.10. *Doubly symmetric SαS process.* Let X be given by

$$X_t = \int_S f_t \, dM, \qquad t \in T,$$

where $F = \{f_t\}_{t \in T}$ itself is a symmetric stochastic process on a probability space (S, μ). As before, M is an SαS random measure on S with control measure μ. We will call such processes *doubly symmetric*. The above representation is not minimal which can be easily seen from condition (iv) of Theorem 3.8. Indeed, assuming the minimality, we have

$$\{f^* > 0\} = \{(f_{t_1}/f^*, f_{t_2}/f^*, \ldots) \in B\} \ \text{mod-}\mu$$

for some $t_1, t_2, \ldots \in T$ and $B \in \mathscr{B}_{\mathbf{R}^\infty}$. Since $F \stackrel{d}{=} -F$, we also have

$$\{f^* < 0\} = \{(f_{t_1}/f^*, f_{t_2}/f^*, \ldots) \in B\} \ \text{mod-}\mu,$$

which gives $f^* = 0$ mod-μ, a contradiction.

4.2. Nonsymmetric case — strictly stable processes. Now we will consider the class $\mathscr{M}_\alpha$ of independently scattered strictly α-stable random measures. Every random measure $M \in \mathscr{M}_\alpha$ is characterized by two "parameters": its control measure μ on S and the skewness function $\beta\colon S \to [-1, 1]$. The space $L(M)$, which is defined as before as the closure of stochastic integrals $I_M(f)$ of simple functions $f\colon S \mapsto \mathbf{R}$ in some (any) $L^p(\Omega, P)$ $(0 \leqslant p < \alpha)$, is isomorphic to the $L^\alpha(S, \mu)$-space (μ is the control measure of M). Specifically, we have

$$E\exp\big(iI_M(f)\big) = \exp\Big\{-\int_S |f|^\alpha \, d\mu + i\tan(\pi\alpha/2)\int_S f^{\langle\alpha\rangle}\beta \, d\mu\Big\} \tag{4.11}$$

for every $f \in L^\alpha(S, \mu)$ (see [14]). Here $x^{\langle\alpha\rangle} := \operatorname{sign}(x)|x|^\alpha$. It is well known that every strictly stable process $X = \{X_t\}_{t \in T}$ separable in probability has a representation

$$X_t \mapsto I_M(f_t) \tag{4.12}$$

for some random measure $M \in \mathscr{M}_\alpha$ with skewness function $\beta = 1$, and $f_t \in L^\alpha(S, \mu)$ (see, e.g., [14]). To obtain minimal integral representations we need to consider representations of strictly stable process X described by the pairs

$(\boldsymbol{F}, \beta)$, where $\boldsymbol{F} = \{f_t\}_{t\in \boldsymbol{T}} \subset L^\alpha(S, \mu)$ and $\beta\colon S \mapsto [-1, 1]$. The restriction to $\beta = 1$ will not work.

LEMMA 4.11. *Consider representation* (4.12) *with an arbitrary pair* $(\boldsymbol{F}, \beta)$. *Let* $U_0\colon \boldsymbol{F} \mapsto L^\alpha(S', \mu')$ *be a linear isometry. Then there exists a function* $\beta'\colon S' \mapsto [-1, 1]$ *such that*

$$X_t \mapsto I_{M'}(U_0 f_t) \tag{4.13}$$

is a representation of $\boldsymbol{X}$, *where* M' *is a strictly* α-*stable random measure with skewness function* β' *and control measure* μ'.

Proof. In view of (4.11) and our assumption on U_0, we need to show that

$$\int_{S'} \left(\sum a_j U_0 f_{t_j}\right)^{\langle\alpha\rangle} \beta' d\mu' = \int_S \left(\sum a_j f_{t_j}\right)^{\langle\alpha\rangle} \beta d\mu \tag{4.14}$$

for some function $\beta'\colon S' \mapsto [-1, 1]$ and all $a_1, \ldots, a_n \in \boldsymbol{R}$, $t_1, \ldots, t_n \in \boldsymbol{T}$.

Choose a sequence $\{f_{t_n}\} \subset \boldsymbol{F}$ dense in L^α and modify the set $\boldsymbol{F}$ replacing each f_t by the pointwise limit of some subsequence from $\{f_{t_n}\}$. Denote the modified set also by $\boldsymbol{F}$. This modification, of each f_t on a null set, does not alter our assumptions. Let f^* be as in (3.5). Then condition (CG) of Section 3 holds with

$$\mathscr{A} = \sigma\{f_{t_n}/f^*\colon n \in \boldsymbol{N}\}$$

and, by Theorem 3.4, U_0 is of the form (3.6). Without loss of generality we may also assume that $\operatorname{supp}(\boldsymbol{F}) = S$ and that $\|f^*\|_{L^\alpha} = 1$. Let $E^{\mathscr{A}}$ denote the conditional expectation given $\mathscr{A}$ with respect to the probability measure $|f^*|^\alpha d\mu$ on $(S, \mathscr{B}_S)$. Define

$$\beta_0 := \operatorname{sign}(f^*) E^{\mathscr{A}}[\beta \operatorname{sign}(f^*)]$$

and let

$$\beta' := \operatorname{sign}(h) \beta_0 \circ \phi.$$

By (3.6) the left-hand side of (4.14) equals

$$\begin{aligned}
\int_{S'} \left(\sum a_j h f_{t_j} \circ \phi\right)^{\langle\alpha\rangle} \beta' d\mu' &= \int_{S'} \left(\sum a_j f_{t_j} \circ \phi\right)^{\langle\alpha\rangle} \beta_0 \circ \phi\, |h|^\alpha d\mu' \\
&= \int_S \left(\sum a_j f_{t_j}\right)^{\langle\alpha\rangle} \beta_0 (|h|^\alpha d\mu') \circ \phi^{-1} \\
&= \int_S \left(\sum a_j \frac{f_{t_j}}{f^*}\right)^{\langle\alpha\rangle} E^{\mathscr{A}}[\beta \operatorname{sign}(f^*)]\, |f^*|^\alpha (|h|^\alpha d\mu') \circ \phi^{-1} \\
&= \int_S \left(\sum a_j \frac{f_{t_j}}{f^*}\right)^{\langle\alpha\rangle} E^{\mathscr{A}}[\beta \operatorname{sign}(f^*)]\, |f^*|^\alpha d\mu \qquad \text{(using (3.7))} \\
&= \int_S \left(\sum a_j \frac{f_{t_j}}{f^*}\right)^{\langle\alpha\rangle} \beta \operatorname{sign}(f^*)\, |f^*|^\alpha d\mu = \int_S \left(\sum a_j f_{t_j}\right)^{\langle\alpha\rangle} \beta\, d\mu,
\end{aligned}$$

which completes the proof. ∎

THEOREM 4.12. *Let X be a strictly stable process whose representation* (4.12) *is determined by a pair* (F, β). *Then the following are equivalent:*

(i) (4.12) *is minimal in the class* $\mathcal{M}_\alpha$.

(ii) F *is rigid in* $L^\alpha(S, \mu)$.

Proof. Assume (i). By Proposition 3.9 there exists a rigid set $F' \subset L^\alpha(S', \mu')$ and a linear isometry $V\colon L^\alpha(S', \mu') \mapsto L^\alpha(S, \mu)$ such that $VF' = F$. Let $U_0\colon F \mapsto L^\alpha(S', \mu')$ be the inverse to V on F. By Lemma 4.11 there exists a representation

$$X_t \mapsto I_{M'}(U_0 f_t), \qquad t \in T,$$

where $M' \in \mathcal{M}_\alpha$ is a random measure on S'. From the definition of minimality and (4.12) there exists a linear isometry $U\colon L^\alpha(S, \mu) \mapsto L^\alpha(S', \mu')$, corresponding to a d.p. transformation from $L(M)$ into $L(M')$, such that $Uf = U_0 f$ for every $f \in F$. Now we see that $UV\colon L^\alpha(S', \mu') \mapsto L^\alpha(S', \mu')$ is the identity on the rigid set F', implying UV is the identity on $L^\alpha(S', \mu')$. Hence V is onto and the rigidity of F follows from such a property of F'.

Assume (ii). Consider another representation $X_t \mapsto I_{M'}(f_t')$ of X, where M' is a strictly stable random measure on S' with skewness function β' and control measure μ', and $f_t' \in L^\alpha(S', \mu')$. Then (4.12) implies that $U_0 f_t := f_t'$ is a linear isometry from F into $L^\alpha(S', \mu')$ satisfying (4.14). Since F is rigid, U_0 has a unique extension $U\colon L^\alpha(S, \mu) \mapsto L^\alpha(S', \mu')$, which, by Theorem 3.1, is of the form

$$Ug = h \cdot g \circ \phi, \qquad g \in L^\alpha(S, \mu).$$

To complete the proof we need to show that U corresponds to a d.p. transformation from $L(M)$ into $L(M')$. This amounts to proving that

$$\int_{S'} (Ug)^{\langle\alpha\rangle} \beta' \, d\mu' = \int_S g^{\langle\alpha\rangle} \beta \, d\mu \quad \text{for every } g \in L^\alpha(S, \mu). \tag{4.15}$$

Let

$$\beta_1(s) := E_{(|h|^\alpha d\mu')} \{[\operatorname{sign}(h)] \beta' \mid \phi = s\}$$

be the conditional "expectation" with respect to the measure $|h|^\alpha \, d\mu'$ (defined by means of the Riesz theorem) given $\phi = s$. From (4.15) we infer that the equation in (4.14) holds for all $f \in \overline{\operatorname{lin}}(F)_{L^\alpha}$; taking such f and a fixed f^* of full support in F we get

$$\int_{S'} [U(f^* + f)]^{\langle\alpha\rangle} \beta' \, d\mu' = \int_{S'} (f^* \circ \phi + f \circ \phi)^{\langle\alpha\rangle} \operatorname{sign}(h) \beta' |h|^\alpha \, d\mu'$$
$$= \int_{S'} (f^* \circ \phi + f \circ \phi)^{\langle\alpha\rangle} \beta_1 \circ \phi \, |h|^\alpha \, d\mu' = \int_S (f^* + f)^{\langle\alpha\rangle} \beta_1 \, d\mu,$$

where the last equality follows by (3.2). Hence

$$\int_S (f^* + f)^{\langle\alpha\rangle} \beta_1 \, d\mu = \int_S (f^* + f)^{\langle\alpha\rangle} \beta \, d\mu \quad \text{for every } f \in \overline{\operatorname{lin}(F)}_{L^\alpha},$$

which yields

$$\int_S \left(1+\frac{f}{f^*}\right)^{\langle\alpha\rangle} (\beta_1-\beta)(f^*)^{\langle\alpha\rangle}\, d\mu = 0 \quad \text{for every } f \in \overline{\operatorname{lin}(F)}_{L^\alpha}.$$

This equality implies $\beta_1 = \beta$ μ-a.e. by an extension of Rudin's theorem proved in [10]. Hence we get

$$\beta \circ \phi = E_{(|h|^\alpha d\mu')} \{[\operatorname{sign}(h)]\, \beta' \mid \phi\}, \tag{4.16}$$

and a verification of (4.15) is easy. Indeed, using (3.2) we have

$$\begin{aligned}\int_S g^{\langle\alpha\rangle} \beta\, d\mu &= \int_S g^{\langle\alpha\rangle} \beta\, (|h|^\alpha d\mu') \circ \phi^{-1} \\ &= \int_{S'} (g\circ\phi)^{\langle\alpha\rangle} E_{(|h|^\alpha d\mu')} \{[\operatorname{sign}(h)]\, \beta' \mid \phi\}\, |h|^\alpha\, d\mu' = \int_{S'} (Ug)^{\langle\alpha\rangle} \beta'\, d\mu'\end{aligned}$$

for every $g \in L^\alpha(S, \mu)$. This completes the proof. ∎

The following corollary shows the consistency between the notions of minimality studied in this section.

Corollary 4.13. *A minimal representation of an SαS process in the class* $\mathcal{M}_\alpha^s$ *is also minimal in* $\mathcal{M}_\alpha$. *If* $X_t \mapsto I_M(f_t)$ *is minimal in* $\mathcal{M}_\alpha$ *and* X *is an SαS process, then* $\beta = 0$.

Proof. In both classes the minimality is equivalent to the rigidity of F. The second claim follows from (4.16) and the fact that ϕ is a null-preserving isomorphism when acting between two rigid sets. ∎

In view of Theorem 4.12, examples of minimal representations for strictly stable processes can be obtained trivially from our previous examples by changing SαS random measures to strictly stable ones.

Remark 4.14. Characterizations of minimal representations of stable process obtained in this paper can be easily carried over to multidimensional strictly stable process (see [5], Section 2).

5. AUXILIARY RESULTS USED IN THE PROOF OF THEOREM 3.8

Proposition 5.1. *Let* (X, μ) *be a measure Borel space and let* Γ *be a set of measurable functions* $f: X \mapsto Y$, *where* Y *is a Borel space. If* $\sigma(\Gamma) \neq \mathscr{B}_X$ mod-μ, *then there exists a null-preserving Borel isomorphism* $\psi: X \mapsto X$ *such that* $\psi \neq \mathrm{id}_X$ mod-μ, $\psi\circ\psi = \mathrm{id}_X$, *and*

$$f \circ \psi = f \quad \mu\text{-a.e., for every } f \in \Gamma. \tag{5.1}$$

Conversely, if (5.1) *holds for some nonsingular map* $\psi: X \mapsto X$ *which is different from the identity* (mod-μ), *then* $\sigma(\Gamma) \neq \mathscr{B}_X$ mod-μ.

Proof. Since X and Y are Borel spaces, there is a countable set $\Gamma_0 \subset \Gamma$ which is dense in Γ in the topology of convergence in μ. Thus $\sigma(\Gamma_0) = \sigma(\Gamma)$

mod-μ and it suffices to establish (5.1) only for $f \in \Gamma_0$ (the nonsingularity of ψ is crucial here). Let $\Gamma_0 = \{f_n\}_{n \in N_0}$, $N_0 \subset N$; define $F: X \mapsto Y^{N_0}$ by

$$F(x) := (f_1(x), f_2(x), \ldots).$$

Then $\sigma(\Gamma) = F^{-1}(\mathscr{B}_{Y^{N_0}})$ mod-μ. This shows that, by changing $(Y, \mathscr{B}_Y)$ to $(Y^{N_0}, \mathscr{B}_{Y^{N_0}})$, one reduces the proof to the case when Γ consists of only one function. Thus, from now on, we assume that $\Gamma = \{f\}$ and $\sigma(\Gamma) = f^{-1}(\mathscr{B}_Y)$.

Sufficiency. Let K_n, $n \geqslant 1$, be a sequence of sets separating points of X and generating $\mathscr{B}_X$. Since

$$\{x: \psi(x) \neq x\} \subset \bigcup_n K_n \cap \psi^{-1} K_n^c,$$

there is an n such that $\mu(K_n \cap \psi^{-1} K_n^c) > 0$. Let $A = K_n \cap \psi^{-1} K_n^c$; we claim that $A \notin f^{-1}(\mathscr{B}_Y)$ mod-μ. Indeed, suppose $\mu(A \cap f^{-1} B^c) = 0$ for some $B \in \mathscr{B}_Y$. Using $A \cap \psi^{-1} A = \emptyset$ and (3.2), we get

$$\mu(\psi^{-1}(A^c \cap f^{-1} B)) \geqslant \mu(A \cap f^{-1} B) > 0.$$

Thus, by the nonsingularity of ψ, $\mu(A^c \cap f^{-1} B) > 0$, which proves that $\mu(A \triangle f^{-1} B) > 0$ for every $B \in \mathscr{B}_Y$.

Necessity. The idea of this proof is simple. Since $f^{-1}(\mathscr{B}_Y) \neq \mathscr{B}_X$, the partition $\{f^{-1}\{y\}: y \in Y\}$ contains sufficiently many sets consisting of more than just one point. On each such set one can define an isomorphism different from the identity. Then a function ψ is obtained by pasting together such isomorphisms. Now we will give details of this argument.

Without loss of generality we may assume that μ is a probability measure. Let $\mu(\cdot \mid f = y)$ be a family of regular conditional probabilities on X such that

$$\mu(A \cap f^{-1} B) = \int_B \mu(A \mid f = y) \, \nu(dy)$$

for every $A \in \mathscr{B}_X$, $B \in \mathscr{B}_Y$, where $\nu = \mu \circ f^{-1}$. Let

$$\mu(\cdot \mid f = y) = p(y) \mu_d(\cdot \mid f = y) + q(y) \mu_c(\cdot \mid f = y)$$

be the decomposition into the discrete (μ_d) and continuous (μ_c) parts; $p(y) + q(y) = 1$, $p(y), q(y) \geqslant 0$; the measurability of $p(y)$, $q(y)$, $\mu_d(\cdot \mid f = y)$, and $\mu(\cdot \mid f = y)$ follows by standard arguments. Denote by D^y the set of atoms of $\mu_d(\cdot \mid f = y)$ when $p(y) > 0$, and $D^y := \emptyset$ when $p(y) = 0$.

First we suppose that $\nu\{y: q(y) > 0\} > 0$. Put

$$Y_0 := \{y \in Y: q(y) > 0\}.$$

By the above assumption X must have the cardinality continuum. By Kuratowski's isomorphism theorem, there exists a Borel isomorphism $I: X \mapsto [0, 1]$.

For every $y \in Y_0$, consider a distribution function

$$F(t\,|\,y) := \mu_c(\{x\colon I(x) \leqslant t\} \mid f = y), \qquad t \in [0, 1].$$

Notice that a map $x \mapsto F(I(x)|y)$, considered as a random variable on the probability space $(X, \mathscr{B}_X, \mu_c(\cdot\,|\,f = y))$, has a uniform distribution on $[0, 1]$ $(y \in Y_0)$. Let

$$C^y := \{x \in X_0\colon F(I(x) \mid f = y) \leqslant 1/2\}, \qquad y \in Y_0.$$

Define

$$A_0 := f^{-1}Y_0 \cap \{x \in X\colon x \in C^{f(x)} \text{ and } x \notin D^{f(x)}\}$$

and

$$A_1 := f^{-1}Y_0 \cap \{x \in X\colon x \notin C^{f(x)} \text{ and } x \notin D^{f(x)}\}.$$

It is easy to verify that $\mu(A_0) = \mu(A_1) > 0$. Moreover, if

$$Y_1 := \{y \in Y_0\colon \mu_c(A_0 \mid f = y) = \mu_c(A_1 \mid f = y) = 1/2,$$
$$\mu_d(A_0 \mid f = y) = \mu_d(A_1 \mid f = y) = 0\},$$

then $\nu(Y_0 \setminus Y_1) = 0$. Both A_0 and A_1 have the cardinality continuum because the conditional measures are continuous. Applying again Kuratowski's isomorphism theorem we find Borel isomorphisms $J_i\colon A_i \mapsto [0, 1]$, $i = 0, 1$. Let

$$G_i(t\,|\,y) := 2\mu_c(\{x \in A_i\colon J_i(x) \leqslant t\} \mid f = y), \qquad y \in Y_1,$$

be the conditional distribution function of J_i given $f = y$. Define ϕ: $A_0 \times Y_1 \mapsto A_1$ by

$$\phi(x, y) := J_1^{-1} \circ G_1^{-1}[G_0(J_0(x)|y) \mid y], \qquad x \in A_0,\ y \in Y_1,$$

where $G_i^{-1}(t\,|\,y) = \inf\{u\colon G_i(u\,|\,y) > t\}$. Then the continuity of the conditional measures implies that, for each $y \in Y_1$, the map $\phi(\cdot, y)\colon A_0 \mapsto A_1$ preserves $\mu_c(A_i \cap (\cdot) \mid f = y)$, and $\phi(x, y)$ is jointly measurable. Put

$$\psi_0(x) := \phi(x, f(x)), \qquad x \in A_0.$$

We will show that $\psi_0\colon A_0 \mapsto A_1$ is invertible μ-a.e. and μ-preserving. Indeed, let

$$\bar{\psi}_0(x) := \xi(x, f(x)), \qquad x \in A_1,$$

where

$$\xi(x, y) := J_0^{-1} \circ G_0^{-1}[G_1(J_1(x)|y) \mid y], \qquad x \in A_1,\ y \in Y_1.$$

Then

$$\begin{aligned}
\mu\{x \in A_0\colon \bar{\psi}_0(\psi_0(x)) \neq x\}
&= \int_{Y_1} \mu\{x \in A_0\colon \xi(\phi(x, f(x)), f(x)) \neq x \mid f = y\}\,\nu(dy) \\
&= \int_{Y_1} q(y)\,\mu_c\{x \in A_0\colon \xi(\phi(x, y), y) \neq x \mid f = y\}\,\nu(dy) = 0,
\end{aligned}$$

by the continuity of the conditional distributions. Similarly we show that

$$\mu\{x \in A_1 \colon \psi_0(\bar{\psi}_0(x)) \neq x\} = 0.$$

Hence there exist $A_i' \subset A_i$ with $A_i' = A_i$ μ-a.e., $i = 0, 1$, such that $\psi_0 \colon A_0' \mapsto A_1'$ is one-to-one and onto. Now, if $B \subset A_1'$, then

$$\begin{aligned}\mu(\psi_0^{-1} B) &= \int_{Y_1} q(y)\mu_c\{x \in A_0 \colon \phi(x, y) \in B \mid f = y\}\, \nu(dy)\\ &= \int_{Y_1} q(y)\mu_c\{x \in A_1 \colon x \in B \mid f = y\}\, \nu(dy) = \mu(B),\end{aligned}$$

which proves that ψ_0 is μ-preserving; thus ψ_0 and ψ_0^{-1} are nonsingular. Define

$$\psi(x) = \begin{cases} \psi_0(x) & \text{for } x \in A_0', \\ \psi_0^{-1}(x) & \text{for } x \in A_1', \\ x & \text{elsewhere.} \end{cases}$$

Since A_0' and A_1' are disjoint sets of positive measure, $\mu\{x \colon \psi(x) \neq x\} > 0$. Finally, (5.1) follows from the fact that the conditional measures are concentrated on the sets $\{f = y\}$ and that $\psi_0^{-1} = \bar{\psi}$ μ-a.e. on A_1'. This ends the proof under the assumption $\nu\{y \colon q(y) > 0\} > 0$.

Now we assume the opposite, i.e.,

$$\mu(\cdot \mid f = y) = \mu_d(\cdot \mid f = y) \quad \nu\text{-a.e.}$$

First we suppose that

$$\mu_d(\cdot \mid f = y) = \delta_{h(y)}(\cdot) \quad \nu\text{-a.e.},$$

where $h \colon Y \mapsto X$ is measurable and such that $f(h(y)) = y$ for every $y \in Y$. We will show that this cannot be the case. Indeed, for every $A \in \mathscr{B}_X$ we have

$$\mu(A \triangle f^{-1}(h^{-1} A)) = \int_Y \delta_{h(y)}(A \triangle f^{-1}(h^{-1} A))\, \nu(dy) = 0,$$

which shows that $f^{-1}(\mathscr{B}_Y) = \mathscr{B}_X$ mod-μ, contradicting the assumption of the proposition. Thus

$$\mu_d(\cdot \mid f = y) = a(y)\delta_{h(y)}(\cdot) + b(y)\delta_{k(y)}(\cdot) + c(y)\lambda(\cdot \mid f = y) \quad \nu\text{-a.e.},$$

where $h, k \colon Y \mapsto X$ are measurable, $f(h(y)) = f(k(y)) = y$, and $h(y) \neq k(y)$ for every $y \in Y$. Here $a(y)$, $b(y)$, $c(y) \geqslant 0$ are such that

$$\nu\{y \colon a(y)b(y) > 0\} > 0, \tag{5.2}$$

and $\lambda(\cdot \mid f = y)$ is a measurable family of discrete probability measures such that, for each $y \in Y$, $\lambda(\{h(y), k(y)\} \mid f = y) = 0$. Define

$$\phi(x, y) := \begin{cases} k(y) & \text{if } x = h(y), \\ h(y) & \text{if } x = k(y), \\ x & \text{otherwise,} \end{cases}$$

and

$$\psi(x) := \phi(x, f(x)), \quad x \in X.$$

Since $\psi(\psi(x)) = x$ for every $x \in X$, ψ is one-to-one and onto, and $\psi^{-1} = \psi$. Furthermore, (5.1) holds trivially and

$$\mu\{x: \psi(x) \neq x\} \geqslant \int_Y [a(y)+b(y)]\, \nu(dy) > 0.$$

It remains to verify that ψ and ψ^{-1} are nonsingular. Let $A \in \mathscr{B}_X$. Since

$$\mu(\psi^{-1} A) = \int_Y \left[a(y) \mathbf{1}_A(k(y)) + b(y) \mathbf{1}_A(h(y)) + c(y) \lambda(A \mid y)\right] \nu(dy)$$

and

$$\mu(A) = \int_Y \left[a(y) \mathbf{1}_A(h(y)) + b(y) \mathbf{1}_A(k(y)) + c(y) \lambda(A \mid y)\right] \nu(dy),$$

using (5.2) we infer that $\mu(\psi^{-1} A) = 0$ if and only if $\mu(A) = 0$. This completes the proof of the proposition. ∎

PROPOSITION 5.2. *Under the assumptions of Proposition* 5.1, $\sigma(\Gamma) = \mathscr{B}_X$ mod-μ *if and only if there exists a sequence* $\{f_n\} \subset \Gamma$ *and a* μ*-null set* X_0 *such that the map*

$$x \mapsto (f_1(x), f_2(x), \ldots)$$

is one-to-one on $X \backslash X_0$.

Proof. Exactly as in the proof of Proposition 5.1, we reduce the problem to the case of Γ consisting of one function, say $\Gamma = \{f\}$.

Suppose that $\sigma\{f\} = \mathscr{B}_X$ mod-μ. Choose a sequence $\{K_n\} \subset \mathscr{B}_X$ separating points of X. By our assumption, for every n, there exists $B_n \in \mathscr{B}_Y$ such that

$$\mu\left(f^{-1}(B_n) \triangle K_n\right) = 0.$$

Then f is one-to-one on $X \backslash X_0$, where $X_0 = \bigcup_n f^{-1}(B_n) \triangle K_n$ is a μ-null set.

The converse follows by Kuratowski's isomorphism theorem. ∎

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