PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 26, Fasc. 1 (2006), pp. 143–153

TOWARDS A GENERAL DOOB–MEYER DECOMPOSITION THEOREM

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Abstract. Both the Doob–Meyer and Graversen–Rao decomposition theorems can be proved following an approach based on predictable compensators of discretizations and weak-L^1 technique, which was developed by K. M. Rao. It is shown that any decomposition obtained by Rao's method gives predictability of compensators without additional assumptions (like submartingality in the original Doob–Meyer theorem or finite energy in the Graversen–Rao theorem).

2000 Mathematics Subject Classification: 60G07, 60G48, 60H05.

Key words and phrases: Doob–Meyer decomposition theorem, predictable compensator, processes with finite energy, martingales.

1. INTRODUCTION

In his seminal papers [11] and [12], Meyer proved that any submartingale belonging to so-called class (D) admits a unique decomposition into a sum of a uniformly integrable martingale and a "natural" (nowadays: "predictable") integrable increasing process. More than twenty years later, Graversen and Rao [3] obtained a Doob–Meyer type decomposition for processes "with finite energy", in general without uniqueness. While the original Doob–Meyer theorem was motivated by needs of potential theory, and only later found interesting probabilistic applications (vide: stochastic integration), the latter result was used in analysis of Markov processes [3] and quite recently proved to be a useful tool in investigations of the structure of Dirichlet processes and their extensions [1].

Both the Doob–Meyer and Graversen–Rao theorems can be proved following an approach based on predictable compensators of discretizations and weak-L^1 technique, which was developed by Rao [14]. In the present paper we

* Supported by Komitet Badań Naukowych under Grant No 1 P03A 022 26 and completed while the author was visiting Université de Rouen and Helsinki University of Technology.

show that any decomposition obtained by Rao's method leads to *predictable* compensators without additional assumptions (like submartingality in the original Doob–Meyer theorem or finite energy in the Graversen–Rao theorem).

The idea of the proof is in a sense similar to that from the paper [8] and is based on the celebrated Komlós theorem [10]. The details are however much subtle and require other advanced tools, like limit theorems for stochastic integrals and tightness in so-called S-topology introduced in [7].

2. THE RESULT

Let $\mathscr{B} = (\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in[0,T]}, P)$ be a stochastic basis, satisfying the "usual" conditions, i.e. the filtration $\{\mathscr{F}_t\}$ is right-continuous and $\mathscr{F}_0$ contains all P-null sets of $\mathscr{F}_T$. By convention, we set $\mathscr{F}_\infty = \mathscr{F}$. The family of stopping times with values in $[0, T]^* = [0, T]\cup\{+\infty\}$ and with respect to the filtration $\{\mathscr{F}_t\}_{t\in[0,T]^*}$ will be denoted by $\mathscr{T}$.

Let $\{X_t\}_{t\in[0,T]}$ be a stochastic process on $(\Omega, \mathscr{F}, P)$, adapted to $\{\mathscr{F}_t\}_{t\in[0,T]}$ (i.e. for each $t\in[0, T]$, X_t is $\mathscr{F}_t$-measurable) and progressively measurable. We say that X is *of class* (D) if the family of random variables $\{X_\tau;\ \tau\in\mathscr{T}\}$ is uniformly integrable (by definition $X_{+\infty} = 0$).

We say that X has *càdlàg* (or *regular*) *trajectories* if its P-almost all trajectories are right-continuous and have limits from the left on $[0, T]$.

For definitions of predictability, martingales etc. we refer to standard textbooks (e.g. [2], [4], [5], [9] or [13]).

Let $\theta_n = \{0 = t_0^n < t_1^n < t_2^n < \ldots < t_{k_n}^n = T\}$, $n = 1, 2, \ldots$, be condensing partitions of $[0, T]$ with

$$\max_{1\leqslant k\leqslant k_n} t_k^n - t_{k-1}^n \to 0 \quad \text{as } n\to\infty.$$

By "discretizations" $\{X_t^n\}_{t\in\theta_n}$ of $\{X_t\}_{t\in[0,T]}$ we mean the processes defined by

$$X_t^n = X_{t_k^n} \quad \text{if } t_k^n \leqslant t < t_{k+1}^n,\ X_T^n = X_T.$$

If random variables $\{X_t\}_{t\in[0,T]}$ are integrable, we can associate with any discretization X^n its "predictable compensator"

$$A_t^n = \begin{cases} 0 & \text{if } 0\leqslant t < t_1^n, \\ \sum_{j=1}^{k} E(X_{t_j^n} - X_{t_{j-1}^n} \mid \mathscr{F}_{t_{j-1}^n}) & \text{if } t_k^n \leqslant t < t_{k+1}^n,\ k = 1, 2, \ldots, k_n - 1, \end{cases}$$

$$A_T^n = \sum_{j=1}^{k_n} E(X_{t_j^n} - X_{t_{j-1}^n} \mid \mathscr{F}_{t_{j-1}^n}).$$

Notice that A_t^n is $\mathscr{F}_{t_{k-1}^n}$-measurable for $t_k^n \leqslant t < t_{k+1}^n$, and so the processes A^n are predictable in a very intuitive manner, both in the discrete and continuous case. It is also clear that for each $n \in N$ the discrete-time process $\{M_t^n\}_{t\in\theta_n}$ given by

$$M_t^n = X_t^n - A_t^n, \qquad t \in \theta_n,$$

is a martingale with respect to the discrete filtration $\{\mathscr{F}_t\}_{t\in\theta_n}$.

THEOREM 1. *Let* $\{X_t\}_{t\in[0,T]}$ *be a càdlàg process* of class (D) *with respect to the stochastic basis* $\mathscr{B}$. *If for some condensing sequence* $\{\theta_n\}_{n\in N}$ *the corresponding random variables* $\{A_T^n\}_{n\in N}$ *are uniformly integrable, then one can find a uniformly integrable martingale* $\{M_t\}_{t\in[0,T]}$ *and a predictable integrable càdlàg process* $\{A_t\}_{t\in[0,T]}$ of class (D) *such that we have the decomposition*

$$X_t = M_t + A_t, \qquad t \in [0, T]. \tag{1}$$

An immediate consequence of predictability of $\{A_t\}$ is contained in the following

COROLLARY 2. *If* $X_t = M_t' + A_t'$, $t \in [0, T]$, *is another decomposition with properties described in Theorem* 1, *then* $N_t = A_t - A_t'$, $t \in [0, T]$, *is a uniformly integrable* continuous *martingale.*

It is clear that if we can attribute to $\{A_t\}$ some additional properties (e.g. it is nondecreasing or has finite variation or zero quadratic variation ...), then the martingale N in Corollary 2 must be zero and we obtain the uniqueness of the decomposition. In less standard cases this idea has been exploited in [3] (for Markov processes) and [1] (for weak Dirichlet processes).

One may ask what are the processes with "exploding" sequences of compensators, i.e. with $\{A_T^n\}_{n\in N}$ not uniformly integrable. A variety of such processes can be constructed using the idea of self-cancellation of jumps, as in the following example.

EXAMPLE 3. Let $\{r_n\}_{n\in N}$ be a Rademacher sequence (i.i.d. with $P(r_n = 1) = P(r_n = -1) = 1/2$). Let $\{t_n\}_{n\in N}$ be a (deterministic) sequence of times decreasing to 0. Define

$$X_t = \sum_n I_{[t_n, t_{n-1})}(t)\, r_n \frac{1}{\sqrt{n}},$$

and consider the natural filtration generated by X. Notice that X has regular trajectories, is adapted and bounded, hence of class (D).

We shall prove that X does not admit any decomposition of the form $X_t = M_t + A_t$, where M is a uniformly integrable martingale and A is a predictable, integrable càdlàg process. On the contrary, suppose we are given such a representation. Then we have

$$\Delta M_{t_n} = \Delta X_{t_n} - \Delta A_{t_n} = r_n \frac{1}{\sqrt{n}} - r_{n+1} \frac{1}{\sqrt{n+1}} - \Delta A_{t_n}.$$

Using the facts that ΔM_{t_n} has null conditional expectation with respect to $\mathscr{F}_{t_n-}$ and ΔA_{t_n} is $\mathscr{F}_{t_n-}$-measurable, we obtain

$$\Delta A_{t_n} = -r_{n+1}\frac{1}{\sqrt{n+1}} \quad \text{and} \quad \Delta M_{t_n} = r_n \frac{1}{\sqrt{n}}.$$

In particular, with probability one

$$\sum_n |\Delta M_{t_n}|^2 = +\infty,$$

which is impossible, since the quadratic variation of a martingale is finite.

3. A REMARK ON THE GRAVERSEN–RAO THEOREM

Following [3] we say that X is a process *of finite energy* if along with a condensing sequence $\{\theta_n\}_{n\in N}$ of partitions

$$\sup_n E\left[\sum_{t_i^n\in\theta_n} (X_{t_i^n}-X_{t_{i-1}^n})^2\right] < +\infty. \tag{2}$$

Of course, if X is of finite energy, $|X_t|^2$ is integrable for every $t \leqslant T$ and $\sum_{s\leqslant T} \Delta X_s^2$ is integrable, where $\Delta X_s = X_s - X_{s-}$. It is also easy to see that any process with finite energy satisfies one of the main assumptions of our Theorem 1: the sequence $\{A_T^n\}$ is bounded in L^2, hence uniformly integrable.

Further, it is not difficult to show that for each $\varepsilon > 0$ there exists a stopping time τ_ε such that $P(\tau_\varepsilon < T) < \varepsilon$ and

$$E \sup_{t\in[0,T]} |X_{\tau_\varepsilon \wedge t}|^2 < +\infty.$$

This property gives some kind of localization in class (D), but in general we do not know whether processes with finite energy form a subclass of class (D) processes.

Thus we are not able to show that the Graversen–Rao decomposition theorem is contained in our Theorem 1. Moreover, we have no examples showing that it is necessary to complete the assumptions of the Graversen–Rao theorem (e.g. by considering processes of class (D)).

What we want to stress is the fact that in the *sketch* of the proof given in [3] one can find convergence of quantities like

$$E\int_0^T A_t\, dC_t,$$

where C_t is an increasing *integrable, possibly unbounded* process. Corresponding limits are taken for granted, without paying any attention to details.

In the next section we rigorously perform similar computations and we find the class (D) property unavoidable.

4. PROOF OF THEOREM 1

We will work with notation introduced in Section 2.

By the uniform integrability of $\{A_T^n\}$ we can find a subsequence $\{A_T^{n_k}\}$ convergent weakly in L^1 to some random variable α. This gives us the desired decomposition

$$X_t = M_t + A_t,$$

where $M_t = E(X_T - \alpha \mid \mathscr{F}_t)$ is a uniformly integrable martingale (we take a càdlàg version of this process) and $A_t = X_t - M_t$ is a càdlàg process.

The essential novelty is contained in the proof of predictability of the process $\{A_t\}$, where we apply the Komlós theorem [10] in a similar way as it was done in [8], in the proof of the classical Doob–Meyer decomposition theorem, and then explore some properties of so-called S-topology introduced in [7].

Just as in the paper [8], we can find a further subsequence $\{n_{k_l}\}$ such that as $N \to \infty$

$$B_T^N = \frac{1}{N} \sum_{l=1}^{N} A_T^{n_{k_l}} \to \alpha = A_T \text{ a.s. and in } L^1. \tag{3}$$

It follows that, as $N \to \infty$,

$$M_T - \frac{1}{N} \sum_{l=1}^{N} M_T^{n_{k_l}} \to 0 \text{ a.s. and in } L^1, \tag{4}$$

where $M_T = X_T - A_T$ and $M_T^n = X_T^n - A_T^n = X_T - A_T^n$.

Next let us consider natural interpolations $\{\tilde{M}_t^n\}_{t \in [0,T]}$ of the discrete-time martingales $\{M_t^n\}_{t \in \theta_n}$ to a uniformly integrable martingale with respect to the full filtration $\{\mathscr{F}_t\}_{t \in [0,T]}$. In other words,

$$\tilde{M}_t^n = E(M_T^n \mid \mathscr{F}_t), \quad t \in [0, T].$$

It is a routine computation to verify that we have a decomposition

$$\tilde{M}_t^n = \tilde{X}_t^n - \tilde{A}_t^n, \quad t \in [0, T],$$

where

$$\tilde{X}_0^n = X_0, \quad \tilde{X}_t^n = E(X_{t_k^n} \mid \mathscr{F}_t) \quad \text{if } t_{k-1}^n < t \leqslant t_k^n,\ k = 1, 2, \ldots, k_n,$$

$$\tilde{A}_0^n = 0, \quad \tilde{A}_t^n = A_{t_k^n}^n \quad \text{if } t_{k-1}^n < t \leqslant t_k^n,\ k = 1, 2, \ldots, k_n.$$

The processes $\tilde{A}^n$ are adapted to the filtration $\{\mathscr{F}_t\}_{t \in [0,T]}$ and their trajectories are *left continuous*, hence they are *predictable* by the very definition of the predictable σ-field.

Notice that for $t \in \theta_n$

$$\tilde{M}_t^n = M_t^n, \qquad \tilde{A}_t^n = A_t^n,$$

and, in particular,

$$\tilde{M}_T^n = M_T^n, \qquad \tilde{A}_T^n = A_T^n.$$

We have also

LEMMA 4. *The sequence $\{\tilde{A}^n\}$ is uniformly of class* (D), *i.e. the family $\{\tilde{A}_\tau^n: n \in N, \tau \in \mathcal{T}\}$ is uniformly integrable.*

Proof. By the very definition,

$$\tilde{A}_\tau^n = \sum_{k=1}^{k_n} A_{t_k^n}^n I(t_{k-1}^n < \tau \leqslant t_k^n).$$

Since τ is a stopping time, the event $\{t_{k-1}^n < \tau \leqslant t_k^n\}$ belongs to $\mathcal{F}_{t_k^n}$. If we define

(5) $$\varrho^n(\tau) = 0 \text{ if } \tau = 0, \qquad \varrho^n(\tau)(\omega) = t_k^n \text{ if } t_{k-1}^n < \tau \leqslant t_k^n,$$

then $\varrho^n(\tau)$ is a stopping time with respect to the discrete filtration $\{\mathcal{F}_t\}_{t \in \theta_n}$, and

$$\tilde{A}_\tau^n = A_{\varrho^n(\tau)}^n.$$

By the discrete Doob–Meyer decomposition, $A_{\varrho^n(\tau)}^n = X_{\varrho^n(\tau)} - M_{\varrho^n(\tau)}^n$, where X is of class (D) and $\{M^n\}$ is a sequence of discrete time martingales with uniformly integrable terminal values $M_T^n = X_T - A_T^n$. ∎

Set

$$\tilde{R}_t^N = \frac{1}{N} \sum_{l=1}^{N} \tilde{M}_t^{n_{k_l}},$$

and observe that (4) implies uniform in probability convergence of martingales $\tilde{R}^N$ to the martingale M:

(6) $$P\big(\sup_{t \in [0,T]} |M_t - \tilde{R}_t^N| > \varepsilon\big) \to 0 \quad \text{as } N \to \infty,\ \varepsilon > 0.$$

Fix a stopping time $\tau \in \mathcal{T}$. Since $\{\tilde{R}_\tau^N\}_{N \in N}$ is uniformly integrable, we obtain from the above that

(7) $$\tilde{R}_\tau^N \to M_\tau \text{ in } L^1.$$

In what follows we shall suppress the subscript k_l in the subsequence n_{k_l}. With $\varrho^n(t)$ defined by (5) we have $\tilde{X}_t^n = E(X_{\varrho^n(t)} \mid \mathcal{F}_t)$ and we can rewrite (7) in the form

$$\frac{1}{N} \sum_{m=1}^{N} \big(E(X_{\varrho^m(\tau)} \mid \mathcal{F}_\tau) - \tilde{A}_\tau^m\big) - (X_\tau - A_\tau)$$
$$= \frac{1}{N} \sum_{m=1}^{N} E(X_{\varrho^m(\tau)} - X_\tau \mid \mathcal{F}_\tau) + \Big(A_\tau - \frac{1}{N} \sum_{m=1}^{N} \tilde{A}_\tau^m\Big) \to 0 \text{ in } L^1.$$

As m tends to infinity, $\varrho^m(\tau) \searrow \tau$, and by the right continuity of $\{X_t\}$, we have $X_{\varrho^m(\tau)} \to X_\tau$ a.s. Since $\{X_t\}$ is of class (D), the latter convergence holds also in L^1, hence

$$E(X_{\varrho^m(\tau)} - X_\tau | \mathscr{F}_\tau) \to 0 \text{ in } L^1.$$

Finally we see that for any stopping time τ

$$\tilde{B}_\tau^N = \frac{1}{N}\sum_{m=1}^{N} \tilde{A}_\tau^m \to A_\tau \text{ in } L^1. \tag{8}$$

This fact allows us to deduce a further remarkable property of the sequence $\{\tilde{A}^m\}$.

LEMMA 5. *For each stopping time $\tau \in \mathscr{T}$, $\tilde{A}_\tau^m$ converges to A_τ weakly in L^1.*

Proof. Fix $\tau \in \mathscr{T}$ and suppose that for some bounded random variable Z and along with some subsequence $\{m_r\}$

$$E\tilde{A}_\tau^{m_r} Z \to c \neq EA_\tau Z. \tag{9}$$

Due to the "subsequence property" of the Komlós theorem, the relation (3) remains unchanged if we replace the subsequence $\{n_{k_l}\}$ with its subsequence $\{m_r\}$. Consequently also (6) and (8) hold, and hence

$$\frac{1}{N}\sum_{r=1}^{N} \tilde{A}_\tau^{m_r} \to A_\tau \text{ in } L^1.$$

This is in contradiction to (9). ∎

Notice that $\tilde{A}_\tau^n$'s are $\mathscr{F}_{\tau-}$-measurable, and so by (8) the same property belongs to A_τ. We have thus checked one of the two conditions equivalent to the predictability of a càdlàg process (see e.g. [4], Theorem 4.33). The other condition requires that $A_\tau = A_{\tau-}$ a.s. on $\{\tau < +\infty\}$ for every totally inaccessible stopping time. We may and do assume that $A_\tau \geqslant A_{\tau-}$ a.s. or $A_\tau \leqslant A_{\tau-}$ a.s. on $\tau < +\infty$ (otherwise set e.g. $G = \{A_\tau \geqslant A_{\tau-}\} \in \mathscr{F}_\tau$; then $\tau' = \tau I_G + (+\infty) I_{G^c}$ is totally inaccessible and satisfies $A_{\tau'} \geqslant A_{\tau'-}$).

Let $\{C_t\}$ be a continuous nonnegative increasing process such that the process $\{I(\tau \leqslant t) - C_t\}_{t\in[0,T]}$ is a uniformly integrable martingale of zero mean. Since τ is totally inaccessible, $P(\tau = 0) = 0$ and we have $C_t = 0$ a.s.

Fix $K > 0$ and define stopping times

$$\eta_K = \inf\{t \in [0, T]\colon C_t > K\} \wedge T. \tag{10}$$

Notice that by continuity of C and $C_0 = 0$ a.s. we have $\eta_K > 0$ a.s.

We shall prove that

$$E\int_0^T \tilde{A}_t^m \, dC_{t\wedge\eta_K} \to E\int_0^T A_t \, dC_{t\wedge\eta_K} \quad \text{as } m \to \infty. \tag{11}$$

At first we shall ensure uniform integrability of the integrals.

LEMMA 6. *Let* $\{D_t\}_{t\in[0,T]}$ *be a* bounded *increasing adapted continuous process,* $D_0 = 0$ *a.s. and let* $\{B_t^i\}_{i\in I}$ *be a family of càdlàg or* càglàd *processes which are* uniformly of class (D). *Then the family of integrals* $\{\int_0^T B_s^i\, dD_s\}_{i\in I}$ *is uniformly integrable.*

Proof. We may assume that $D_T \leqslant 1$. Let $\Phi\colon \boldsymbol{R}^+ \to \boldsymbol{R}^+$ be a convex increasing function such that $\Phi(u)/u \to +\infty$ as $u \to +\infty$ and

$$\sup_{i,\tau} E\Phi(|B_\tau^i|) < +\infty.$$

Consider stopping times $\{\eta_{k/n}\colon k = 1, 2, \ldots, n,\ n\in N\}$ defined for D by (10) and observe that

$$\sup_{i,n} E\Phi\Big(\Big|\sum_{k=1}^{n} B_{\eta_{k/n}}^i (D_{\eta_{k/n}} - D_{\eta_{(k-1)/n}})\Big|\Big) \leqslant \sup_{i,n} E\Phi\Big(\sum_{k=1}^{n} |B_{\eta_{k/n}}^i| n^{-1}\Big)$$

$$\leqslant \sup_{i,n} n^{-1} \sum_{k=1}^{n} E\Phi(|B_{\eta_{k/n}}^i|) < +\infty.$$

Since a.s. $\int_0^T B_s^i\, dD_s = \lim_{n\to\infty} \sum_{k=1}^n B_{\eta_{k/n}}^i (D_{\eta_{k/n}} - D_{\eta_{(k-1)/n}})$, the proof is complete. ■

In fact, we have proved uniform integrability of the larger family:

COROLLARY 7. *The following family*

$$\Big\{\int_0^T B_s^i\, dD_s\colon i\in I\Big\} \cup \Big\{\sum_{k=1}^{n} B_{\eta_{k/n}}^i (D_{\eta_{k/n}} - D_{\eta_{(k-1)/n}})\colon i\in I,\ n\in N\Big\}$$

is uniformly integrable.

Let us return to the proof of (11). Assume for brevity that $K\in N$. Suppose that for some $\delta > 0$ and along with some subsequence $\{m'\}$

$$|E\int_0^T \tilde{A}_t^{m'}\, dC_{t\wedge\eta_K} - E\int_0^T A_t\, dC_{t\wedge\eta_K}| > \delta. \tag{12}$$

We have, by Corollary 7,

$$E\int_0^T A_t\, dC_{t\wedge\eta_K} = \lim_{n\to\infty} \sum_{k=1}^{K\cdot n} EA_{\eta_{k/n}} (C_{\eta_{k/n}} - C_{\eta_{(k-1)/n}}).$$

On the other hand, by Lemma 5 for fixed $n\in N$, $k \leqslant K\cdot n$ we have

$$EA_{\eta_{k/n}} (C_{\eta_{k/n}} - C_{\eta_{(k-1)/n}}) = \lim_{m'\to\infty} E\tilde{A}_{\eta_{k/n}}^{m'} (C_{\eta_{k/n}} - C_{\eta_{(k-1)/n}}).$$

It follows that one can find a subsequence $m_n' \to \infty$ of $\{m'\}$ such that

$$E\int_0^T A_t\, dC_{t\wedge\eta_K} = \lim_{n\to\infty} \sum_{k=1}^{K\cdot n} E\tilde{A}_{\eta_{k/n}}^{m_n'} (C_{\eta_{k/n}} - C_{\eta_{(k-1)/n}}) \tag{13}$$

$$= \lim_{n\to\infty} \int_0^T \pi_n(\tilde{A}^{m_n'})_t\, d\pi_n(C)_t,$$

where $\pi_n(B)$ is the discretization of the process B at random times $0 \leqslant \eta_{1/n} \leqslant \eta_{2/n} \leqslant \ldots \leqslant \eta_K \leqslant T$. We shall prove that along the subsequence $\{m'_n\}$

$$E\int_0^T \tilde{A}_t^{m'_n}\, dC_{t\wedge\eta_K} - E\int_0^T \pi_n(\tilde{A}^{m'_n})_t\, d\pi_n(C)_t \to 0.$$

Given (13), this will contradict (12).

In view of Corollary 7 it is enough to prove that for an arbitrary sequence $\{m_n\}$

$$\int_0^T \tilde{A}_t^{m_n}\, dC_{t\wedge\eta_K} - \int_0^T \pi_n(\tilde{A}^{m_n})_t\, d\pi_n(C)_t \to 0 \text{ in probability.}$$

Let $\bar{A}^n$ be the càdlàg version of the process $\tilde{A}^n$. Obviously, we have

$$\int_0^T \tilde{A}_t^{m_n}\, dC_{t\wedge\eta_K} = \int_0^T \bar{A}_{t-}^{m_n}\, dC_{t\wedge\eta_K}.$$

We have also for $k > 0$

$$\pi_n(\tilde{A}^{m_n})_{\eta_{k/n}} = \bar{A}^{m_n}_{\eta_{k/n}-},$$

and so

$$\int_0^T \pi_n(\tilde{A}^{m_n})_t\, d\pi_n(C)_t = \int_0^T \bar{A}_{t-}^{m_n}\, d\pi_n(C)_t.$$

We were thus able to reduce the problem to the convergence

$$\int_0^T \bar{A}_{t-}^{m_n}\, d\big(C_{t\wedge\eta_K} - \pi_n(C)_t\big) \to 0 \text{ in probability.}$$

We are going to apply results of [6]. In this context we need to recall the notion of S-tightness, i.e. uniform tightness in so-called S-topology on the Skorokhod space $D([0, T]: \mathbf{R}^1)$ introduced in [7] (see Proposition 3.1 there).

Let $\{X^\alpha\}$ be a family of stochastic processes with càdlàg trajectories on $[0, T]$. For a càdlàg function $x \in D([0, 1]: \mathbf{R}^1)$ denote by $N_a^b(x)$ the number of up-crossings given levels $a < b$, $a, b \in \mathbf{R}^1$, on the interval $[0, T]$. Set also $\|x\|_\infty = \sup_{t\in[0,T]} |x(t)|$. The family $\{X^\alpha\}$ is said to be *S-tight* if the family $\{\|X^\alpha\|_\infty\}$ is bounded in probability and for each pair $a < b$ of reals the family $\{N_a^b(X^\alpha)\}$ is bounded in probability.

LEMMA 8. *The family $\{\bar{A}^n\}$ is S-tight.*

Proof. Any trajectory of $\bar{A}^n$ can be obtained by change of time of the corresponding trajectory of A^n (this change of time is related to the discretization θ_n and it eliminates the value 0 taken by A^n on $[0, t_1^n)$). Hence we have

$$\|\bar{A}^n\|_\infty = \|A^n\|_\infty, \quad N_a^b(\bar{A}^n) \leqslant N_a^b(A^n), \quad a < b,\ a, b \in \mathbf{R}^1.$$

To prove S-tightness of the family $\{A^n\}$ we observe first that due to the discrete nature of processes A^n it is sufficient to compute the quantities $\|A^n\|_\infty$ and $N_a^b(A^n)$ over the finite set θ_n. Further, on θ_n we have $A_t^n = X_t^n - M_t^n$, where $\{X^n\}$ is a restriction of the càdlàg process X and M^n is a martingale with respect to the discrete filtration $\{\mathcal{F}_t\}_{t\in\theta_n}$. Since

$$\sup_n E|M_T^n| \leqslant E|X_T| + \sup_n E|A_T^n| < +\infty,$$

we obtain S-tightness of $\{M^n\}$ by standard martingale inequalities. And S-tightness of the family $\{X^n\}$ of discretizations of a càdlàg-process is obvious. ■

Given S-tightness of the sequence $\{\bar{A}^n\}$ we are completely in the framework considered in [6]. We cannot however simply apply Theorem 3.11 of [7] and then Theorem 1 of [6], for we do not control the convergence of $\bar{A}_0^n = A_{t_1^n}^n$. Instead we can use Theorem 7 of [6] which states that any limit in distribution of our sequence of stochastic integrals is again a stochastic integral with respect to the limit of the sequence $C_{t\wedge\eta_K} - \pi_n(C)_t$, which is 0. We have proved (11).

If (11) is established, the rest of the proof is straightforward. We have

$$\begin{aligned} EA_{\tau-} I(\tau \leqslant \eta_K) &= E\int_0^T A_{t-}\, dI(\tau \leqslant t\wedge\eta_K) = E\int_0^T A_{t-}\, dC_{t\wedge\eta_K} \\ &= E\int_0^T A_t\, dC_{t\wedge\eta_K} \quad \text{(for } C \text{ is continuous)} \\ &= \lim_{m\to\infty} E\int_0^T \tilde{A}_t^m\, dC_{t\wedge\eta_K} \quad \text{(by (11))} \\ &= \lim_{m\to\infty} E\int_0^T \tilde{A}_t^m\, dI(\tau \leqslant t\wedge\eta_K) = \lim_{m\to\infty} E\tilde{A}_\tau^m I(\tau \leqslant \eta_K) \\ &= EA_\tau I(\tau \leqslant \eta_K) \quad \text{(by Lemma 5).} \end{aligned}$$

We have assumed that $A_\tau \geqslant A_{\tau-}$ or $A_\tau \leqslant A_{\tau-}$, so we obtain

$$A_{\tau-} I(\tau \leqslant \eta_K) = A_\tau I(\tau \leqslant \eta_K) \text{ a.s.,} \qquad K > 0.$$

Since C_t is integrable, $P(\eta_K < T) \to 0$ as $K \to \infty$. Hence $A_\tau = A_{\tau-}$ a.s. and the theorem follows.

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Received on 25.6.2006