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A FAMILY OF GENERALIZED GAMMA CONVOLUTED VARIABLES

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Abstract. This paper consists of three parts: in the first part, we describe a family of generalized gamma convoluted (abbreviated as GGC) variables. In the second part, we use this description to prove that several r.v.'s, related to the length of excursions away from 0 for a recurrent linear diffusion on $\mathbb{R}_+$, are GGC. Finally, in the third part, we apply our results to the case of Bessel processes with dimension $d = 2(1-\alpha)$, where $0 < d < 2$ or $0 < \alpha < 1$.

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1. NOTATION AND INTRODUCTION

1.1. Let $l : \mathbb{R}_+ \to \mathbb{R}_+$ denote a Borel function such that

$$\int_0^\infty \frac{l(z)}{z}\, dz < \infty. \tag{1.1}$$

Without loss of generality, we assume that

$$\int_0^\infty \frac{l(z)}{z}\, dz = 1. \tag{1.2}$$

With l we associate an r.v. Y on $\mathbb{R}_+$ whose probability density f_Y is given by

$$f_Y(u) = \int_0^\infty e^{-uz}\, l(z) dz \quad (u \geqslant 0). \tag{1.3}$$

Indeed, due to (1.2), we get

$$\int_0^\infty f_Y(u) du = \int_0^\infty du \int_0^\infty e^{-uz}\, l(z) dz = \int_0^\infty \frac{l(z)}{z}\, dz = 1. \tag{1.4}$$

To emphasize the relation between Y and l, we shall (sometimes) write Y_l.

We denote by $\varphi_l \equiv \varphi_{Y_l}$ the Laplace transform of Y_l:

$$\varphi_l(\lambda) = \varphi_{Y_l}(\lambda) = E(e^{-\lambda Y_l}) = \int_0^\infty e^{-\lambda u} f_{Y_l}(u)du = \int_0^\infty \frac{l(z)}{\lambda + z}\, dz. \tag{1.5}$$

Thus, since f_{Y_l} is the Laplace transform of l, φ_l is the Stieltjes transform of l.

1.2. A reminder about GGC variables. Let μ denote a positive σ-finite measure on $\mathbb{R}_+$. We recall (see Bondesson [3]) that a positive r.v. Y is a GGC variable with Thorin measure μ if

$$E(e^{-\lambda Y}) = \exp\left\{ -\int_0^\infty (1 - e^{-\lambda x})\Big(\int_0^\infty e^{-xz}\mu(dz)\Big)\frac{dx}{x}\right\} \quad (\lambda \geqslant 0). \tag{1.6}$$

Such an r.v. is self-decomposable, hence infinitely divisible.

The GGC r.v.'s Y whose Thorin measure μ has a finite total mass, equal to m, are characterized by (see [6])

$$E(e^{-\lambda Y}) = \exp\left\{ -m\int_0^\infty (1 - e^{-\lambda x})\, E(e^{-xG})\, \frac{dx}{x}\right\}, \tag{1.7}$$

where G is an $\mathbb{R}_+$-valued r.v. such that $E\big(\log^+(1/G)\big) < \infty$. Such an r.v. is a gamma-m mixture, i.e. it satisfies[1]:

$$Y \overset{\text{(law)}}{=} \gamma_m \cdot Z, \tag{1.8}$$

where γ_m is a gamma variable with parameter m, independent of the $\mathbb{R}_+$-valued variable Z. We note that any r.v. which is a gamma-m mixture is also a gamma-m' mixture for any $m' > m$, since we have the identity

$$\gamma_m \overset{\text{(law)}}{=} \gamma_{m'} \cdot \beta_{m,\, m'-m}, \tag{1.9}$$

where $\gamma_{m'}$ is a gamma variable with parameter m' and $\beta_{m,\,m'-m}$ is a beta variable with parameters $(m,\, m'-m)$ independent of $\gamma_{m'}$.

We also recall (see [3], p. 51) that the parameter m of a GGC r.v. Y, with Thorin measure with total mass m, may be obtained from the formula

$$m = \sup\left\{\delta \geqslant 0;\ \lim_{u \downarrow 0_+} \frac{f_Y(u)}{u^{\delta - 1}} = 0\right\}. \tag{1.10}$$

---

[1] It would be more correct to say that: the law of such an r.v. is a gamma-m mixture; however, such an abuse is usual, and should not lead to confusion.

2. A FAMILY OF GGC VARIABLES

The aim of this part is to present a sufficient condition on l which implies that the associated variable Y_l is GGC.

DEFINITION 1. A function l which satisfies (1.1) belongs to the *class* $\mathcal{C}$ if there exist $a \geqslant 0$, $b > a$, $\sigma \geqslant 0$ and $\theta : \mathbb{R}_+ \to \mathbb{R} \cup (+\infty)$ a Borel, decreasing function, which is identically equal to $+\infty$ on $[0, a[$, such that

$$l(z) = \exp\left\{\sigma + \int_b^z \frac{\theta(y)}{y}\, dy\right\}. \tag{2.1}$$

Of course, if (2.1) is satisfied with $a > 0$, then the function l is identically 0 on $[0, a[$. On the other hand, if l is identically 0 on $[0, a[$ and differentiable on $]a, \infty[$, then l belongs to the class $\mathcal{C}$ if and only if the function

$$y \to y\,(\log\, l)'(y) := \theta(y) \tag{2.2}$$

is decreasing on $[a, \infty[$.

The following properties are elementary:

(2.3) If $l \in \mathcal{C}$, then, for every $u > 0$, $x \to l(ux) \in \mathcal{C}$.

(2.4) If $l_1, l_2 \in \mathcal{C}$, then $l_1 \cdot l_2 \in \mathcal{C}$.

(2.5) For every α real, $x \to x^\alpha$ satisfies (2.1) but not (1.1).

(2.6) For every $k < 0$ and $\gamma \geqslant 0$, $x \to (x+\gamma)^k \in \mathcal{C}$.

THEOREM 2. *Assume that l satisfies* (1.2) *and belongs to* $\mathcal{C}$, *and let* Y_l *denote the r.v. associated with l. Then*: Y_l *is a GGC r.v. whose Thorin measure* μ *has total mass* m *smaller than or equal to* 1. *In other terms*, *there exists an r.v.* G *taking values in* $\overline{\mathbb{R}}_+$ *and satisfying* $E\big(\log^+(1/G)\big) < \infty$ *and* $m \leqslant 1$ *such that*

$$E(e^{-\lambda Y_l}) = \exp\left\{-m\int_0^\infty (1-e^{-\lambda x})E(e^{-xG})\frac{dx}{x}\right\} \quad (\lambda \geqslant 0). \tag{2.7}$$

P r o o f. Our proof consists of three parts.

1. It suffices to show that Y_l is GGC since, if so, then the total mass m of its Thorin measure equals, by (1.3) and (1.10):

$$m = \sup\left\{\delta \geqslant 0;\ \lim_{u\downarrow 0_+} \frac{1}{u^{\delta-1}}\int_0^\infty e^{-uz}\, l(z)dz = 0\right\}$$

and, of course, $m \leqslant 1$ since, for $\delta = 1$:

$$\frac{1}{u^{\delta-1}}\int_0^\infty e^{-uz}\, l(z)dz = \int_0^\infty e^{-uz}\, l(z)dz \underset{u\downarrow 0_+}{\longrightarrow} \int_0^\infty l(z)dz > 0.$$

2. To show that Y_l is GGC, we shall use the following characterization (see [3], Theorem 6.1.1, p. 90) of these r.v.'s:

Y is GGC if and only if its Laplace transform φ_Y is hyperbolically completely monotone, that is, it satisfies: for every $u>0$, the function H_u, defined by

$$H_u(w)=\varphi_Y(uv)\cdot\varphi_Y\left(\frac{u}{v}\right),\quad \text{where } w=v+\frac{1}{v}, \tag{2.8}$$

is a completely monotone function, i.e., it is the Laplace transform of a positive measure carried by $\mathbb{R}_+$.

In our framework, this criterion becomes: for every $u>0$, H_u is completely monotone with, by (1.5),

$$H_u(w)=\int_0^\infty\int_0^\infty \frac{l(x)l(y)}{(x+uv)(y+u/v)}\,dx\,dy \qquad (w=v+1/v), \tag{2.9}$$

and so

$$H_u(w)=\int_0^\infty\int_0^\infty \frac{l(ux)l(uy)}{(x+v)(y+1/v)}\,dx\,dy \tag{2.10}$$

(after the change of variables $x=ux',\ y=uy'$).

Our aim is to show that the hypothesis $l\in\mathcal{C}$ implies that H_u is completely monotone, and since $x\to l(ux)$ belongs to $\mathcal{C}$ if $l\in\mathcal{C}$ (by (2.3)), it suffices to see that the function H defined by

$$H(w):=\int_0^\infty\int_0^\infty \frac{l(x)l(y)}{(x+v)(y+1/v)}\,dx\,dy \qquad (w=v+1/v) \tag{2.11}$$

is completely monotone.

3. We show now that H, defined by (2.11), is completely monotone.

(i) We write

$$\begin{aligned} H(w)&=\int_0^\infty\int_0^\infty \frac{l(x)l(y)}{(x+v)(y+1/v)}\,dx\,dy \\ &=\frac{1}{2}\int_0^\infty\int_0^\infty l(x)l(y)\left[\frac{1}{(x+v)(y+1/v)}+\frac{1}{(x+1/v)(y+v)}\right]dx\,dy \end{aligned} \tag{2.12}$$

(by symmetry). Hence

$$\begin{aligned} H(w)=\frac{1}{2}\int_0^\infty\int_0^\infty l(x)l(y)\bigg[&\frac{x^2-1}{xy-1}\cdot\frac{1}{x^2+xw+1} \\ &+\frac{y^2-1}{xy-1}\cdot\frac{1}{y^2+yw+1}\bigg]dx\,dy \end{aligned}$$

(after reducing both reciprocals to the same denominator and decomposing into simple elements). Consequently,

$$H(w) = \frac{1}{2}\int_0^\infty\int_0^\infty l(x)l(y)dx\,dy$$
$$\times\left[\frac{x^2-1}{xy-1}\int_0^\infty \exp\big(-b(x^2+xw+1)\big)db + \frac{y^2-1}{xy-1}\int_0^\infty \exp\big(-b(y^2+yw+1)\big)db\right]$$
$$= \frac{1}{2}\int_0^\infty\int_0^\infty l(x)l(y)dx\,dy\left[\frac{x^2-1}{xy-1}\cdot\frac{1}{x}\int_0^\infty \exp\big(-bw-b(x+1/x)\big)db\right.$$
$$\left.+\int_0^\infty \frac{y^2-1}{xy-1}\,\frac{1}{y}\exp\big(-bw-b(y+1/y)\big)db\right]$$

(after making the change of variables $bx = b'$, $by = b'$). Therefore

$$H(w) = \int_0^\infty e^{-bw}db\left[\int_0^\infty\int_0^\infty l(x)l(y)\frac{x^2-1}{(xy-1)x}\,\exp\big(-b(x+1/x)\big)dx\,dy\right] \tag{2.13}$$

after interverting the orders of integration.

We note that the preceding computation is a little formal: we have transformed an absolutely convergent integral into an integral which is no longer absolutely convergent; however, this does not matter for our purpose, as we shall soon gather the different terms in another way.

(ii) Thus, we need to show, by (2.13), that, for every $b \geqslant 0$:

$$I_b := \int_0^\infty\int_0^\infty l(x)l(y)\,\frac{x^2-1}{(xy-1)x}\,\exp\big(-b(x+1/x)\big)dx\,dy \geqslant 0. \tag{2.14}$$

- Let us show (2.14). For this purpose, we define the four domains (Fig. 1)

$$\mathcal{N}_1 = \left\{0 < x \leqslant 1,\ y > \frac{1}{x}\right\},\quad \mathcal{N}_2 = \left\{x \geqslant 1,\ y < \frac{1}{x}\right\},$$
$$\mathcal{P}_1 = \left\{x \geqslant 1,\ y > \frac{1}{x}\right\},\qquad \mathcal{P}_2 = \left\{0 < x \leqslant 1,\ y < \frac{1}{x}\right\}.$$

Let us define

$$\psi(x,y) := l(x)l(y)\,\frac{x^2-1}{(xy-1)x}\,\exp\big(-b(x+1/x)\big). \tag{2.15}$$

It is clear that ψ is negative on $\mathcal{N}_1$ and $\mathcal{N}_2$ and positive on $\mathcal{P}_1$ and $\mathcal{P}_2$. We note

$$N_i := \iint_{\mathcal{N}_i} |\psi(x,y)|dx\,dy \qquad (i=1,2),$$
$$P_i := \iint_{\mathcal{P}_i} \psi(x,y)\,dx\,dy \qquad (i=1,2).$$

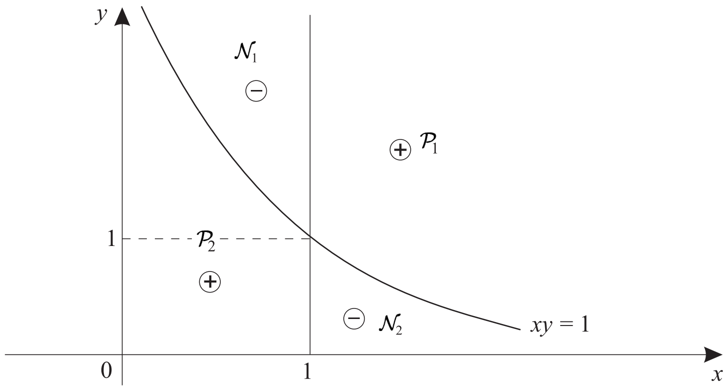

FIGURE 1

• To prove (2.14) it suffices to see that $N_i \leqslant P_i$ $(i = 1, 2)$. To compute N_1 and P_2 ($\mathcal{N}_1,\ \mathcal{P}_2 \subset \{(x,y) \in \mathbb{R}_+^2;\ x \leqslant\ 1\}$), we make the change of variables for $x \in]0,1],\ t \geqslant 2$:

$$x = \frac{t - \sqrt{t^2-4}}{2};$$

so we have

$$\frac{1}{x} = \frac{t + \sqrt{t^2-4}}{2}, \qquad x + \frac{1}{x} = t, \qquad \frac{x^2-1}{x^2}\,dx = dt.$$

We obtain

$$\text{(2.16)} \qquad N_1 = \int_2^\infty dt \int_{(t+\sqrt{t^2-4})/2}^\infty dy\, l\Big(\frac{t-\sqrt{t^2-4}}{2}\Big) l(y)\, \frac{e^{-bt}}{y - (t+\sqrt{t^2-4})/2}$$
$$= \int_2^\infty dt \int_0^\infty dz\, l\Big(\frac{t-\sqrt{t^2-4}}{2}\Big) l\Big(\Big(\frac{t+\sqrt{t^2-4}}{2}\Big)(1+z)\Big) \frac{e^{-bt}}{z}$$

(after making the change of variable $y = (1+z)\big((t+\sqrt{t^2-4})/2\big)$) and

$$\text{(2.17)} \qquad P_2 = \int_2^\infty dt \int_0^1 dz\, l\Big(\frac{t-\sqrt{t^2-4}}{2}\Big) l\Big(\Big(\frac{t+\sqrt{t^2-4}}{2}\Big)(1-z)\Big) \frac{e^{-bt}}{z}.$$

To compute N_2 and P_1 ($\mathcal{N}_2,\ \mathcal{P}_1 \subset \{(x,y) \in \mathbb{R}_+^2;\ x \geqslant 1\}$) for $x \geqslant 1,\ t \geqslant 2$ we make the change of variable:

$$x = \frac{t + \sqrt{t^2-4}}{2};$$

so we have

$$\frac{1}{x} = \frac{t-\sqrt{t^2-4}}{2}, \qquad x+\frac{1}{x} = t, \qquad \frac{x^2-1}{x^2}\,dx = dt.$$

We obtain

$$N_2 = \int_2^\infty dt \int_0^1 dz\, l\Big(\frac{t+\sqrt{t^2-4}}{2}\Big) l\Big(\Big(\frac{t-\sqrt{t^2-4}}{2}\Big)(1-z)\Big)\frac{e^{-bt}}{z}, \tag{2.18}$$

$$P_1 = \int_2^\infty dt \int_0^\infty dz\, l\Big(\frac{t+\sqrt{t^2-4}}{2}\Big) l\Big(\Big(\frac{t-\sqrt{t^2-4}}{2}\Big)(1+z)\Big)\frac{e^{-bt}}{z}. \tag{2.19}$$

We shall now use the hypothesis that l belongs to $\mathcal{C}$ to show that

$$P_1 \geqslant N_1 \quad \text{and} \quad P_2 \geqslant N_2,$$

which will complete the proof of our theorem.

- Comparing (2.19) and (2.16), it suffices to prove that $P_1 \geqslant N_1$ to show that

$$\Big(\frac{t+\sqrt{t^2-4}}{2}\Big) l\Big(\Big(\frac{t-\sqrt{t^2-4}}{2}\Big)(1+z)\Big) \geqslant l\Big(\frac{t-\sqrt{t^2-4}}{2}\Big) l\Big(\Big(\frac{t+\sqrt{t^2-4}}{2}\Big)(1+z)\Big),$$

i.e.

$$l\left(\frac{1}{x}\right) l(c\,x) \geqslant l(x) l\left(\frac{c}{x}\right) \quad \text{with } x \leqslant 1 \text{ and } c \geqslant 1. \tag{2.20}$$

If $a \geqslant 1$ (a being featured in the definition of $\mathcal{C}$), the relation (2.20) is trivially satisfied since $l(x) = 0$ for $x \leqslant a$ (and $x \leqslant 1$).

We now examine the case $0 \leqslant a < 1$.

If $x \leqslant a$, the relation (2.20) is again trivially satisfied. Thus, let us assume that $1 \geqslant x \geqslant a$. The relation (2.20) is equivalent to

$$\log\, l\left(\frac{1}{x}\right) - \log\, l(x) \geqslant \log\, l\left(\frac{c}{x}\right) - \log\,(c\,x)$$

or also to

$$\int_x^{1/x} \frac{\theta(y)}{y}\,dy - \int_{cx}^{c/x} \frac{\theta(y)}{y}\,dy \geqslant 0 \tag{2.21}$$

(since $\log\, l(x) = \sigma + \int_b^x \big(\theta(y)/y\big)\,dy$ by (2.1)). Thus, (2.21) is equivalent to

$$\int_x^{1/x} \frac{\theta(y)}{y}\,dy - c\int_x^{1/x} \frac{\theta(cy)}{cy}\,dy = \int_x^{1/x} \frac{\theta(y)-\theta(cy)}{y}\,dy \geqslant 0, \tag{2.22}$$

which is satisfied since θ is decreasing (and $c \geqslant 1$). We have shown that $P_1 \geqslant N_1$.

We now show that $P_2 \geqslant N_2$.

This time, using (2.17) and (2.18) it suffices to show that

$$l\left(\frac{t-\sqrt{t^2-4}}{2}\right) l\left(\frac{t+\sqrt{t^2-4}}{2}(1-z)\right) \geqslant l\left(\frac{t+\sqrt{t^2-4}}{2}\right) l\left(\frac{t-\sqrt{t^2-4}}{2}(1-z)\right)$$

or, equivalently,

$$l(x)\, l\left(\frac{c}{x}\right) \geqslant l\left(\frac{1}{x}\right) l(cx) \quad \text{with } x \leqslant 1 \text{ and } c \leqslant 1. \tag{2.23}$$

The relation (2.23) is trivial for $x \leqslant a$ (since $cx \leqslant a$ and $l(cx) = 0$). It remains to examine the case $x \geqslant a$, $a \leqslant 1$. The relation (2.23) is then equivalent to

$$\int_{cx}^{c/x} \frac{\theta(y)}{y}\, dy - \int_{x}^{1/x} \frac{\theta(y)}{y}\, dy \geqslant 0, \quad \text{i.e.,} \quad \int_{x}^{1/x} \frac{\theta(cy) - \theta(y)}{y}\, dy \geqslant 0.$$

The latter relation is obvious since θ is decreasing (and $c < 1$). This completes the proof of Theorem 2. ∎

REMARK 3. Recall (see (2.8) above) that a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be *hyperbolically completely monotone* (HCM) if, for every $u > 0$, the function of w:

$$v + \frac{1}{v} = w \to \varphi(uv)\varphi\left(\frac{u}{v}\right) \quad \text{(with } v \geqslant 0\text{)},$$

is completely monotone. Thus, by (1.5), our Theorem 2 may be stated as follows: if l belongs to $\mathcal{C}$, then its Stieltjes transform is HCM.

3. APPLICATION TO SOME R.V.'S RELATED TO RECURRENT LINEAR DIFFUSIONS

3.1. Our notation and hypotheses are now those of Salminen et al. [11] to which we refer the reader. $(X_t,\ t \geqslant 0)$ denotes an $\mathbb{R}_+$-valued diffusion which is recurrent; we denote its speed measure (assumed to have no atoms) by σ, and its scale function by S. $(L_t,\ t \geqslant 0)$ denotes the (continuous) local time at 0 and $(\tau_u,\ u \geqslant 0)$ its right-continuous inverse:

$$\tau_u := \inf\{t \geqslant 0;\ L_t > u\}. \tag{3.1}$$

$(\tau_u,\ u \geqslant 0)$ is a subordinator whose Lévy measure admits a density (see [11]) which we shall denote by ν:

$$E\big(\exp(-\lambda\tau_u)\big) = \exp\Big\{-u\int_0^\infty (1-e^{-\lambda x})\nu(x)dx\Big\}. \tag{3.2}$$

In fact, ν may be expressed in the form

(3.3) $$\nu(x) = \int_0^\infty e^{-xz} K(dz),$$

where K, the Krein measure (see Kotani and Watanabe [8] and Knight [7]), satisfies

(3.4) $$\int_0^\infty \frac{K(dz)}{z(1+z)} < \infty \quad \text{and} \quad \int_0^\infty \frac{K(dz)}{z} = \infty.$$

3.2. Let, for every $t \geqslant 0$:

(3.5) $$g_t := \sup\{s \leqslant t;\ X_s = 0\}, \quad d_t := \inf\{s \geqslant t;\ X_s = 0\},$$

and denote by $\mathfrak{e}_p$ $(p > 0)$ an exponentially distributed variable with parameter p, i.e. with density $f_{\mathfrak{e}_p}(u) = p\, e^{-pu} 1_{u \geqslant 0}$; $\mathfrak{e}_p$ is assumed to be independent of $(X_t,\ t \geqslant 0)$. We define

(3.6) $$Y_p^{(1)} := \mathfrak{e}_p - g_{\mathfrak{e}_p}, \quad Y_p^{(2)} := d_{\mathfrak{e}_p} - \mathfrak{e}_p, \quad Y_p^{(3)} := d_{\mathfrak{e}_p} - g_{\mathfrak{e}_p}.$$

It is shown in [11], Theorem 18, that for $i = 1, 2, 3$, $Y_p^{(i)}$ is infinitely divisible. More precisely, concerning $Y_p^{(3)}$, it is shown that $Y_p^{(3)}$ is a gamma-2 mixture, and consequently (see Kristiansen [9]) that $Y_p^{(3)}$ is infinitely divisible.

The aim of the following Theorem 4 is to improve, if possible, the results we have just recalled. More precisely, we shall prove that under certain hypotheses the r.v.'s $Y_p^{(i)}$ $(i = 1, 2, 3)$ are GGC r.v.'s whose Thorin measures have total masses $m \leqslant 1$. Thus, these variables are:

- GGC, hence self-decomposable, and *a fortiori* infinitely divisible;
- gamma-m mixtures, with $m \leqslant 1$, and not only gamma-2 mixtures (see identity (1.9)).

THEOREM 4. *We assume that Krein's measure K (defined by (3.3)) admits a differentiable density k.*

1. *Assume that*

(3.7) $$\frac{k'}{k}(x) = \frac{1}{x} + \frac{\theta(p+x)}{p+x} \quad \textit{with } \theta \textit{ decreasing};$$

then $Y_p^{(1)}$ is a GGC r.v. whose Thorin measure is a subprobability.

2. *Assume that*

(3.8) $$\frac{k'}{k}(x) = \frac{1}{x+p} + \frac{\theta(x)}{x} \quad \textit{with } \theta \textit{ decreasing};$$

then $Y_p^{(2)}$ is a GGC r.v. whose Thorin measure is a subprobability.

3. *Assume that*

$$
\text{(3.9)}\qquad
\begin{aligned}
&\frac{k'}{k}(z) = \frac{\theta(z)}{z} && \textit{for } z < p,\\
&\frac{k'(z) - k'(z-p)}{k(z) - k(z-p)} = \frac{\theta(z)}{z} && \textit{for } z \geqslant p,
\end{aligned}
$$

with θ decreasing; then $Y_p^{(3)}$ is a GGC r.v. whose Thorin measure is a subprobability.

P r o o f. We denote by $f_{Y_p^{(i)}}$ the density of $Y_p^{(i)}$. From [11], p. 115, we have

$$
f_{Y_p^{(1)}}(u) = C_1(p) \int_p^\infty e^{-uz} \frac{k(z-p)}{z-p}\, dz, \tag{3.10}
$$

$$
f_{Y_p^{(2)}}(u) = C_2(p) \int_0^\infty e^{-uz}\, \frac{k(z)}{z+p}\, dz, \tag{3.11}
$$

$$
f_{Y_p^{(3)}}(u) = C_3(p) \int_0^\infty e^{-uz} \big(k(z) - 1_{\{z \geqslant p\}}\, k(z-p)\big) dz, \tag{3.12}
$$

where $C_i(p),\ i = 1, 2, 3$, are three normalising constants. We shall now use Theorem 2 with, successively:

$$
l^{(1)}(x) = C_1(p)\, \frac{k(x-p)}{x-p}\, 1_{x \geqslant p}, \tag{3.13}
$$

$$
l^{(2)}(x) = C_2(p)\, \frac{k(x)}{x+p}, \tag{3.14}
$$

$$
l^{(3)}(x) = C_3(p)\, \big(k(x) - 1_{x \geqslant p}\, k(x-p)\big). \tag{3.15}
$$

We have already noted that, for $i = 1, 2, 3$,

$$
\int_0^\infty \frac{l^{(i)}(x)}{x}\, dx < \infty.
$$

Indeed:

$$
\begin{aligned}
\int_0^\infty \frac{l^{(1)}(x)}{x}\, dx &= C_1(p) \int_p^\infty \frac{k(x-p)}{x(x-p)}\, dx = C_1(p) \int_0^\infty \frac{k(x)}{x(x+p)}\, dx\\
&< \infty \quad \text{(by (3.4))},\\
\int_0^\infty \frac{l^{(2)}(x)}{x}\, dx &= C_2(p) \int_0^\infty \frac{k(x)}{x(x+p)}\, dx < \infty \quad \text{(by (3.4))},\\
\int_0^\infty \frac{l^{(3)}(x)}{x}\, dx &= C_3(p) \int_0^\infty k(x) \left(\frac{1}{x} - \frac{1}{x+p}\right) dx\\
&= p\, C_3(p) \int_0^\infty \frac{k(x)}{x(x+p)}\, dx < \infty \quad \text{(by (3.4))}.
\end{aligned}
$$

Finally, it remains to observe that hypothesis (3.7) (respectively, (3.8) and (3.9)) implies that $l^{(1)} \in \mathcal{C}$ (respectively, $l^{(2)} \in \mathcal{C}$ and $l^{(3)} \in \mathcal{C}$). ■

4. APPLICATION TO RECURRENT BESSEL PROCESSES

4.1. The notation is the same as in the preceding part, but now $(X_t,\ t \geqslant 0)$ is a Bessel process with dimension $d = 2(1-\alpha)$ with $0 < d < 2$ or, equivalently, $0 < \alpha < 1$.

THEOREM 5. *For any* $\alpha \in]0,1[$ *and for any* $p > 0$ *the r.v.'s*

$$Y_p^{(1)} = \mathfrak{e}_p - g_{\mathfrak{e}_p}, \quad Y_p^{(2)} = d_{\mathfrak{e}_p} - \mathfrak{e}_p, \quad Y_p^{(3)} = d_{\mathfrak{e}_p} - g_{\mathfrak{e}_p}$$

are GGC r.v.'s whose Thorin measures have the same total mass $1-\alpha = d/2$ (*less than* 1).

Proof. We have already noted that since, by (3.3),

$$\nu(a) = \int_0^\infty e^{-az}\, K(dz)$$

and, by [5], p. 213,

$$\nu(a) = \frac{1}{2^\alpha \Gamma(\alpha)}\, \frac{1}{a^{\alpha+1}} 1_{\{a>0\}},$$

the density k of Krein's measure equals here

$$k(z) = \frac{1}{2^\alpha \Gamma(\alpha)\Gamma(\alpha+1)}\, z^\alpha 1_{\{z>0\}}. \tag{4.1}$$

1. We begin by proving Theorem 5 for the r.v. $Y^{(2)}$. (To simplify the notation, we write $Y^{(2)}$ instead of $Y_p^{(2)}$.)

To see that $Y^{(2)}$ is GGC, it suffices, by Theorem 2, to show that $l^{(2)} \in \mathcal{C}$, where

$$l^{(2)}(x) = C\, \frac{x^\alpha}{x+p} \qquad \text{(from (4.1) and (3.14))}. \tag{4.2}$$

Thus

$$x(\log l^{(2)})'(x) = \alpha - \frac{x}{x+p} = \alpha - 1 + \frac{p}{x+p}$$

is a decreasing function of x, hence $l^{(2)} \in \mathcal{C}$ from (2.2). It remains to see that the total mass of the Thorin measure of $Y^{(2)}$ equals $1-\alpha$. Now, by (1.10), this total mass m equals

$$m := \sup\left\{\delta \geqslant 0;\ \lim_{u\downarrow 0_+} \frac{1}{u^{\delta-1}}\, f_{Y^{(2)}}(u) = 0\right\} \tag{4.3}$$
$$= \sup\left\{\delta \geqslant 0;\ \lim_{u\downarrow 0_+} \frac{C}{u^{\delta-1}} \int_0^\infty e^{-ux} \frac{x^\alpha}{x+p}\, dx = 0\right\}.$$

However, since the function $x \to x^\alpha/(x+p)$ decreases for x large enough and is equivalent to $x^{\alpha-1}$ when $x \to \infty$, the Tauberian theorem implies

$$f_{Y^{(2)}}(u) \underset{u\to 0}{\sim} \frac{C'}{u^\alpha}. \tag{4.4}$$

It is then clear that (4.3) and (4.4) imply $m = 1-\alpha$.

2. We now prove Theorem 5 for the r.v. $Y^{(1)}$. For this purpose, we shall use a more direct method than that relying on Theorem 2. Indeed, from (1.5), (3.13) and (4.1) we have

$$E(e^{-\lambda Y^{(1)}}) = \int_0^\infty \frac{l^{(1)}(z)}{\lambda+z}\, dz = C\int_p^\infty \frac{1}{\lambda+z}(z-p)^{\alpha-1} dz \tag{4.5}$$
$$= C\int_0^\infty \frac{1}{\lambda+p+z}\, z^{\alpha-1} dz = C\int_0^\infty z^{\alpha-1} dz \int_0^\infty e^{-(\lambda+p+z)u} du$$
$$= C\int_0^\infty e^{-(\lambda+p)u} du \int_0^\infty e^{-zu} z^{\alpha-1} dz$$
$$= C\Gamma(\alpha)\int_0^\infty e^{-(\lambda+p)u} \frac{du}{u^\alpha} = (\lambda+p)^{\alpha-1} C\Gamma(\alpha)\Gamma(1-\alpha)$$
$$= \left(1+\frac{\lambda}{p}\right)^{\alpha-1}$$

since the Laplace transform $E(e^{-\lambda Y^{(1)}})$ equals 1 for $\lambda = 0$. Thus

$$Y^{(1)} \overset{\text{(law)}}{=} \frac{1}{p}\,\gamma_{1-\alpha}, \tag{4.6}$$

where $\gamma_{1-\alpha}$ is a gamma r.v. with parameter $1-\alpha$, i.e., with density

$$f_{\gamma_{1-\alpha}}(u) := \frac{e^{-u}}{\Gamma(1-\alpha)}\, u^{-\alpha}\, 1_{u\geqslant 0}.$$

It follows clearly from (4.5) that

$$E(e^{-\lambda Y^{(1)}}) = \exp\left\{-(1-\alpha)\,\log\left(1+\frac{\lambda}{p}\right)\right\} \tag{4.7}$$
$$= \exp\left\{-(1-\alpha)\int_0^\infty (1-e^{-\lambda x})\,\frac{dx}{x}\, e^{-xp}\right\}.$$

Thus, by (1.7), formula (4.7) shows that $Y^{(1)}$ is a GGC variable with Thorin measure $(1-\alpha)\delta_p$.

3. We now prove Theorem 5 for the r.v. $Y_p^{(3)}$. In fact, the result that $Y^{(3)}$ is a GGC variable whose Thorin measure has total mass equal to $1-\alpha$ has already been proved in [2] (with $p=1$, but this involves no loss of generality). The proof we shall give now is a totally different one from that of [2]. We also assume here, for simplicity, that $p=1$ and we write $Y^{(3)}$ instead of $Y_1^{(3)}$. Following the arguments of the proof of Theorem 2, we need to show, by (2.16)–(2.19) that, for every $x\in]0,1]$:

$$\begin{aligned}\Delta(x)=\int_0^\infty &\left\{l\left(\frac{1}{x}\right)l\big(x(1+z)\big)-l(x)l\left(\frac{1}{x}(1+z)\right)\right\}\frac{dz}{z}\\ &+\int_0^1\left\{l(x)l\left(\frac{1}{x}(1-z)\right)-l\left(\frac{1}{x}\right)l\big(x(1-z)\big)\right\}\frac{dz}{z}\geqslant 0,\end{aligned}\tag{4.8}$$

where the function $l\ (=l^{(3)})$ equals, by (4.1) and (3.15),

$$l(y)=y^\alpha-1_{y\geqslant 1}\,(y-1)^\alpha\qquad(y\geqslant 0).\tag{4.9}$$

Thus, we need to show (4.8). For this purpose, we have to compute the integrals featured in (4.8), hence, given (4.9), to discuss, owing to the positions of $x(1+z)$, $x^{-1}(1+z)$, $x^{-1}(1-z)$ and $x(1-z)$ with respect to 1 (see Fig. 2). We consider the first integral in (4.8) for $x(1+z)\geqslant 1$ (hence, *a fortiori* $x^{-1}(1+z)\geqslant 1$ since $x\leqslant 1$). The first term equals

$$\begin{aligned}\Delta_1(x)&=\int_{(1/x)-1}^\infty\left\{\left[\frac{1}{x^\alpha}-\left(\frac{1}{x}-1\right)^\alpha\right]\Big(x^\alpha(1+z)^\alpha-\big(x(1+z)-1\big)^\alpha\Big)\right.\\ &\qquad\left.-x^\alpha\left[\left(\frac{1}{x^\alpha}(1+z)^\alpha\right)-\left(\frac{1}{x}(1+z)-1\right)^\alpha\right]\right\}\frac{dz}{z}\\ &=\int_{(1/x)-1}^\infty\big(1-(1-x)^\alpha\big)\left[(1+z)^\alpha-\left(1+z-\frac{1}{x}\right)^\alpha\right]\\ &\qquad-\big((1+z)^\alpha-(1+z-x)^\alpha\big)\frac{dz}{z}\\ &=\int_{(1/x)-1}^\infty\left\{\left[(1+z-x)^\alpha-\left(1+z-\frac{1}{x}\right)^\alpha\right]\right.\\ &\qquad\left.-(1-x)^\alpha\left[(1+z)^\alpha-\left(1+z-\frac{1}{x}\right)^\alpha\right]\right\}\frac{dz}{z}\\ &:=\Delta_1^{(1)}(x)-\Delta_1^{(2)}(x).\end{aligned}$$

Let us examine $\Delta_1^{(1)}(x)$:

$$\Delta_1^{(1)}(x) = \int_{(1/x)-1}^{\infty} \left[(1+z-x)^\alpha - \left(1+z-\frac{1}{x}\right)^\alpha \right] \frac{dz}{z}$$
$$= \int_{(1/x)-1}^{\infty} \frac{dz}{z} \int_{1+z-(1/x)}^{1+z-x} \alpha\, u^{\alpha-1} du.$$

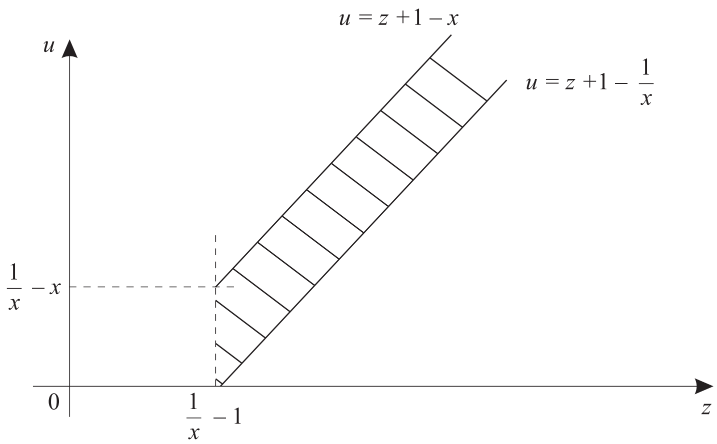

FIGURE 2

Now, we apply Fubini's theorem:

$$\Delta_1^{(1)}(x) = \int_0^{(1/x)-1} \alpha\, u^{\alpha-1} du \int_{(1/x)-1}^{u+(1/x)-1} \frac{dz}{z} + \int_{(1/x)-1}^{\infty} \alpha\, u^{\alpha-1} du \int_{u+x-1}^{u+(1/x)-1} \frac{dz}{z}$$
$$= \int_0^{(1/x)-1} \alpha\, u^{\alpha-1} \log\left(\frac{ux+1-x}{1-x}\right) du + \int_{(1/x)-1}^{\infty} \alpha\, u^{\alpha-1} \log\left(\frac{u+x^{-1}-1}{u+x-1}\right) du.$$

Thus we compute each term of $\Delta(x)$ and we obtain, after some simple, although tedious, computations:

$$\Delta(x) = \overline{\Delta}^1(x) + \overline{\Delta}^2(x) + \overline{\Delta}^3(x),$$

where

$$\overline{\Delta}^1(x) := \int_0^{(1/x)-1} \log\left(\frac{1}{1-x}\right) \alpha\, u^{\alpha-1} du + \int_0^{(1/x)-x} \log\left(\frac{ux+1-x}{1-x}\right) \alpha\, u^{\alpha-1} du,$$

$$\overline{\Delta}^2(x) := \int\limits_{(1/x)-x}^{\infty} \log\left(\frac{u+x^{-1}-1}{u+x-1}\right)\alpha\, u^{\alpha-1}du$$
$$- \int\limits_{(1/x)-1}^{\infty} \log\left(\frac{u/(1-x)+x^{-1}-1}{u/(1-x)-1}\right)\alpha\, u^{\alpha-1}du,$$
$$\overline{\Delta}^3(x) := \int\limits_{0}^{x(1-x)} \log\left(\frac{1-x-u}{(1-x)^2}\right)\alpha\, u^{\alpha-1}du$$
$$- \int\limits_{0}^{(1/x)-1} \log\left(\frac{u/(1-x)+x^{-1}-1}{x^{-1}-1}\right)\alpha\, u^{\alpha-1}du.$$

We note that, since $x \in]0,1]$, we have

$$x(1-x) \leqslant 1-x \leqslant \frac{1}{x}-1 \leqslant \frac{1}{x}-x$$

and that all the integrals in $\overline{\Delta}^i(x)$ are positive. For example, we have

$$\overline{\Delta}^2(x) = \int\limits_{(1/x)-x}^{\infty} \log\left(\frac{ux+(1-x)}{ux+(1-x)^2}\right)\alpha\, u^{\alpha-1}du$$

and this last integral is positive since $(1-x)^2 \leqslant 1-x$. Gathering thus all the terms in $\Delta(x)$, we obtain

$$(4.10)\quad \Delta(x) = \int\limits_0^{\infty} \log\left(\frac{ux+1-x}{ux+(1-x)^2}\right)\alpha\, u^{\alpha-1}du$$
$$- \int\limits_{(1/x)-1}^{(1/x)-x} \log\left(\frac{1-x}{x(u+x-1)}\right)\alpha\, u^{\alpha-1}du + \int\limits_0^{x(1-x)} \log\left(\frac{1-x-u}{(1-x)^2}\right)\alpha\, u^{\alpha-1}du$$
$$= \alpha(1-x)^{\alpha} \int\limits_{1/x}^{(1/x)+1} \log\left(\frac{1}{x(v-1)}\right)\left[\left(\frac{xv-1}{1+v-xv}\right)^{\alpha-1}\frac{x^{2-\alpha}}{(1+x-xv)^2}\right.$$
$$\left. + \left(\frac{xv-1}{x(v-1)}\right)^{\alpha-1}\frac{1-x}{x(v-1)^2} - v^{\alpha-1}\right]dv$$

after making the changes of variables:

$$\frac{ux+1-x}{ux+(1-x)^2} = \frac{1}{x(v-1)},$$
$$u = (1-x)v,$$
$$\frac{1-x-u}{(1-x)^2} = \frac{1}{x(v-1)}$$

in the first, the second, and the third integral of (4.10), respectively.

Thus, to conclude, it remains to show that, for every $v \in [x^{-1},\ x^{-1}+1]$,

(4.11)

$$v^{\alpha-1} \leqslant \left(\frac{xv-1}{1+x-xv}\right)^{\alpha-1} \frac{1}{x^{\alpha-1}} \frac{x}{(1+x-xv)^2} + \left(\frac{xv-1}{x(v-1)}\right)^{\alpha-1} \frac{1-x}{x(v-1)^2}$$

or, equivalently, that

$$\left(\frac{xv-1}{xv}\right)^{1-\alpha} \leqslant \frac{x}{(1+x-xv)^{\alpha+1}} + \frac{1-x}{x} \frac{1}{(v-1)^{\alpha+1}}. \tag{4.12}$$

Now, this last inequality is obvious. Indeed, since

$$f_1(v) := \left(\frac{xv-1}{xv}\right)^{1-\alpha} \quad \text{and} \quad f_2(v) := \frac{x}{(1+x-xv)^{\alpha+1}}$$

are increasing, it suffices to verify that

$$f_1\left(\frac{1}{x}+1\right) \leqslant f_2\left(\frac{1}{x}\right).$$

We have

$$f_1\left(\frac{1}{x}+1\right) = \left(\frac{x}{1+x}\right)^{1-\alpha} \leqslant 1 \leqslant f_2\left(\frac{1}{x}\right) = \frac{1}{x^\alpha}$$

since $x \in]0,1]$. This shows that $Y^{(3)}$ is GGC.

Finally, it is not difficult to prove that the total mass of the Thorin measure equals $1-\alpha$. Indeed, since $l^{(3)}(x) = C\big(x^\alpha - 1_{x\geqslant 1}(x-1)^\alpha\big)$, we have

$$l^{(3)}(x) \underset{x\to\infty}{\sim} C\, x^{\alpha-1}.$$

Hence, by the Tauberian theorem,

$$f_{Y^{(3)}}(u) \underset{u\to 0}{\sim} \frac{C}{u^\alpha},$$

and we finally use (1.10). ■

4.2. Description of the r.v.'s $\mathbb{G}_\alpha^{(i)}$ $(i=1,2,3;\ 0<\alpha<1)$. In the sequel, it will be convenient to assume that $p=1$ and we write simply $Y^{(i)}$ for the r.v.'s $Y_1^{(i)}$ $(i=1,2,3)$. Theorem 5 implies, by (1.7), the existence of r.v.'s $\mathbb{G}_\alpha^{(i)}$ $(i=1,2,3;\ \alpha\in]0,1[)$ such that $E\big(\log^+(1/\mathbb{G}_\alpha^{(i)})\big) < \infty$ and

(4.13)

$$E\big(\exp(-\lambda Y^{(i)})\big) = \exp\left\{-(1-\alpha)\int_0^\infty (1-e^{-\lambda x})\frac{dx}{x}\, E\big(\exp(-x\mathbb{G}_\alpha^{(i)})\big)\right\}.$$

The aim of this section is to identify the (laws of the) r.v.'s $\mathbb{G}_\alpha^{(i)}$ and to describe some of their properties.

(i) The case $i=1$.

Formula (4.6) implies that the r.v. $\mathbb{G}_\alpha^{(1)}$ is a.s. equal to 1, i.e. its distribution is δ_1, the Dirac measure at 1. In particular, this distribution does not depend on α.

(ii) The case $i=3$.

In [2] a complete study of the r.v.'s $\mathbb{G}_\alpha^{(3)}$, denoted by $\mathbb{G}_\alpha$ in [2], has been undertaken. We refer the reader to formula (1.17), p. 318, in [2] (note that in formula (1.50), p. 322, exponent α is missing). In particular, it is shown there that the density $f_{\mathbb{G}_\alpha^{(3)}}$ of $\mathbb{G}_\alpha^{(3)}$ equals

$$f_{\mathbb{G}_\alpha^{(3)}}(u)=\frac{\alpha\sin(\pi\alpha)}{(1-\alpha)\pi}\frac{u^{\alpha-1}(1-u)^{\alpha-1}}{(1-u)^{2\alpha}-2(1-u)^\alpha u^\alpha\cos(\pi\alpha)+u^{2\alpha}}1_{[0,1]}(u). \tag{4.14}$$

Thus, $\mathbb{G}_{1/2}^{(3)}$ is arc-sine distributed:

$$f_{\mathbb{G}_{1/2}^{(3)}}(u)=\frac{1}{\pi}\,\frac{1}{\sqrt{u(1-u)}}\,1_{[0,1]}(u) \tag{4.15}$$

and the r.v.'s $\mathbb{G}_\alpha^{(3)}$ converge in law, as $\alpha\to 0$ and $\alpha\to 1$, respectively, towards $\mathbb{G}_0^{(3)}$ and $\mathbb{G}_1^{(3)}$, where

$$\mathbb{G}_0^{(3)}\overset{\text{(law)}}{=}\frac{1}{1+\exp(\pi C)}\qquad\text{with } C \text{ a standard Cauchy r.v.,} \tag{4.16}$$

$$\mathbb{G}_1^{(3)}\overset{\text{(law)}}{=}U\qquad\text{with } U \text{ uniform on } [0,1]. \tag{4.17}$$

(iii) The case $i=2$.

THEOREM 6. *For every* $\alpha\in]0,1[$

(1) (i) *we have*

$$Y^{(2)}\overset{(law)}{=}\mathfrak{e}\cdot\frac{\gamma_{1-\alpha}}{\gamma_\alpha}\overset{(law)}{=}\mathfrak{e}\,\frac{\beta_{1-\alpha,\alpha}}{1-\beta_{1-\alpha,\alpha}}, \tag{4.18}$$

where $\mathfrak{e},\gamma_{1-\alpha},\gamma_\alpha$ *are independent, with respective laws the standard exponential and the gamma distributions with respective parameters* $(1-\alpha)$ *and* α, *and where* $\mathfrak{e}$ *and* $\beta_{1-\alpha,\alpha}$ *are independent with respective distributions the standard exponential and the beta distribution with parameters* $(1-\alpha,\alpha)$;

(ii) *for* $\lambda\geqslant 0$ *we have*

$$E\big(\exp(-\lambda Y^{(2)})\big)=\frac{\lambda^\alpha-1}{\lambda-1}\quad(=\alpha\ \textit{if}\ \lambda=1). \tag{4.19}$$

(2) $Y^{(2)}$ *is a gamma-*$(1-\alpha)$ *mixture, i.e.*

$$Y^{(2)}=\gamma_{1-\alpha}\cdot D_{1-\alpha}^{(2)}, \tag{4.20}$$

where $\gamma_{1-\alpha}$ *is a gamma* $(1-\alpha)$ *variable, independent of the positive r.v.* $D^{(2)}_{1-\alpha}$. *Furthermore*:

$$D^{(2)}_{1-\alpha} \stackrel{(law)}{=} \frac{\mathfrak{e}}{\gamma_\alpha}, \tag{4.21}$$

$$E\big(\exp(-\lambda\, D^{(2)}_{1-\alpha})\big) = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-y}\, \frac{y^\alpha}{\lambda + y}\, dy = \alpha \int_0^\infty \frac{e^{-\lambda y}}{(1+y)^{\alpha+1}}\, dy. \tag{4.22}$$

The density $f_{D^{(2)}_{1-\alpha}}$ *of* $D^{(2)}_{1-\alpha}$ *equals*

$$f_{D^{(2)}_{1-\alpha}}(u) = \frac{\alpha}{(1+u)^{\alpha+1}}\, 1_{[0,\infty[}(u). \tag{4.23}$$

(3) (i) *The density* $f_{\mathbb{G}^{(2)}_\alpha}$ *of* $\mathbb{G}^{(2)}_\alpha$ *equals*

$$f_{\mathbb{G}^{(2)}_\alpha}(u) = \frac{\alpha \sin(\pi\alpha)}{(1-\alpha)\pi}\, \frac{u^{\alpha-1}}{u^{2\alpha} - 2u^\alpha \cos(\pi\alpha) + 1}\, 1_{[0,\infty[}(u). \tag{4.24}$$

(ii) *The r.v.'s* $\mathbb{G}^{(2)}_\alpha$ *are related to the r.v.'s* $\mathbb{G}^{(3)}_\alpha$ *via the identity in law*:

$$\frac{\mathbb{G}^{(2)}_\alpha}{1+\mathbb{G}^{(2)}_\alpha} \stackrel{(law)}{=} \mathbb{G}^{(3)}_\alpha \quad \textit{or, equivalently,} \quad \mathbb{G}^{(2)}_\alpha \stackrel{(law)}{=} \frac{\mathbb{G}^{(3)}_\alpha}{1-\mathbb{G}^{(3)}_\alpha}. \tag{4.25}$$

(iii) *We have the identity*

$$\mathbb{G}^{(2)}_\alpha \stackrel{(law)}{=} \frac{1}{\mathbb{G}^{(2)}_\alpha}. \tag{4.26}$$

(iv) *As* $\alpha \to 0$ *and* $\alpha \to 1$, $\mathbb{G}^{(2)}_\alpha$ *converges in law towards, respectively,*

$$\mathbb{G}^{(2)}_0 \stackrel{(law)}{=} \exp(\pi C) \quad \textit{and} \quad \mathbb{G}^{(2)}_1 \stackrel{(law)}{=} \frac{U}{1-U} \tag{4.27}$$

with C *a standard Cauchy r.v. and* U *uniform on* $[0,1]$.

(4) *Let* $\mu \in]0,1[$ *and* T_μ *denote the positive stable r.v. with index* μ *whose law is characterized by*

$$E\big(\exp(-\lambda T_\mu)\big) = \exp(-\lambda^\mu) \quad (\lambda > 0).$$

Then

$$\mathbb{G}^{(2)}_\alpha \stackrel{(law)}{=} \left(\frac{T_{1-\alpha}}{T'_{1-\alpha}}\right)^{(1-\alpha)/\alpha}, \tag{4.28}$$

where $T'_{1-\alpha}$ *is an independent copy of* $T_{1-\alpha}$.

An equivalent way of writing (4.28) *is*

$$\mathbb{G}_\alpha^{(2)} \overset{(law)}{=} \left(\frac{M_{1-\alpha}}{M'_{1-\alpha}}\right)^{1/\alpha}, \tag{4.29}$$

where $M_{1-\alpha}$ *and* $M'_{1-\alpha}$ *are two independent Mittag-Leffler r.v.'s with parameter* $1-\alpha$, *whose common law is characterized by*

$$\begin{gathered} E\big(\exp(\lambda\, M_{1-\alpha})\big) = \sum_{n\geqslant 0} \frac{\lambda^n}{\Gamma\big(1+n(1-\alpha)\big)}, \\ E[M_{1-\alpha}^n] = \frac{\Gamma(n+1)}{\Gamma\big(1+n(1-\alpha)\big)}, \qquad M_{1-\alpha} \overset{(law)}{=} \left(\frac{1}{T_{1-\alpha}}\right)^{1-\alpha} \end{gathered} \tag{4.30}$$

(see [4], p. 114, Exercise 4.19).

P r o o f. We prove (4.18). Denoting by $(R_t,\ t \geqslant 0)$ a Bessel process with dimension $2(1-\alpha)$ $(0<\alpha<1)$ starting from 0, we have by scaling:

$$Y^{(2)} = d_{\mathfrak{e}} - \mathfrak{e} \overset{(\text{law})}{=} \mathfrak{e}(d_1 - 1) \overset{(\text{law})}{=} \mathfrak{e}\left(\frac{R_1^2}{2\gamma_\alpha}\right)$$

(see [2]), where R_1^2 is the value of R_t^2 for $t=1$. Hence

$$Y^{(2)} \overset{(\text{law})}{=} \mathfrak{e}\ \frac{\gamma_{1-\alpha}}{\gamma_\alpha} = \mathfrak{e}\ \frac{\beta_{1-\alpha,\,\alpha}}{1-\beta_{1-\alpha,\,\alpha}}$$

(from the classical "beta-gamma algebra").

We prove (4.19). We have from (4.1) and (4.2):

$$l^{(2)}(x) = \frac{\sin(\pi\alpha)}{\pi}\,\frac{x^\alpha}{1+x}, \qquad x \geqslant 0,$$

noting that

$$\int_0^\infty \frac{l^{(2)}(x)}{x}\,dx = \frac{\sin(\pi\alpha)}{\pi}\int_0^\infty \frac{x^{\alpha-1}}{1+x}\,dx = \frac{\sin(\pi\alpha)}{\pi}\,B(\alpha, 1-\alpha) = 1$$

(see [10], pp. 3 and 13). Hence, by (1.3), $f_{Y^{(2)}}$, the density of $Y^{(2)}$, equals

$$f_{Y^{(2)}}(u) = \frac{\sin(\pi\alpha)}{\pi}\int_0^\infty e^{-ux}\frac{x^\alpha}{1+x}\,dx$$

(we might also have derived this formula from (4.18)).

We now compute the Laplace transform of $Y^{(2)}$:

$$\begin{aligned}
E\big(\exp(-\lambda Y^{(2)})\big) &= \frac{\sin(\pi\alpha)}{\pi}\int_0^\infty e^{-\lambda u}du\int_0^\infty e^{-ux}\frac{x^\alpha}{1+x}\,dx \\
&= \frac{\sin(\pi\alpha)}{\pi}\int_0^\infty \frac{x^\alpha}{(1+x)(\lambda+x)}\,dx \\
&= \frac{1}{\lambda-1}\,\frac{\sin(\pi\alpha)}{\pi}\int_0^\infty x^\alpha\left[\frac{1}{1+x}-\frac{1}{\lambda+x}\right]dx \\
&= \lim_{A\to\infty}\frac{1}{\lambda-1}\frac{\sin(\pi\alpha)}{\pi}\left[\int_0^A\frac{x^\alpha}{1+x}\,dx-\lambda^\alpha\int_0^{A/\lambda}\frac{x^\alpha}{1+x}\,dx\right] \\
&= \lim_{A\to\infty}\frac{1}{\lambda-1}\frac{\sin(\pi\alpha)}{\pi}\left[\int_0^A\left(x^{\alpha-1}-\frac{x^{\alpha-1}}{1+x}\right)dx\right. \\
&\qquad\left.-\lambda^\alpha\int_0^{A/\lambda}\left(x^{\alpha-1}-\frac{x^{\alpha-1}}{1+x}\right)dx\right] \\
&= \lim_{A\to\infty}\frac{1}{\lambda-1}\frac{\sin(\pi\alpha)}{\pi}\left[\frac{A^\alpha}{\alpha}-\int_0^\infty\frac{x^{\alpha-1}}{1+x}\,dx\right. \\
&\qquad\left.-\frac{\lambda^\alpha}{\alpha}\,(A/\lambda)^\alpha+\lambda^\alpha\int_0^\infty\frac{x^{\alpha-1}}{1+x}\,dx\right] \\
&= \frac{\lambda^\alpha-1}{\lambda-1}\,\frac{\sin(\pi\alpha)}{\pi}\int_0^\infty\frac{x^{\alpha-1}}{1+x}\,dx \\
&= \frac{\lambda^\alpha-1}{\lambda-1}\,\frac{\sin(\pi\alpha)}{\pi}\,B(\alpha,1-\alpha)=\frac{\lambda^\alpha-1}{\lambda-1}
\end{aligned}$$

since (see [10], p. 3) $B(\alpha,1-\alpha)=\Gamma(\alpha)\Gamma(1-\alpha)=\pi/\sin(\pi\alpha)$.

Let us show (4.24). By taking the logarithmic derivative of (4.19):

$$\begin{aligned}
E\big(\exp(-\lambda Y^{(2)})\big) &= \frac{\lambda^\alpha-1}{\lambda-1} \\
&= \exp\left\{-(1-\alpha)\int_0^\infty(1-e^{-\lambda x})\frac{dx}{x}\,E\big(\exp(-x\,\mathbb{G}_\alpha^{(2)})\big)\right\},
\end{aligned}$$

we obtain

$$E\left[\frac{1}{\lambda+\mathbb{G}_\alpha^{(2)}}\right]=\frac{1}{1-\alpha}\left[\frac{1}{1-\lambda}-\frac{\alpha\lambda^{\alpha-1}}{\lambda^\alpha-1}\right]. \tag{4.31}$$

Thus, we have just computed the Stieltjes transform of the r.v. $\mathbb{G}_\alpha^{(2)}$. The inversion formula for the Stieltjes transform (see [12], p. 345) leads us to

$$\begin{aligned} f_{\mathbb{G}_\alpha^{(2)}}(u) &= \frac{1}{2i\pi(1-\alpha)} \lim_{\eta\to 0}\Bigg[\frac{1}{1-\lambda(-u-i\eta)} - \frac{\alpha(-u-i\eta)^{\alpha-1}}{(-u-i\eta)^\alpha - 1} \\ &\qquad - \frac{1}{1-\lambda(-u+i\eta)} + \frac{\alpha(-u+i\eta)^{\alpha-1}}{(-u+i\eta)^\alpha - 1}\Bigg] \quad (u>0) \\ &= \frac{-\alpha}{2i\pi(1-\alpha)}\left[\frac{-u^{\alpha-1}\,e^{-i\pi\alpha}}{u^\alpha\,e^{-i\pi\alpha}-1} + \frac{u^{\alpha-1}\,e^{i\pi\alpha}}{u^\alpha\,e^{i\pi\alpha}-1}\right] \quad (u>0) \\ &= \frac{-\alpha}{2i\pi(1-\alpha)}\left[\frac{-u^{2\alpha-1}+u^{\alpha-1}\,e^{-i\pi\alpha}+u^{2\alpha-1}-u^{\alpha-1}\,e^{i\pi\alpha}}{u^{2\alpha}-u^\alpha\,e^{i\pi\alpha}-u^\alpha\,e^{-i\pi\alpha}+1}\right] \quad (u>0) \\ &= \frac{\alpha\sin(\pi\alpha)}{(1-\alpha)\pi}\;\frac{u^{\alpha-1}}{u^{2\alpha}-2u^\alpha\cos(\pi\alpha)+1}\;1_{(u>0)}. \end{aligned}$$

We now show (4.25). For every h Borel and positive, we have

$$E\left[h\left(\frac{\mathbb{G}_\alpha^{(2)}}{1+\mathbb{G}_\alpha^{(2)}}\right)\right] = \frac{\alpha\sin(\pi\alpha)}{(1-\alpha)\pi}\int_0^\infty h\left(\frac{u}{1+u}\right)\frac{u^{\alpha-1}}{u^{2\alpha}-2u^\alpha\cos(\pi\alpha)+1}\,du$$

using (4.24). Thus, making the change of variable $u/(1+u)=x$, we get

$$\begin{aligned} &E\left[h\left(\frac{\mathbb{G}_\alpha^{(2)}}{1+\mathbb{G}_\alpha^{(2)}}\right)\right] \\ &= \frac{\alpha\sin(\pi\alpha)}{(1-\alpha)\pi}\int_0^1 h(x)\frac{dx}{(1-x)^2}\;\frac{x^{\alpha-1}/(1-x)^{\alpha-1}}{x^{2\alpha}/(1-x)^{2\alpha}-\big(2\cos(\pi\alpha)x^\alpha\big)/(1-x)^\alpha+1} \\ &= \frac{\alpha\sin(\pi\alpha)}{(1-\alpha)\pi}\int_0^1 h(x)\frac{x^{\alpha-1}(1-x)^{\alpha-1}}{x^{2\alpha}-2x^\alpha(1-x)^\alpha\cos(\pi\alpha)+(1-x)^{2\alpha}}\,dx \\ &= E\big[h(\mathbb{G}_\alpha^{(3)})\big] \quad \text{(by (4.14))}. \end{aligned}$$

We now prove (4.26). It is shown in [2], p. 319, (1.27), that

$$\mathbb{G}_\alpha^{(3)} \overset{\text{(law)}}{=} 1-\mathbb{G}_\alpha^{(3)}, \tag{4.32}$$

which is, indeed, obvious. Thus, from (4.25) we get

$$\mathbb{G}_\alpha^{(2)} \overset{\text{(law)}}{=} \frac{\mathbb{G}_\alpha^{(3)}}{1-\mathbb{G}_\alpha^{(3)}} \overset{\text{(law)}}{=} \frac{1-\mathbb{G}_\alpha^{(3)}}{\mathbb{G}_\alpha^{(3)}} = \frac{(1+\mathbb{G}_\alpha^{(2)}-\mathbb{G}_\alpha^{(2)})/(1+\mathbb{G}_\alpha^{(2)})}{\mathbb{G}_\alpha^{(2)}/(1+\mathbb{G}_\alpha^{(2)})} \overset{\text{(law)}}{=} \frac{1}{\mathbb{G}_\alpha^{(2)}}.$$

The relation (4.27) follows immediately from (4.25), (4.16) and (4.17).

We prove (4.28). It is shown in [2], p. 320, that

$$\mathbb{G}_\alpha^{(3)} \overset{\text{(law)}}{=} \frac{(T_{1-\alpha})^{(1-\alpha)/\alpha}}{(T'_{1-\alpha})^{(1-\alpha)/\alpha}+(T_{1-\alpha})^{(1-\alpha)/\alpha}},\quad \mathbb{G}_\alpha^{(3)} \overset{\text{(law)}}{=} \frac{(M_{1-\alpha})^{1/\alpha}}{(M_{1-\alpha})^{1/\alpha}+(M'_{1-\alpha})^{1/\alpha}}. \tag{4.33}$$

Thus, from (4.25) and (4.33) we get

$$(4.34)\quad \mathbb{G}_\alpha^{(2)} \overset{\text{(law)}}{=} \frac{\mathbb{G}_\alpha^{(3)}}{1-\mathbb{G}_\alpha^{(3)}}$$

$$= \frac{(T_{1-\alpha})^{(1-\alpha)/\alpha}/\big((T'_{1-\alpha})^{(1-\alpha)/\alpha}+(T_{1-\alpha})^{(1-\alpha)/\alpha}\big)}{(T'_{1-\alpha})^{(1-\alpha)/\alpha}/\big((T'_{1-\alpha})^{(1-\alpha)/\alpha}+(T_{1-\alpha})^{(1-\alpha)/\alpha}\big)} = \left(\frac{T_{1-\alpha}}{T'_{1-\alpha}}\right)^{(1-\alpha)/\alpha}.$$

We note that (4.34) implies (4.26) and that (4.29) may be obtained from (4.34) in the same manner.

We now prove point (2) of Theorem 6. The formula (4.21), $D_{1-\alpha}^{(2)} \overset{\text{(law)}}{=} \mathfrak{e}/\gamma_\alpha$, is an immediate consequence of (4.18) and (4.20):

$$Y^{(2)} \overset{\text{(law)}}{=} \mathfrak{e}\cdot\frac{\gamma_{1-\alpha}}{\gamma_\alpha} \overset{\text{(law)}}{=} \gamma_{1-\alpha}D_{1-\alpha}^{(2)}$$

after observing that, in the latter formula, we may "simplify by $\gamma_{1-\alpha}$" (see [4] or [6], point 1.4.6, for a justification of this "simplification"). The value of the density of $D_{1-\alpha}^{(2)}$, which is given by (4.23), now follows easily from $D_{1-\alpha}^{(2)} \overset{\text{(law)}}{=} \mathfrak{e}/\gamma_\alpha$. Finally, we have

$$(4.35)\quad E\big(\exp(-\lambda D_{1-\alpha}^{(2)})\big) = E\left(\exp\left(-\lambda\frac{\mathfrak{e}}{\gamma_\alpha}\right)\right)$$

$$= \frac{1}{\Gamma(\alpha)}\int_0^\infty\int_0^\infty \exp\left(-\lambda\frac{x}{y}-x-y\right)y^{\alpha-1}dx\,dy = \frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-y}y^\alpha dy\int_0^\infty e^{-z(\lambda+y)}dz$$

after making the change of variable $x/y=z$. Consequently,

$$(4.36)\qquad E\big(\exp(-\lambda D_{1-\alpha}^{(2)})\big) = \frac{1}{\Gamma(\alpha)}\int_0^\infty \frac{y^\alpha}{\lambda+y}\,e^{-y}dy.$$

The formula

$$(4.37)\qquad E\big(\exp(-\lambda D_{1-\alpha}^{(2)})\big) = \alpha\int_0^\infty e^{-\lambda y}\frac{dy}{(1+y)^{\alpha+1}}$$

follows immediately from (4.23) and it is easy to verify that

$$\frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-y}\frac{y^\alpha}{\lambda+y}\,dy = \alpha\int_0^\infty e^{-\lambda y}\frac{dy}{(1+y)^{\alpha+1}}.$$

Indeed,

$$\frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-y}\frac{y^\alpha}{\lambda+y}\,dy = \frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-y}y^\alpha dy\int_0^\infty e^{-z(\lambda+y)}dz$$
$$= \frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-\lambda z}dz\int_0^\infty e^{-y(1+z)}y^\alpha dy$$
$$= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)}\int_0^\infty e^{-\lambda z}\frac{dz}{(1+z)^{\alpha+1}} = \alpha\int_0^\infty e^{-\lambda z}\frac{dz}{(1+z)^{\alpha+1}}.$$

This completes the proof of Theorem 6. ∎

REMARK 7. 1. From the relation $Y^{(2)} \overset{\text{(law)}}{=} \gamma_{1-\alpha}D^{(2)}_{1-\alpha}$ we deduce that

$$E\big(\exp(-\lambda Y^{(2)})\big) = E\big(\exp(-\lambda\gamma_{1-\alpha}\cdot D^{(2)}_{1-\alpha})\big) = E\bigg(\frac{1}{(1+\lambda D^{(2)}_{1-\alpha})^{1-\alpha}}\bigg)$$
$$= \alpha\int_0^\infty \left(\frac{1+\lambda x}{1+x}\right)^{\alpha-1}\frac{dx}{(1+x)^2} \quad \text{(by (4.23))}$$
$$= \frac{\alpha}{\lambda-1}\int_1^\lambda y^{\alpha-1}dy$$

after making the change of variable $(1+\lambda x)/(1+x) = y$. Consequently,

$$E\big(\exp(-\lambda Y^{(2)})\big) = \frac{\lambda^\alpha - 1}{\lambda - 1}.$$

This is another way to obtain (4.19).

2. Here is now another way to obtain (4.22). It is clear from (4.24) that

$$E(|\log\, \mathbb{G}^{(2)}_\alpha|) < \infty$$

and, since $\mathbb{G}^{(2)}_\alpha \overset{\text{(law)}}{=} 1/\mathbb{G}^{(2)}_\alpha$, that $E(\log\, \mathbb{G}^{(2)}_\alpha) = 0$. Thus, from Theorem 2.1 (ii) in [6] we have

$$f_{Y^{(2)}}(u) = \frac{u^{-\alpha}}{\Gamma(1-\alpha)}\, E\big(\exp(-u\, D^{(2)}_{1-\alpha})\big)$$

(this is formula (2.7) in [6], with $t = 1-\alpha$, $E(\log G) = 0$ and $G \overset{\text{(law)}}{=} 1/G$). Hence, since

$$f_{Y^{(2)}}(u) = \frac{\sin(\pi\alpha)}{\pi}\int_0^\infty e^{-ux}\frac{x^\alpha}{1+x}\,dx = \frac{u^{-\alpha}\sin(\pi\alpha)}{\pi}\int_0^\infty e^{-y}\frac{y^\alpha}{u+y}\,dy$$

(after the change of variable $ux = y$), we obtain

$$E\big(\exp(-u\, D^{(2)}_{1-\alpha})\big) = \frac{\sin(\pi\alpha)}{\pi}\,\Gamma(1-\alpha)\int_0^\infty e^{-y}\frac{y^\alpha}{u+y}\,dy = \frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-y}\frac{y^\alpha}{u+y}\,dy.$$

3. Furthermore, we remark that from Theorem 2.1 of [6] we get

$$f_{D^{(2)}_{1-\alpha}}(u) = u^{-\alpha-1} f_{D^{(2)}_{1-\alpha}}(1/u).$$

This formula follows also from (4.23).

4. Finally, we also observe from Theorem 2.1 in [6], as a consequence of $\mathbb{G}^{(2)}_\alpha \overset{(\mathrm{law})}{=} 1/\mathbb{G}^{(2)}_\alpha$ and $E(\log\, \mathbb{G}^{(2)}_\alpha) = 0$, that

$$f_{Y^{(2)}}(u) = E\big[\big(Y^{(2)}/u\big)^{\alpha/2} J_{-\alpha}(2\sqrt{u\,Y^{(2)}})\big], \tag{4.38}$$

where $J_{-\alpha}$ denotes the Bessel function with index $(-\alpha)$.

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