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LIMIT THEORY FOR PLANAR GILBERT TESSELLATIONS*

BY

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Abstract. A Gilbert tessellation arises by letting linear segments (cracks) in $\mathbb{R}^2$ unfold in time with constant speed, starting from a homogeneous Poisson point process of germs in randomly chosen directions. Whenever a growing edge hits an already existing one, it stops growing in this direction. The resulting process tessellates the plane. The purpose of the present paper is to establish a law of large numbers, variance asymptotics and a central limit theorem for geometric functionals of such tessellations. The main tool applied is the *stabilization theory* for geometric functionals.

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1. INTRODUCTION AND MAIN RESULTS

Let $\mathcal{X} \subseteq \mathbb{R}^2$ be a finite point set and let μ denote a non-degenerate probability measure on $[0, \pi)$. Each $x \in \mathcal{X}$ is independently marked with a unit length random vector $\hat{\alpha}_x$ making an angle $\alpha_x \in [0, \pi)$ distributed according to μ with the x-axis. In the case when $\mathcal{X}$ is a realization of a point process we additionally assume that the marks are independent of the ground process. In the sequel we will refer to the marking described above as to the *usual marking* and to μ as to the *marking measure*. The collection $\bar{\mathcal{X}} = \{(x, \alpha_x)\}_{x \in \mathcal{X}}$ determines a crack growth process (tessellation) according to the following rules. Initially, at the time $t = 0$, the growth process consists of the points (seeds) in $\mathcal{X}$. Subsequently, each point $x \in \mathcal{X}$ gives rise to two segments growing linearly at constant unit rate in the directions of $\hat{\alpha}_x$ and $-\hat{\alpha}_x$ from x. Thus, prior to any collisions, by the time $t > 0$ the seed has developed into the edge with endpoints $x - t\hat{\alpha}_x$ and $x + t\hat{\alpha}_x$, consisting of two segments, say the *upper* one $[x, x + t\hat{\alpha}_x]$ and the *lower* one $[x, x - t\hat{\alpha}_x]$.

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Whenever a growing segment is blocked by an existing edge, it stops growing in that direction, without affecting the behaviour of the second constituent segment though. Since the possible number of collisions is bounded, eventually we obtain a tessellation of the plane. The resulting random tessellation process is variously called: the *Gilbert model/tessellation*, the *crack growth process*, the *crack tessellation*, and the *random crack network*; see, for example, [8], [12] and the references therein.

Let $G(\bar{\mathcal{X}})$ denote the tessellation determined by $\bar{\mathcal{X}}$. We shall write $\xi^+(\bar{x}, \bar{\mathcal{X}})$, $x \in \mathcal{X}$, for the total length covered by the upper segment emanating from x in $G(\bar{\mathcal{X}})$, and likewise we let $\xi^-(\bar{x}, \bar{\mathcal{X}})$ stand for the total length of the lower segment from x. Note that we use $\bar{x}$ for a marked version of x, according to our general convention of putting bars over marked objects. For future use we adopt the convention that if $\bar{x}$ does not belong to $\bar{\mathcal{X}}$, we extend the definition of $\xi^\pm(\bar{x}, \bar{\mathcal{X}})$ by adding $\bar{x}$ to $\bar{\mathcal{X}}$ and endowing it with a mark drawn according to the usual rules. Observe that for some x the values of $\xi^\pm$ may be infinite. However, in most cases in the sequel $\mathcal{X}$ will be a realization of the homogeneous Poisson point process $\mathcal{P} = \mathcal{P}_\tau$ of intensity $\tau > 0$ in growing windows of the plane. We shall use the so-called *stabilization* property of the functionals ξ^+ and ξ^-, as discussed in detail below, to show that the construction of $G(\bar{\mathcal{X}})$ above can be extended to the whole plane yielding a well-defined process $G(\bar{\mathcal{P}})$, where, as usual, $\bar{\mathcal{P}}$ stands for a version of $\mathcal{P}$ marked according to the usual rules. This yields well-defined and a.s. finite whole-plane functionals $\xi^+(\cdot, \bar{\mathcal{P}})$ and $\xi^-(\cdot, \bar{\mathcal{P}})$.

Conceptually somewhat similar growth processes whereby seeds are the realization of a time marked Poisson point process in an expanding window of $\mathbb{R}^2$ and which subsequently grow radially in all directions until meeting another such growing seed have received considerable attention in [1]–[5], [11], [17], where it has been shown that the number of seeds satisfies a law of large numbers and central limit theorem as the window size increases. In this paper we wish to prove analogous limit results for natural functionals (total edge length, sum of power-weighted edge lengths, number of cracks with lengths exceeding a given threshold, etc.) of the crack tessellation process defined by Poisson points in expanding windows of $\mathbb{R}^2$. We will formulate this theory in terms of random measures keeping track not only of the cumulative values of the afore-mentioned functionals but also of their spatial profiles.

One interesting subclass of the birth-and-growth processes described above is the class of Voronoi tessellations [13], [21] where all seeds are born at time 0. In this model as well as in the Gilbert model, growth starts at time 0 and is stopped at a contact point when two growing objects touch, but is continued in other directions. In a sense, we may treat the Gilbert model as a lower-dimensional analogue of the Voronoi tessellation. In opposition to Voronoi and Gilbert tessellations we find the so-called lilypond models which have recently attracted considerable attention [6], [7], [9], [10] and where the entire (rather than just directional) growth is blocked upon a collision of a growing object (a ball, a segment etc.) with another one.

To proceed, consider a function $\phi : [\mathbb{R}_+ \cup \{+\infty\}]^2 \to \mathbb{R}$ with at most polynomial growth, i.e. for some $0 < q < +\infty$

$$\phi(r_1, r_2) = O\big((r_1 + r_2)^q\big). \tag{1.1}$$

With $Q_\lambda := [0, \sqrt{\lambda}]^2$ standing for the square of area λ in $\mathbb{R}^2$, we consider the *empirical measure*

$$\mu_\lambda^\phi := \sum_{x \in \mathcal{P} \cap Q_\lambda} \phi\left(\xi^+(\bar{x}, \bar{\mathcal{P}}), \xi^-(\bar{x}, \bar{\mathcal{P}})\right) \delta_{x/\sqrt{\lambda}}. \tag{1.2}$$

Thus, μ_λ^ϕ is a random (signed) measure on $[0,1]^2$ for all $\lambda > 0$. The large λ asymptotics of these measures is the principal object of study in this paper. Recalling that τ stands for the intensity of $\mathcal{P} = \mathcal{P}_\tau$, we define

$$e(\tau) := \mathbb{E}\phi\left(\xi^+(\bar{\mathbf{0}}, \bar{\mathcal{P}}), \xi^-(\bar{\mathbf{0}}, \bar{\mathcal{P}})\right), \tag{1.3}$$

where **0** is marked independently of $\bar{\mathcal{P}}$ and according to μ.

Notice that because of translation invariance of the functionals ξ^+ and ξ^- the random variables $\xi^+(\bar{\mathbf{0}}, \bar{\mathcal{P}})$ and $\xi^-(\bar{\mathbf{0}}, \bar{\mathcal{P}})$ may be interpreted as lengths of typical upper and lower segments.

The first main result of this paper is the following law of large numbers:

THEOREM 1.1. *For any continuous function $f : [0,1]^2 \to \mathbb{R}$ we have*

$$\lim_{\lambda \to \infty} \frac{1}{\lambda} \int_{[0,1]^2} f d\mu_\lambda^\phi = \tau e(\tau) \int_{[0,1]^2} f(x) dx$$

in L^p, $p \geqslant 1$.

Note that for $f(x) = 1$ the above theorem implies that

$$\lim_{\lambda \to \infty} \frac{1}{\lambda} \sum_{x \in \mathcal{P} \cap Q_\lambda} \phi\left(\xi^+(\bar{x}, \bar{\mathcal{P}}), \xi^-(\bar{x}, \bar{\mathcal{P}})\right) = \tau e(\tau).$$

Since the expected cardinality of $\mathcal{P} \cap Q_\lambda$ is $\tau\lambda$, we may interpret $e(\tau)$ as the asymptotic *mass per point* in μ_λ^ϕ. To characterize the second order asymptotics of random measures μ_λ^ϕ we consider the pair-correlation functions

$$c_\phi[x] := \mathbb{E}\phi^2\left(\xi^+(\bar{x}, \bar{\mathcal{P}}), \xi^-(\bar{x}, \bar{\mathcal{P}})\right), \quad x \in \mathbb{R}^2, \tag{1.4}$$

and

$$\begin{aligned} c_\phi[x, y] := \mathbb{E}\phi&\left(\xi^+(\bar{x}, \bar{\mathcal{P}} \cup \{\bar{y}\}), \xi^-(\bar{x}, \bar{\mathcal{P}} \cup \{\bar{y}\})\right) \\ &\times \phi\left(\xi^+(\bar{y}, \bar{\mathcal{P}} \cup \{\bar{x}\}), \xi^-(\bar{y}, \bar{\mathcal{P}} \cup \{\bar{x}\})\right) - [e(\tau)]^2, \end{aligned} \tag{1.5}$$

where x and y are marked independently of each other and of $\bar{\mathcal{P}}$. In fact, it easily follows by translation invariance that $c_\phi[x]$ above does not depend on x whereas $c_\phi[x, y]$ only depends on $y - x$. In terms of these functions we define the asymptotic *variance per point*:

$$V(\tau) = c_\phi[\mathbf{0}] + \tau \int_{\mathbb{R}^2} c_\phi[\mathbf{0}, x]dx. \tag{1.6}$$

Notice that in a special case when function $\phi(\cdot, \cdot)$ is homogeneous of degree k (i.e. for $c \in \mathbb{R}$ we have $\phi(cr_1, cr_2) = c^k \phi(r_1, r_2)$) one can simplify (1.3) and (1.6). Then the following remark is a direct consequence of standard scaling properties of Gilbert's tessellation construction and those of homogeneous Poisson point processes, whereby upon multiplying the intensity parameter τ by some factor ρ we get all lengths in $\mathcal{G}(\bar{\mathcal{P}})$ re-scaled by factor $\rho^{-1/2}$.

REMARK 1.1. *For $\phi : [\mathbb{R}_+ \cup \{+\infty\}]^2 \to \mathbb{R}$ with at most polynomial growth and homogeneous of degree k we have*

$$\begin{aligned} e(\tau) &= \tau^{-k/2} e(1), \\ V(\tau) &= \tau^{-k} V(1). \end{aligned} \tag{1.7}$$

In other words, $e(\cdot)$ and $V(\cdot)$ are homogeneous of degree $-k/2$ and $-k$, respectively.

Our second theorem gives the variance asymptotics for μ_λ^ϕ.

THEOREM 1.2. *The integral in* (1.6) *converges and $V(\tau) > 0$ for all $\tau > 0$. Moreover, for each continuous $f : [0, 1]^2 \to \mathbb{R}$*

$$\lim_{\lambda \to \infty} \frac{1}{\lambda} \operatorname{Var}\Big[\int_{[0,1]^2} f d\mu_\lambda^\phi \Big] = \tau V(\tau) \int_{[0,1]^2} f^2(x)dx.$$

Our final result is the central limit theorem:

THEOREM 1.3. *For each continuous $f : [0, 1]^2 \to \mathbb{R}$ the family of random variables*

$$\left\{ \frac{1}{\sqrt{\lambda}} \int_{[0,1]^2} f d\mu_\lambda^\phi \right\}_{\lambda > 0}$$

converges in law to $\mathcal{N}\big(0, \tau V(\tau) \int_{[0,1]^2} f^2(x)dx\big)$ as $\lambda \to \infty$. Even more, we have

$$\sup_{t \in \mathbb{R}} \left| \mathbf{P}\left\{ \frac{\int_{[0,1]^2} f d\mu_\lambda^\phi}{\sqrt{\operatorname{Var}\big[\int_{[0,1]^2} f d\mu_\lambda^\phi\big]}} \leqslant t \right\} - \Phi(t) \right| \leqslant \frac{C (\log \lambda)^6}{\sqrt{\lambda}} \tag{1.8}$$

for all $\lambda > 1$, where C is a finite constant.

Principal examples of functional ϕ where the above theory applies are:

1. $\phi(l_1, l_2) = l_1 + l_2$. Then the total mass of μ_λ^ϕ coincides with the total length of edges emitted in $G(\bar{\mathcal{P}})$ by points in $\mathcal{P} \cap Q_\lambda$. Clearly, the so-defined ϕ is homogeneous of order one, and thus Remark 1.1 applies.

2. More generally, $\phi(l_1, l_2) = (l_1 + l_2)^\alpha$, $\alpha \geqslant 0$. Again, the total mass of μ_λ^ϕ is seen here to be the sum of power-weighted lengths of edges emitted in $G(\bar{\mathcal{P}})$ by points in $\mathcal{P} \cap Q_\lambda$. The so-defined ϕ is homogeneous of order α.

3. $\phi(l_1, l_2) = \mathbf{1}_{\{l_1 + l_2 \geqslant \theta\}}$, where θ is some fixed threshold parameter. In this set-up, the total mass of μ_λ^ϕ is the number of edges in $G(\bar{\mathcal{P}})$ emitted by points in $\mathcal{P} \cap Q_\lambda$ and of lengths exceeding threshold θ. This is not a homogeneous functional.

The main tool used in our argument below is the concept of *stabilization* expressing in geometric terms the property of rapid decay of dependencies enjoyed by the functionals considered. The formal definition of this notion and the proof that it holds for Gilbert tessellations are given in Section 2. Next, in Section 3 the proofs of our Theorems 1.1, 1.2 and 1.3 are given.

2. STABILIZATION PROPERTY FOR GILBERT TESSELLATIONS

2.1. Concept of stabilization. Consider a generic real-valued translation-invariant geometric functional ξ defined on pairs $(x, \mathcal{X})$ for finite point configurations $\mathcal{X} \subset \mathbb{R}^2$ and with $x \in \mathcal{X}$. For notational convenience we extend this definition for $x \notin \mathcal{X}$ as well, by putting $\xi(x, \mathcal{X}) := \xi(x, \mathcal{X} \cup \{x\})$ then. More generally, ξ can also depend on i.i.d. marks attached to points of $\mathcal{X}$, in which case the marked version of $\mathcal{X}$ is denoted by $\bar{\mathcal{X}}$.

For an input i.i.d. marked point process $\bar{\mathcal{P}}$ on $\mathbb{R}^2$, where $\mathcal{P}$ in this paper is always taken to be homogeneous Poisson of intensity τ, we say that the functional ξ *stabilizes* at $x \in \mathbb{R}^2$ on input $\bar{\mathcal{P}}$ iff there exists an a.s. finite random variable $R[x, \bar{\mathcal{P}}]$ with the property that

$$\xi\big(\bar{x}, \bar{\mathcal{P}} \cap B(x, R[x, \bar{\mathcal{P}}])\big) = \xi\Big(\bar{x}, \big(\bar{\mathcal{P}} \cap B(x, R[x, \bar{\mathcal{P}}])\big) \cup \bar{A}\Big) \tag{2.1}$$

for each finite $A \subset B(x, R[x, \bar{\mathcal{P}}])^c$, with $\bar{A}$ standing for its marked version and with $B(x, R)$ denoting ball of radius R centered at x. Note that here and henceforth we abuse the notation and refer to intersections of marked point sets with domains in the plane – these are to be understood as consisting of those marked points whose spatial locations fall into the domain considered. When (2.1) holds, we say that $R[x, \bar{\mathcal{P}}]$ is a *stabilization radius* for $\bar{\mathcal{P}}$ at x. By translation invariance we see that if ξ stabilizes at one point, it stabilizes at all points of $\mathbb{R}^2$, and we say that ξ *stabilizes* on (marked) point process $\bar{\mathcal{P}}$. In addition, we say that ξ *stabilizes exponentially* on input $\bar{\mathcal{P}}$ with rate $C > 0$ iff there exists a constant $M > 0$ such that

$$\mathbf{P}\{R[x, \bar{\mathcal{P}}] > r\} \leqslant Me^{-Cr} \tag{2.2}$$

for all $x \in \mathbb{R}^2$ and $r > 0$. Stabilizing functionals are ubiquitous in stochastic geometry; we refer the reader to [1] and [14]–[20] for further details, where prominent examples are discussed including random geometric graphs (nearest neighbor graphs, sphere of influence graphs, Delaunay graphs), random sequential packing and variants thereof, Boolean models and functionals thereof, as well as many others.

2.2. Finite input Gilbert tessellations. Let $\mathcal{X} \subset \mathbb{R}^2$ be a finite point set in the plane and let μ be a non-degenerate probability measure on $[0, \pi)$. As already mentioned in the Introduction, each $x \in \mathcal{X}$ is independently marked with a unit length random vector $\hat{\alpha}_x = [\cos(\alpha_x), \sin(\alpha_x)]$ making an angle $\alpha_x \in [0, \pi)$ distributed according to μ with the x-axis, and the so-marked configuration is denoted by $\bar{\mathcal{X}}$. In order to formally define the Gilbert tessellation $G(\bar{\mathcal{X}})$ as already informally presented above, we consider an auxiliary *partial tessellation mapping* $G(\bar{\mathcal{X}}) : \mathbb{R}_+ \to \mathcal{F}(\mathbb{R}^2)$, where $\mathcal{F}(\mathbb{R}^2)$ is the space of closed sets in $\mathbb{R}^2$ and where, roughly speaking, $G(\bar{\mathcal{X}})(t)$ is to be interpreted as the portion of tessellation $G(\bar{\mathcal{X}})$, identified with the set of its edges, constructed by the time t in the course of the construction sketched above.

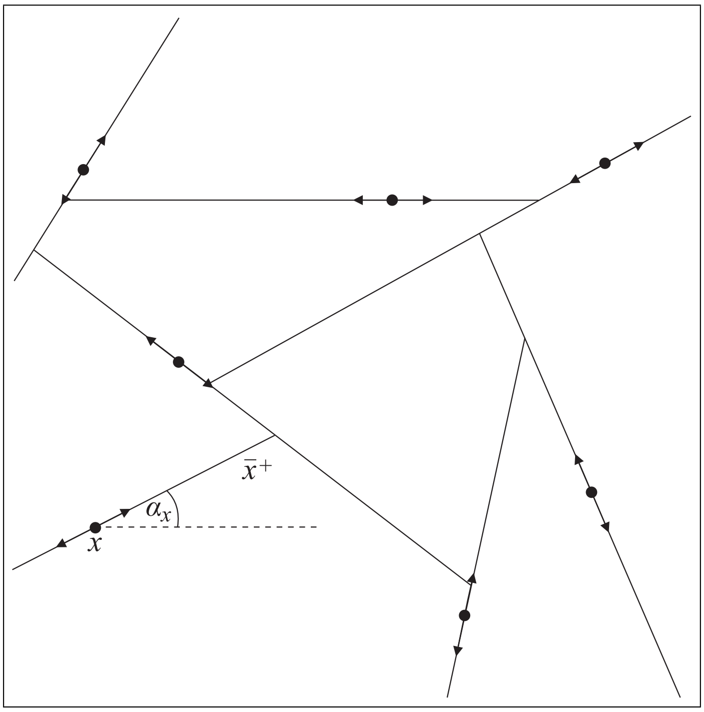

FIGURE 1. Finite input Gilbert tessellation

We proceed as follows. For each $\bar{x} = (x, \alpha_x) \in \bar{\mathcal{X}}$ at the time moment 0 the point x emits in directions $\hat{\alpha}_x$ and $-\hat{\alpha}_x$ two segments, referred to as the $\bar{x}^+$- and

$\bar{x}^-$-branches, respectively. Each branch keeps growing with constant rate 1 in its fixed direction until it meets on its way another branch already present, in which case we say it gets *blocked*, and it stops growing thereupon. The moment when this happens is called the *collision time*. For $t \geqslant 0$ we denote by $G(\bar{\mathcal{X}})(t)$ the union of all branches as grown by the time t. Note that, with $\mathcal{X} = \{x_1, \ldots, x_m\}$, the overall number of collisions admits a trivial bound given by the number of all intersection points of the family of straight lines $\big\{\{x_j + s\hat{\alpha_j},\ s \in \mathbb{R}\};\ j = 1, 2, \ldots, m\big\}$ which is $m(m-1)/2$. Thus, eventually there are no more collisions and all growth unfolds linearly. It is clear from the definition that $G(\bar{\mathcal{X}})(s) \subset G(\bar{\mathcal{X}})(t)$ for $s < t$. The limit set $G(\bar{\mathcal{X}})(+\infty) = \bigcup_{t \in \mathbb{R}_+} G(\bar{\mathcal{X}})(t)$ is denoted by $G(\bar{\mathcal{X}})$ and referred to as the *Gilbert tessellation*. Obviously, since the number of collisions is finite, the so-defined $G(\bar{\mathcal{X}})$ is a closed set arising as a finite union of (possibly infinite) linear segments. For $\bar{x} \in \bar{\mathcal{X}}$ we denote by $\xi^+(\bar{x}, \bar{\mathcal{X}})$ the length of the upper branch $\bar{x}^+$ emanating from x and, likewise, we write $\xi^-(\bar{x}, \bar{\mathcal{X}})$ for the length of the corresponding lower branch.

For future reference it is convenient to consider for each $x \in \mathcal{X}$ the *branch history* functions $\bar{x}^+(\cdot),\ \bar{x}^-(\cdot)$ defined by requiring that $\bar{x}^\pm(t)$ be the *growth tip* of the respective branch $\bar{x}^\pm$ at the time $t \in \mathbb{R}_+$. Thus, prior to any collision in the system, we have just $\bar{x}^\pm(t) = x \pm \hat{\alpha}_x t$, that is to say, all branches grow linearly with their respective speeds $\pm\hat{\alpha}_x$. Next, when some $\bar{y}^\pm,\ y \in \mathcal{X}$, gets blocked by some other $\bar{x}^\pm,\ x \in \mathcal{X}$, at time t, i.e. $\bar{y}^\pm(t) = \bar{x}^\pm(s)$ for some $s \leqslant t$, the blocked branch stops growing and its growth tip remains immobile ever since. Eventually, after all collisions have occurred, the branches not yet blocked continue growing linearly to ∞.

2.3. Stabilization for Gilbert tessellations. We are now in a position to argue that the functionals ξ^+ and ξ^- arising in Gilbert tessellation are exponentially stabilizing on Poisson input $\mathcal{P} = \mathcal{P}_\tau$ with usual marking according to a nondegenerate probability measure μ on $[0, \pi)$. The following is the main theorem of this subsection.

THEOREM 2.1. *The functionals ξ^+ and ξ^- stabilize exponentially on input $\bar{\mathcal{P}}$.*

Before proceeding to the proof of Theorem 2.1 we formulate some auxiliary lemmas.

LEMMA 2.1. *Let $\mathcal{X}$ be a finite point set in $\mathbb{R}^2$ and $\bar{\mathcal{X}}$ the marked version thereof, according to the usual rules. Further, let $y \notin \mathcal{X}$. Then for any $t \geqslant 0$ we have*

$$G(\bar{\mathcal{X}})(t) \triangle G(\bar{\mathcal{X}} \cup \{\bar{y}\})(t) \subset B(y, t)$$

with $\triangle$ standing for the symmetric difference.

Proof. For a point set $\mathcal{Y} \subset \mathbb{R}^2$ and $x \in \mathcal{Y}$ we will use the notation $(\bar{x}, \bar{\mathcal{Y}})^+$ and $(\bar{x}, \bar{\mathcal{Y}})^-$ to denote, respectively, the upper and lower branch outgrowing from

$\bar{x}$ in $G(\bar{\mathcal{Y}})$. Also, we use the standard extension of this notation for branch-history functions. Note first that, by the construction of $G(\bar{\mathcal{Y}})$ and by the triangle inequality, we have

(2.3)
$$(\bar{x}, \bar{\mathcal{Y}})^{\varepsilon}(s') \in B(y, s') \Rightarrow \forall_{s>s'} (\bar{x}, \bar{\mathcal{Y}})^{\varepsilon}(s) \in B(y, s),\ s' \geqslant 0, \varepsilon \in \{-1, +1\}.$$

This is a formal version of the obvious statement that, regardless of the collisions, each branch grows with speed at most one throughout its entire history.

Next, write $\mathcal{X}' = \mathcal{X} \cup \{y\}$ and $\Delta(t) = G(\bar{\mathcal{X}})(t) \triangle G(\bar{\mathcal{X}}')(t)$ for $t \geqslant 0$. Further, let $t_1 < t_2 < t_3 < \ldots < t_n$ be the joint collection of collision times for configurations $\bar{\mathcal{X}}$ and $\bar{\mathcal{X}}'$.

Choose arbitrary $p \in \Delta(t)$. Then there exist unique $\mathcal{Y} = \mathcal{Y}(p) \in \{\mathcal{X}, \mathcal{X}'\}$ and $x \in \mathcal{Y}$ as well as $\varepsilon \in \{+, -\}$ with the property that $p = (\bar{x}, \bar{\mathcal{Y}})^{\varepsilon}(u)$ for some $u \leqslant t$. We also write $\mathcal{Y}'$ for the second element of $\{\mathcal{X}, \mathcal{X}'\}$, i.e. $\{\mathcal{Y}, \mathcal{Y}'\} = \{\mathcal{X}, \mathcal{X}'\}$. With this notation, there is a unique $i = i(p)$ with t_i marking the collision time in $\mathcal{Y}'$ where the branch $(\bar{x}, \mathcal{Y}')^{\varepsilon}$ gets blocked in $G(\bar{\mathcal{Y}}')$; clearly, $u > t_i$ then and for $s < t_i$ we have $(\bar{x}, \bar{\mathcal{Y}})^{\varepsilon}(s) \notin \Delta(t)$.

We should show that $p \in B(y, t)$. We proceed inductively with respect to i. For $i = 0$ we have $x = y$ and $\mathcal{Y} = \mathcal{X}'$. Since $(\bar{y}, \bar{\mathcal{X}}')^{\varepsilon}(0) = y \in B(y, 0)$, the observation (2.3) implies that $p = (\bar{y}, \bar{\mathcal{X}})^{\varepsilon}(u) \in B(y, u) \subset B(y, t)$. Further, consider the case $i > 0$ and assume with no loss of generality that $\mathcal{Y}(p) = \mathcal{X}$, the argument in the converse case being fully symmetric. The fact that $p \in G(\bar{\mathcal{X}})(t) \triangle G(\bar{\mathcal{X}}')(t)$ and that $p = (\bar{x}, \bar{\mathcal{X}})^{\varepsilon}(u)$ implies the existence of a point $z \in \mathcal{X}'$ such that a branch emitted from z does block $\bar{x}^{\varepsilon}$ in $G(\bar{\mathcal{X}}')$ (by definition, necessarily at the time t_i) but does not block it in $G(\bar{\mathcal{X}})$. In particular, we see that $(\bar{z}, \bar{\mathcal{X}}')^{\delta}(s) = (\bar{x}, \bar{\mathcal{X}})^{\varepsilon}(t_i)$ and $(\bar{z}, \bar{\mathcal{X}}')^{\delta}(s') \in \Delta(s')$ for some δ, s, s' such that $\delta \in \{+, -\}$ and $s' < s \leqslant t_i$. By the inductive hypothesis we get $(\bar{z}, \bar{\mathcal{X}}')^{\delta}(s') \in B(y, s')$. Using again the observation (2.3) we conclude thus that $(\bar{x}, \bar{\mathcal{X}})^{\varepsilon}(t_i) = (\bar{z}, \bar{\mathcal{X}}')^{\delta}(s) \in B(y, s)$, and hence $p = (\bar{x}, \bar{\mathcal{X}})^{\varepsilon}(u) \in B(y, u) \subset B(y, t)$. This shows that $p \in B(y, t)$, as required. Since p was chosen arbitrary, this completes the proof of the lemma. ■

Our second auxiliary lemma is

LEMMA 2.2. *For an arbitrary finite point configuration $\mathcal{X} \subset \mathbb{R}^2$ and $\bar{x} \in \bar{\mathcal{X}}$ we have*

(2.4)
$$\xi^{+}(\bar{x}, \bar{\mathcal{X}}) = \xi^{+}\Big(\bar{x}, \bar{\mathcal{X}} \cap B\big(x, 2\xi^{+}(\bar{x}, \bar{\mathcal{X}})\big)\Big),$$
$$\xi^{-}(\bar{x}, \bar{\mathcal{X}}) = \xi^{-}\Big(\bar{x}, \bar{\mathcal{X}} \cap B\big(x, 2\xi^{-}(\bar{x}, \bar{\mathcal{X}})\big)\Big).$$

Proof. We only show the first equality in (2.4), the proof of the second one being fully analogous. Define $A(\bar{\mathcal{X}}, \bar{x}) = \bar{\mathcal{X}} \setminus B\big(x, 2\xi^{+}(\bar{x}, \bar{\mathcal{X}})\big)$. Clearly, $A(\bar{\mathcal{X}}, \bar{x})$ is finite and we will proceed by induction on its cardinality.

If $|A(\bar{\mathcal{X}},\bar{x})| = 0$, our claim is trivial. Assume now that $|A(\bar{\mathcal{X}},\bar{x})| = n$ for some $n \geqslant 1$ and let $\bar{y} = (y,\alpha_y) \in A(\bar{\mathcal{X}},\bar{x})$. Put $t = \xi^+(\bar{x},\bar{\mathcal{X}})$ and $\bar{\mathcal{X}}' = \bar{\mathcal{X}}\setminus\{\bar{y}\}$. Applying Lemma 2.1 we see that $G(\bar{\mathcal{X}})(t)\triangle G(\bar{\mathcal{X}}')(t) \subset B(y,t)$. We claim that $\xi^+(\bar{x},\bar{\mathcal{X}}) = \xi^+(\bar{x},\bar{\mathcal{X}}')$. Assume by contradiction that $\xi^+(\bar{x},\bar{\mathcal{X}}) \neq \xi^+(\bar{x},\bar{\mathcal{X}}')$. Then for arbitrarily small $\epsilon > 0$ we have $\big(G(\bar{\mathcal{X}})(t)\triangle G(\bar{\mathcal{X}}')(t)\big) \cap B(x,t+\epsilon) \neq \emptyset$. On the other hand, since $\|x-y\| > 2t$ as $y \notin B(x,2t)$, for $\varepsilon_0 > 0$ small enough we get $B(x,t+\epsilon_0) \cap B(y,t) = \emptyset$. Thus, we are led to

$$\emptyset \neq \big(G(\bar{\mathcal{X}})(t)\triangle G(\bar{\mathcal{X}}')(t)\big) \cap B(x,t+\varepsilon_0) \subset B(y,t) \cap B(x,t+\varepsilon_0) = \emptyset,$$

which is a contradiction. Consequently, we conclude that $\xi^+(\bar{x},\bar{\mathcal{X}}) = \xi^+(\bar{x},\bar{\mathcal{X}}')$, as required. Since $|A(\bar{\mathcal{X}}',\bar{x})| = n-1$, the inductive hypothesis yields $\xi^+(\bar{x},\bar{\mathcal{X}}') = \xi^+\big(\bar{x},\bar{\mathcal{X}}' \cap B(x,2\xi^+(\bar{x},\bar{\mathcal{X}}'))\big) = \xi^+\big(\bar{x},\bar{\mathcal{X}}' \cap B(\bar{x},2t)\big)$. Moreover, $\bar{\mathcal{X}}' \cap B(x,2t) = \bar{\mathcal{X}} \cap B(x,2t)$. Putting these together we obtain

$$\xi^+(\bar{x},\bar{\mathcal{X}}) = \xi^+(\bar{x},\bar{\mathcal{X}}') = \xi^+\big(\bar{x},\bar{\mathcal{X}}' \cap B(x,2t)\big) = \xi^+\big(\bar{x},\bar{\mathcal{X}} \cap B(x,2t)\big),$$

which completes the proof. ∎

In full analogy to Lemma 2.2 we obtain

LEMMA 2.3. *For a finite point configuration $\mathcal{X} \subset \mathbb{R}^2$ and $x \in \mathcal{X}$ we have*

$$\xi^+(\bar{x},\bar{\mathcal{X}}) = \xi^+(\bar{x},\bar{\mathcal{X}} \cup \bar{A}_1) \quad \textit{and} \quad \xi^-(\bar{x},\bar{\mathcal{X}}) = \xi^-(\bar{x},\bar{\mathcal{X}} \cup \bar{A}_2)$$

for arbitrary $A_1 \subset B\big(x,2\xi^+(\bar{x},\bar{\mathcal{X}})\big)^c$, $A_2 \subset B\big(x,2\xi^-(\bar{x},\bar{\mathcal{X}})\big)^c$.

Combining Lemmas 2.2 and 2.3 we conclude

COROLLARY 2.1. *Assume that finite marked configurations $\bar{\mathcal{X}}$ and $\bar{\mathcal{Y}}$ coincide on $B\big(x,2\xi^+(\bar{x},\bar{\mathcal{X}})\big)$. Then*

$$\xi^+\Big(\bar{x},\bar{\mathcal{X}} \cap B\big(x,2\xi^+(\bar{x},\bar{\mathcal{X}})\big)\Big) = \xi^+(\bar{x},\bar{\mathcal{X}}) = \xi^+(\bar{x},\bar{\mathcal{Y}}).$$

Analogous relations hold for ξ^-.

In the sequel we will also need the following remark:

REMARK 2.1. *For a non-degenerate measure μ on $[0,\pi)$ there exist constants $\delta > 0$ and $\epsilon \in (0,\pi/2)$ such that for all $x \in [0,\pi)$*

$$\mu(x+\pi/2-\epsilon, x+\pi/2+\epsilon) > \delta,$$

where μ is treated as a measure on the circle and points x and $x+\pi$ are identified.

We are now ready to proceed with the proof of Theorem 2.1.

Proof of Theorem 2.1. We are going to show that the functional ξ^+ stabilizes exponentially on input process $\bar{\mathcal{P}}$. The corresponding statement for ξ^- follows in full analogy. Consider auxiliary random variables $\xi^+_\varrho,\ \varrho \geqslant 0$, given by

$$\xi^+_\varrho = \xi^+\big(\bar{x}, \bar{\mathcal{P}} \cap B(x,\varrho)\big),$$

which is clearly well defined in view of the a.s. finiteness of $\bar{\mathcal{P}} \cap B(x,\varrho)$. We claim that there exist constants $M, C > 0$ such that for $\varrho \geqslant t \geqslant 0$

$$\mathbf{P}(\xi^+_\varrho > t) \leqslant M e^{-Ct}. \tag{2.5}$$

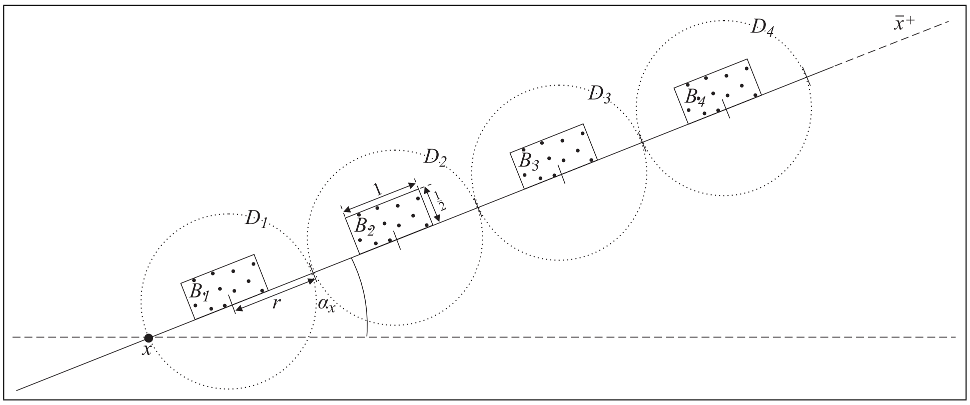

FIGURE 2

Indeed, let $\varrho \geqslant 0$. For the marking measure μ choose ϵ as in Remark 2.1 and take $r = \frac{1}{2}(1 + \mathrm{tg}\epsilon + 1/\cos\epsilon)$. Consider the branch $\bar{x}^+ := \big(\bar{x}, \bar{\mathcal{P}} \cap B(x,\varrho)\big)^+$ and planar regions B_i and $D_i,\ i \geqslant 1$, along the branch as represented in Figure 2. Say that the event $\mathcal{E}_i$ occurs iff

- the region B_i contains exactly one point y of $\mathcal{P}$ and the angular mark α_y lies within $(\alpha_x + \pi/2 - \epsilon, \alpha_x + \pi/2 + \epsilon)$, and
- there are no further points of $\mathcal{P}$ falling into D_i.

Notice that the choice of r ensures that with probability one on $\mathcal{E}_i$ the branch $\bar{x}^+$ does not extend past B_i, either getting blocked in B_i or in an earlier stage of its growth. Let p stand for the common positive value of $\mathbf{P}(\mathcal{E}_i),\ i \geqslant 0$. By standard properties of Poisson point process the events $\mathcal{E}_i$ are stochastically independent. We conclude that, for $\mathbb{N} \ni n \leqslant \varrho/(2r)$,

$$\mathbf{P}(\xi^+_\varrho \geqslant 2rn) \leqslant \mathbf{P}\Big(\bigcap_{i=1}^{n} \mathcal{E}_i^c\Big) = (1-p)^n,$$

which decays exponentially, whence the desired relation (2.5) follows.

Our next step is to define a random variable $R^+ = R^+[x, \bar{\mathcal{P}}]$ and to show it is a stabilization radius for ξ^+ at x for input process $\bar{\mathcal{P}}$. We shall also establish exponential decay of tails of R^+. For $\varrho > 0$ we put $R^+_\varrho = 2\xi^+_\varrho$. Further, we set

$\hat{\varrho} = \inf\{m \in \mathbb{N} \mid R_m^+ \leqslant m\}$. As $\mathbf{P}\big(\bigcap_{m\in\mathbb{N}}\{R_m^+ \geqslant m\}\big) \leqslant \inf_{m\in\mathbb{N}} \mathbf{P}(R_m^+ \geqslant m)$, which is 0 by (2.5), we readily conclude that so defined $\hat{\varrho}$ is a.s. finite. Take

$$R^+ := R^+_{\hat{\varrho}}. \tag{2.6}$$

Consequently, since $R^+ \leqslant \hat{\varrho}$ by definition, for any finite $A \subset B(x, R^+)^c$ we get a.s. by Lemma 2.3 and Corollary 2.1

$$\begin{aligned}
&\xi^+\Big(\bar{x}, \big(\bar{\mathcal{P}} \cap B(x, R^+)\big) \cup A\Big)\\
&= \xi^+\Big(\bar{x}, \bar{\mathcal{P}} \cap B(x, \hat{\varrho}) \cap B\Big(x, 2\xi^+\big(\bar{x}, \bar{\mathcal{P}} \cap B(x, \hat{\varrho})\big)\Big) \cup A\Big)\\
&= \xi^+\Big(\bar{x}, \bar{\mathcal{P}} \cap B(x, \hat{\varrho}) \cap B\Big(x, 2\xi^+\big(\bar{x}, \bar{\mathcal{P}} \cap B(x, \hat{\varrho})\big)\Big)\Big) = \xi^+\big(\bar{x}, \bar{\mathcal{P}} \cap B(x, R^+)\big).
\end{aligned}$$

Thus, R^+ is a stabilization radius for ξ^+ on $\bar{\mathcal{P}}$, as required. Further, taking into account that $R_k^+ = R^+$ for all $k \geqslant \hat{\varrho}$ by Corollary 2.1, we have for $m \in \mathbb{N}$

$$\begin{aligned}
\mathbf{P}(R^+ \geqslant m) &= \mathbf{P}(\lim_{k\to\infty} R_k^+ \geqslant m) = \lim_{k\to\infty} \mathbf{P}(R_k^+ \geqslant m)\\
&= \lim_{k\to\infty} \mathbf{P}(\xi_k^+ \geqslant m/2) \leqslant M e^{-Cm/2},
\end{aligned} \tag{2.7}$$

whence the desired exponential stabilization follows. ∎

Using the just proved stabilization property of ξ^+ and ξ^- we can now define

$$\xi^+(\bar{x}, \bar{\mathcal{P}}) := \xi^+\big(\bar{x}, \bar{\mathcal{P}} \cap B(x, R^+)\big) = \lim_{\varrho\to\infty} \xi^+\big(\bar{x}, \bar{\mathcal{P}} \cap B(x, \varrho)\big) = R^+/2 \tag{2.8}$$

and likewise for ξ^-. Clearly, the knowledge of these *infinite volume* functionals allows us to define the whole-plane Gilbert tessellation $G(\bar{\mathcal{P}})$.

3. PROOFS OF THE MAIN RESULTS

Theorems 1.1, 1.2 and 1.3 are now an easy consequence of the exponential stabilization Theorem 2.1. Indeed, observe first that, by (1.1), (2.8) and (2.7), the geometric functional

$$\xi(\bar{x}, \bar{\mathcal{X}}) := \phi\big(\xi^+(\bar{x}, \bar{\mathcal{X}}), \xi^-(\bar{x}, \bar{\mathcal{X}})\big)$$

satisfies the p-th bounded moment condition ([20], (4.6)) for all $p > 0$. Hence, Theorem 1.1 follows by Theorem 4.1 in [20]. Further, Theorem 1.2 holds true by Theorem 4.2 in [20]. Finally, Theorem 1.3 follows by Theorem 4.3 in [20] and Theorem 2.2 and Lemma 4.4 in [14].

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