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EXPLICIT SOLUTIONS OF THE EXTENDED SKOROKHOD PROBLEMS IN TIME-DEPENDENT BOUNDED REGIONS WITH ORTHOGONAL REFLECTION FIELDS

BY

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Abstract. We consider the extended Skorokhod problem for $\mathbb{R}^n$-valued càdlàg functions with the constraining set that changes in time and the reflection field naturally defined by the standard orthonormal basis. We find an explicit formula for the solution of such an extended Skorokhod problem in the case where the evolving constraining set is a region sandwiched between two graphs. We obtain the best Lipschitz constant for the extended Skorokhod map of this type.

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1. INTRODUCTION

To define the Skorokhod problem (SP) or the extended Skorokhod problem (ESP) in $\mathbb{R}^n$ one needs three things: an $\mathbb{R}^n$-valued càdlàg function ψ of a nonnegative variable t, a closed subset G of $\mathbb{R}^n$, and a set-valued function d assigning to each point $\mathbf{x}$ on the boundary of G a non-empty closed convex cone in $\mathbb{R}^n$ with the vertex at the origin and a closed graph $\{(\mathbf{x}, d(\mathbf{x})) : \mathbf{x} \in \partial G\}$. We will use $D[0,\infty)$ to denote real-valued right continuous functions with left limits defined on $[0,\infty)$, traditionally called càdlàg functions. $D([0,\infty),\mathbb{R}^n)$ will denote càdlàg functions taking values in $\mathbb{R}^n$ and $D_G([0,\infty),\mathbb{R}^n)$ will denote a subspace of $D([0,\infty),\mathbb{R}^n)$ consisting of functions ψ such that $\psi(0) \in G$. The convergence in $D([0,\infty),\mathbb{R}^n)$ will mean the uniform convergence on compact sets. The subspaces of $D([0,\infty),\mathbb{R}^n)$ and $D_G([0,\infty),\mathbb{R}^n)$ consisting of piecewise constant functions with a finite number of jumps will be denoted by $S([0,\infty),\mathbb{R}^n)$ and $S_G([0,\infty),\mathbb{R}^n)$, respectively. We will use $\overline{\mathrm{co}}(A)$ to denote the closed convex hull of a set A and $|\eta|(t)$ will denote the total variation of η on $[0,t]$.

A pair of functions $(\phi, \eta) \in D_G\big([0,\infty), \mathbb{R}^n\big) \times D\big([0,\infty), \mathbb{R}^n\big)$ is a *solution of the ESP* for ψ with respect to $\big(G, d(\cdot)\big)$ if $\phi = \psi + \eta$ and for every $t \geqslant 0$ the following conditions are satisfied:

$$\phi(t) \in G, \tag{1.1}$$

$$\eta(t) - \eta(s) \in \overline{\mathrm{co}}\Big[\bigcup_{u \in (s,t]} d\big(\phi(u)\big)\Big] \quad \text{for every } s \in [0,t], \tag{1.2}$$

$$\eta(t) - \eta(t-) \in \overline{\mathrm{co}}\big[d\big(\phi(t)\big)\big]. \tag{1.3}$$

The function ϕ in the solution of the ESP is called the *extended Skorokhod map* for ψ and the mapping $\Gamma_G : D\big([0,\infty), \mathbb{R}^n\big) \to D\big([0,\infty), \mathbb{R}^n\big)$ defined by $\Gamma_G(\psi) = \phi$ is called the *extended Skorokhod map* or, shortly, the ESM.

The pair (ϕ, η) is said to be a *solution of the SP* for ψ with respect to $\big(G, d(\cdot)\big)$ if instead of (1.2) and (1.3) the following stronger conditions are satisfied:

$$|\eta|(t) < \infty, \tag{1.4}$$

$$|\eta|(t) = \int_0^t I_{\phi(s) \in \partial G} d|\eta|(s), \tag{1.5}$$

$$\eta(t) = \int_0^t \gamma(s) d|\eta|(s) \tag{1.6}$$

for some function γ such that $\gamma(t) \in d^1\big(\phi(t)\big)$ $d|\eta|$-almost everywhere.

If the ESM $\Gamma_G(\psi) = \phi$ provides the solution of the SP, then it is called the *Skorokhod map* or, shortly, the SM.

Intuitively speaking, given an unrestricted process ψ, η provides the minimum force necessary to keep the path of its constrained version ϕ within the constraining region G. Whenever the change $\psi(t+\Delta t) - \psi(t)$ would place $\phi(t) + \psi(t+\Delta t) - \psi(t)$ outside of G, the vector $\eta(t+\Delta t) - \eta(t)$ would push it back into G along the direction prescribed by $d\big(\phi(t+\Delta t)\big)$.

The SM and the ESM are important tools in studying stochastic equations with reflections as well as in some queueing and network models. Historically, the SP appeared first in [19] in the real-valued case and was further studied in [6], [12], [15], [18]. In [8] and [9] an extensive study of the SP on convex polyhedra was presented. The ESP was introduced in [16]. Over the last two decades numerous efforts have been made to obtain some form of explicit solution to the SP. Some of them can be found in [4], [5], [11], [14], and [24]. Recent developments in the area of the SP include the explicit formula for the real-valued SM obtained in [13] in the case of a closed interval as a constraining domain. These results were extended in [3] to ESP on the interval whose endpoints change in time. Similar results were obtained by the author in [20] and [21]. In Theorem 2.11 of [21] we have shown that, for any $\alpha \in D[0,\infty)$, $\beta \in D[0,\infty)$ such that $\alpha \leqslant \beta$, the solution of ESP for any $\psi \in D[0,\infty)$ on $[\alpha,\beta]$ is a pair $(\psi - \Xi, -\Xi)$, where

(1.7)
$$\Xi_{\alpha,\beta}(\psi)(t) = I_{\{\tau^\beta \leqslant \tau_\alpha\}} I_{[\tau^\beta,\infty)}(t) H_{\alpha,\beta}(\psi)(t) + I_{\{\tau_\alpha < \tau^\beta\}} I_{[\tau_\alpha,\infty)}(t) L_{\alpha,\beta}(\psi)(t).$$

In the above formula,

$$(1.8)\ \ \tau_\alpha = \inf\{t > 0 \mid \alpha(t) - \psi(t) > 0\}, \quad \tau^\beta = \inf\{t > 0 \mid \psi(t) - \beta(t) > 0\},$$

$$(1.9) \qquad H_{\alpha,\beta}(\psi)(t) = \sup_{0 \leqslant s \leqslant t} \big[\big(\psi(s) - \beta(s)\big) \wedge \inf_{s \leqslant r \leqslant t} \big(\psi(r) - \alpha(r)\big)\big],$$

and

(1.10)
$$L_{\alpha,\beta}(\psi)(t) = -H_{-\beta,-\alpha}(-\psi)(t) = \inf_{0 \leqslant s \leqslant t} \big[\big(\psi(s) - \alpha(s)\big) \vee \sup_{s \leqslant r \leqslant t} \big(\psi(r) - \beta(r)\big)\big].$$

We will be needing the following well-known properties of solutions of the ESP in the real-valued case.

REMARK 1.1. *Let $\psi \in D[0,\infty)$, $\alpha \in D[0,\infty)$, and $\beta \in D[0,\infty)$ be such that $\alpha \leqslant \beta$. If (ϕ,η) is a solution of the ESP for ψ on $[\alpha,\beta]$ and $s \in [0,t]$, then the following conditions hold*:

(1.11) *if $\eta(t) > \eta(s)$ then there is $r \in (s,t]$ such that $\phi(r) = \alpha(r)$;*
(1.12) *if $\eta(t) < \eta(s)$ then there is $r \in (s,t]$ such that $\phi(r) = \beta(r)$.*

Proof. The statements follow immediately from properties (1) and (2) of Definition 2.2 of [3] or properties (i) and (ii) of Definition 1.1 of [21]. ■

REMARK 1.2. *Let $\psi \in D[0,\infty)$, $\alpha \in D[0,\infty)$, and $\beta \in D[0,\infty)$ be such that $\alpha \leqslant \beta$. If (ϕ,η) is a solution of the ESP for ψ on $[\alpha,\beta]$ and $r \in [0,\infty)$, then the following conditions hold*:

(1.13) *if $\eta(t) > \eta(t-)$ then $\phi(t) = \alpha(t)$;*
(1.14) *if $\eta(t) < \eta(t-)$ then $\phi(t) = \beta(t)$.*

Proof. These statements follow immediately from properties (1) and (3) of Definition 2.2 of [3] or properties (i) and (iii) of Definition 1.1 of [21]. ■

2. EXTENDED SKOROKHOD PROBLEMS WITH EVOLVING CONSTRAINTS

In this paper we are interested in the constraining domains in $\mathbb{R}^n$ that change with time, and so we shall need to introduce the convergence for sets. This will be defined in the sense of the Hausdorff metric. For any two sets $G_1, G_2 \subset \mathbb{R}^n$ their *Hausdorff distance* is defined by

$$d_H(G_1, G_2) = \big(\sup_{x \in G_1} d(x, G_2)\big) \vee \big(\sup_{x \in G_2} d(x, G_1)\big),$$

where $d(x, G) = \inf_{y \in G} \|x - y\|$ and where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^n$.

It is well known that the set of all non-empty compact subsets of $\mathbb{R}^n$ forms a complete metric space with d_H. It will be discussed in the proof of Proposition 4.2.

DEFINITION 2.1. A closed set G in $\mathbb{R}^n$ will be called a *stratum* if it admits the representation

$$G = \{\mathbf{x} : x^i \in [a^i, b^i], i < n, x^n \in [A(x^1, \dots, x^{n-1}), B(x^1, \dots, x^{n-1})]\}, \tag{2.1}$$

where $a^i \leqslant b^i$ for $i = 1, 2, \dots, n-1$ and A, B are two real-valued continuous functions on $[a^1, b^1] \times \ldots \times [a^{n-1}, b^{n-1}]$ such that $A(x) \leqslant B(x)$ for every x. Given such a representation we will shortly write

$$G = S([a^1, b^1] \times \ldots \times [a^{n-1}, b^{n-1}], [A, B]). \tag{2.2}$$

It is necessary that we extend the domains of functions A and B to $\mathbb{R}^{n-1}$. These extensions, denoted by $\bar{A}$ and $\bar{B}$, will be defined by

$$\bar{A}(x^1, x^2, \dots, x^{n-1}) = A\big(\pi_{\alpha^1,\beta^1}(x^1), \pi_{\alpha^2,\beta^2}(x^2), \dots, \pi_{\alpha^{n-1},\beta^{n-1}}(x^{n-1})\big),$$

$$\bar{B}(x^1, x^2, \dots, x^{n-1}) = B\big(\pi_{\alpha^1,\beta^1}(x^1), \pi_{\alpha^2,\beta^2}(x^2), \dots, \pi_{\alpha^{n-1},\beta^{n-1}}(x^{n-1})\big).$$

Because of the special nature of the last coordinate it will sometimes be convenient to use $n = d + 1$. For the sake of brevity we shall also use $D = [a^1, b^1] \times \ldots \times [a^d, b^d]$. In the special case when A and B are constant functions, G will be called a *block*. In other words, a block is a cross product of n intervals.

The projections $\pi_{a,b} : \mathbb{R} \to [a, b]$ were used in [13] and [20] to construct the SM in $\mathbb{R}$. They were defined by

$$\pi_{a,b} = \begin{cases} a & \text{if } x \leqslant a, \\ x & \text{if } a \leqslant x \leqslant b, \\ b & \text{if } x \geqslant b. \end{cases} \tag{2.3}$$

In the vector-valued case we will need similar projections onto blocks and strata. Given a block $D = [a^1, b^1] \times \ldots \times [a^n, b^n]$ we define $\pi_D : \mathbb{R}^n \to D$ by

$$\pi_D(x) = \big(\pi_{a^1,b^1}(x^1), \pi_{a^2,b^2}(x^2), \dots, \pi_{a^n,b^n}(x^n)\big). \tag{2.4}$$

Finally, the projection on a stratum $G = S([a^1, b^1] \times \ldots \times [a^{n-1}, b^{n-1}], [A, B])$ will be defined by

(2.5)

$$\pi_G(x) = \big(\pi_{a^1,b^1}(x^1), \dots, \pi_{a^{n-1},b^{n-1}}(x^{n-1}), \pi_{\bar{A}(x^1,\dots,x^{n-1}),\bar{B}(x^1,\dots,x^{n-1})}(x^n)\big),$$

which we will shortly write as

$$\pi_G(x) = \big(\pi_D(x), \pi_{A(\pi_D(x)),B(\pi_D(x))}(x^n)\big). \tag{2.6}$$

We shall use $\mathcal{G}$ to denote the space of all strata in $\mathbb{R}^n$. For any $K \geqslant 0$ we shall use $\mathcal{G}_K$ to denote the space of all strata in $\mathbb{R}^n$ such that A and B satisfy the Lipschitz condition with constant K.

DEFINITION 2.2. A family $\{G_t : t \geqslant 0\}$ of closed subsets of $\mathbb{R}^n$ will be called *càdlàg* if the function $t \mapsto G_t$ is càdlàg with respect to the Hausdorff metric d_H.

To represent a càdlàg family of strata we shall use the following notation:

$$G_t = S([\alpha_t^1, \beta_t^1] \times \ldots \times [\alpha_t^{n-1}, \beta_t^{n-1}], [A_t, B_t]), \tag{2.7}$$

where $\alpha_t^i \leqslant \beta_t^i$ for $i = 1, 2, \ldots, d$ and $A_t \leqslant B_t$.

DEFINITION 2.3. A family of pairs $\big\{\big(G_t, d_t(\cdot)\big) : t \geqslant 0\big\}$ will be called an *orthogonal evolving stratum constraining system* if G_t is a stratum for every $t \geqslant 0$, $\{G_t : t \geqslant 0\}$ is càdlàg, and

$$d_t(x) = \Big\{ \sum_{i \in I_t^+} r^i \mathbf{e}_i - \sum_{i \in I_t^-} r^i \mathbf{e}_i : r_i \geqslant 0 \text{ for } i \in I_t^+(x) \cup I_t^-(x) \Big\}, \tag{2.8}$$

where

$$I_t^+(x) = \{i : 1 \leqslant i < n \text{ and } x^i = \alpha_t^i, \text{ or } i = n \text{ and } x^n = A(x^1, x^2, \ldots, x^{n-1})\},$$

$$I_t^-(x) = \{i : 1 \leqslant i < n \text{ and } x^i = \beta_t^i, \text{ or } i = n \text{ and } x^n = B(x^1, x^2, \ldots, x^{n-1})\}.$$

In the special case when G_t is a block for every t, the orthogonal evolving stratum constraining system will be called an *orthogonal evolving block constraining system*.

Note that in the orthogonal evolving stratum constraining system, it is the stratum that varies in time. The constraining field d, on the other hand, remains steady. For any t, if x is a point on the boundary of G_t that lies on a particular side of G_t, then $d(x)$ contains the one-dimensional cone generated by one vector from the standard orthonormal basis in $\mathbb{R}^n$ that corresponds to that side. For instance, if $x^k = \beta_t^k$ for some $1 \leqslant k \leqslant n-1$, then $d(x) \supset -\mathbb{R}^+\mathbf{e}_k$; if $x^n = B_t(x^1, x^2, \ldots, x^{n-1})$, then $d(x) \supset -\mathbb{R}^+\mathbf{e}_n$. Thus, it is the constraining field d that is orthogonal.

DEFINITION 2.4. Given an orthogonal evolving stratum constraining system $\big\{\big(G_t, d_t(\cdot)\big) : t \geqslant 0\big\}$ and a càdlàg function $\psi \in D_{G_0}\big([0,\infty), \mathbb{R}^n\big)$, the pair $(\phi, \eta) \in D_{G_0}\big([0,\infty), \mathbb{R}^n\big) \times D_{\{0\}}\big([0,\infty), \mathbb{R}^n\big)$ is the *solution of the evolving ESP* for ψ with respect to $\big(G_t, d_t(\cdot)\big)$ if the following conditions hold for every $t \geqslant 0$:

(i) $\phi(t) = \psi(t) + \eta(t)$;
(ii) $\phi(t) \in G_t$;
(iii) $\eta(t) - \eta(s) \in \overline{\mathrm{co}}\big[\bigcup_{u\in(s,t]} d_u\big(\phi(u)\big)\big]$ for every $s \in [0,t]$;
(iv) $\eta(t) - \eta(t-) \in d_t\big(\phi(t)\big)$.

THEOREM 2.1. *Let $\{(G_t, d_t(\cdot)) : t \geqslant 0\}$ be an orthogonal evolving stratum constraining system with $G_t = S([\alpha_t^1, \beta_t^1] \times \ldots \times [\alpha_t^{n-1}, \beta_t^{n-1}], [A_t, B_t])$. Then, the evolving ESP for any $\psi \in D_{G_0}\big([0,\infty), \mathbb{R}^n\big)$ on $\big(G_t, d(\cdot)\big)$ has a unique solution (ϕ, η) given by $\eta = \big(-\Xi_{\alpha^1,\beta^1}(\psi^1), -\Xi_{\alpha^2,\beta^2}(\psi^2), \ldots, -\Xi_{\alpha^n,\beta^n}(\psi^n)\big)$ and $\phi = \psi + \eta$, where*

$$\alpha_t^n = A_t\big(\psi^1(t) - \Xi_{\alpha_t^1,\beta_t^1}, \psi^2(t) - \Xi_{\alpha_t^2,\beta_t^2}, \ldots, \psi^{n-1}(t) - \Xi_{\alpha_t^{n-1},\beta_t^{n-1}}\big), \tag{2.9}$$

$$\beta_t^n = B_t\big(\psi^1(t) - \Xi_{\alpha_t^1,\beta_t^1}, \psi^2(t) - \Xi_{\alpha_t^2,\beta_t^2}, \ldots, \psi^{n-1}(t) - \Xi_{\alpha_t^{n-1},\beta_t^{n-1}}\big), \tag{2.10}$$

and, for every $i = 1, 2, \ldots, n$,

$$\begin{aligned}\Xi_{\alpha^i,\beta^i}(\psi^i)(t) = {} & I_{\{\tau^{\beta^i} \leqslant \tau_{\alpha^i}\}} I_{[\tau^{\beta^i},\infty)} H_{\alpha^i,\beta^i}(\psi^i)(t) \\ & + I_{\{\tau^{\alpha^i} < \tau_{\beta^i}\}} I_{[\tau^{\alpha^i},\infty)} L_{\alpha^i,\beta^i}(\psi^i)(t).\end{aligned} \tag{2.11}$$

P r o o f. For any fixed $i \in \{1, 2, \ldots, n-1\}$ consider the ESP for ψ^i on $[\alpha^i, \beta^i]$. By (1.7), it has a unique solution $\big(\psi^i - \Xi_{\alpha^i,\beta^i}(\psi^i), -\Xi_{\alpha^i,\beta^i}(\psi^i)\big)$, where

$$\begin{aligned}\Xi_{\alpha^i,\beta^i}(\psi^i)(t) = {} & I_{\{\tau^{\beta^i} \leqslant \tau_{\alpha^i}\}} I_{[\tau^{\beta^i},\infty)}(t) H_{\alpha^i,\beta^i}(\psi^i)(t) \\ & + I_{\{\tau_{\alpha^i} < \tau^{\beta^i}\}} I_{[\tau_{\alpha^i},\infty)}(t) L_{\alpha^i,\beta^i}(\psi^i)(t).\end{aligned} \tag{2.12}$$

Let α^n and β^n be defined as in (2.9) and (2.10), respectively. They are well defined because, for each $i = 1, 2, \ldots, n-1$ and every $t \geqslant 0$,

$$\psi^i(t) - \Xi_{\alpha^i,\beta^i}(\psi^i)(t) \in [\alpha_t^i, \beta_t^i].$$

Then, $\alpha^n \in D[0,\infty)$, $\beta^n \in D[0,\infty)$, and $\alpha^n \leqslant \beta^n$. Now consider the ESP for ψ^n on $[\alpha^n, \beta^n]$. By (1.7), it has a unique solution $\big(\psi^n - \Xi_{\alpha^n,\beta^n}(\psi^n), -\Xi_{\alpha^n,\beta^n}(\psi^n)\big)$.

Let $\eta = \big(-\Xi_{\alpha^1,\beta^1}(\psi^1), \ldots, -\Xi_{\alpha^{n-1},\beta^{n-1}}(\psi^{n-1}), -\Xi_{\alpha^n,\beta^n}(\psi^n)\big)$ and $\phi = \psi + \eta$. We will show that (ϕ, η) is a solution of the evolving ESP for ψ with respect to $\{(G_t, d_t(\cdot)) : t \geqslant 0\}$. We only need to show properties (ii)–(iv) of Definition 2.4, as the property (i) holds by the definition of ϕ. Let $t \in [0,\infty)$. To show property (ii) consider the i-th coordinate of ϕ. Note that $\phi^i(t) = \psi^i(t) - \Xi_{\alpha^i,\beta^i}(\psi^i)(t) \in [\alpha_t^i, \beta_t^i]$ for every $1 \leqslant i \leqslant n$ because ϕ^i is the ESM for ψ^i on $[\alpha^i, \beta^i]$. In particular, $\phi^n(t) \in \big[A_t\big(\phi^1(t), \ldots, \phi^{n-1}(t)\big), B_t\big(\phi^1(t), \ldots, \phi^{n-1}(t)\big)\big]$, and so $\phi(t) \in G_t$.

In order to show property (iii) let $s \in [0,t]$ and consider $\eta(t) - \eta(s)$. Let $J^+_{s,t} = \{1 \leqslant i \leqslant n : \eta^i(t) - \eta^i(s) > 0\}$, $J^-_{s,t} = \{1 \leqslant i \leqslant n : \eta^i(t) - \eta^i(s) < 0\}$. For every $i \in J^+_{s,t}$, by (1.11), $\phi^i(u_i) = \alpha^i(u_i)$ for some $u_i \in (s,t]$, and therefore $\big(\eta^i(t) - \eta^i(s)\big)\,\mathbf{e}_i \in \{r\mathbf{e}_i : r \geqslant 0\} \subset d_{u_i}\big(\phi(u_i)\big)$. Similarly, by (1.12), $\phi^i(u_i) = \beta^i(u_i)$ for some $u_i \in (s,t]$, so $\big(\eta^i(t) - \eta^i(s)\big)\,\mathbf{e}_i \in \{-r\mathbf{e}_i : r \geqslant 0\} \subset d_{u_i}\big(\phi(u_i)\big)$ for every $i \in J^-_{s,t}$. Thus we have $\eta(t) - \eta(s) = \sum_{i \in J^+_{s,t}} \big(\eta^i(t) - \eta^i(s)\big)\,\mathbf{e}_i + \sum_{i \in J^-_{s,t}} \big(\eta^i(t) - \eta^i(s)\big)\,\mathbf{e}_i \in \overline{\mathrm{co}}\big[\bigcup_{u \in (s,t]} d_u\big(\phi(u)\big)\big]$.

To prove property (iv) let $J^+_t = \{1 \leqslant i \leqslant n : \eta^i(t) - \eta^i(t-) > 0\}$ and let $J^-_t = \{1 \leqslant i \leqslant n : \eta^i(t) - \eta^i(t-) < 0\}$. By (1.13), $\phi^i(t) = \alpha^i(t)$ for $i \in J^+_t$ and, by (1.14), $\phi^i(t) = \beta^i(t)$ for $i \in J^-_t$. Hence $J^+_t = I^+_t\big(\phi(t)\big)$ and $J^-_t = I^-_t\big(\phi(t)\big)$. Thus $\eta(t) - \eta(t-) = \sum_{i \in J^+_t} \big(\eta^i(t) - \eta^i(t-)\big)\,\mathbf{e}_i + \sum_{i \in J^-_t} \big(\eta^i(t) - \eta^i(t)\big)\,\mathbf{e}_i \in \big\{ \sum_{i \in I^+_t(\phi(t))} r^i\mathbf{e}_i - \sum_{i \in I^-_t(\phi(t))} r^i\mathbf{e}_i : r_i \geqslant 0\big\} = d_t\big(\phi(t)\big)$. ∎

Clearly, both α^n and β^n in (2.9) and (2.10) are not only functions of t but also depend on ψ. However, it is important to understand that they only depend on $\psi^1, \psi^2, \ldots, \psi^{n-1}$ and not on ψ^n.

EXAMPLE 2.1. Consider the ESP for a function $\psi \in S\big([0,\infty), \mathbb{R}^n\big)$ with an orthogonal evolving stratum constraining system $\big(G_t, d_t(\cdot)\big)$ such that

$$\psi(t) = \psi(t_k) \quad \text{and} \quad G_t = G_{t_k} \quad \text{for every } t \in [t_k, t_{k+1}), k = 0, 1, \ldots, m,$$

where $0 = t_0 < t_1 < t_2 < \ldots < t_m < \infty$ and $t_{m+1} = \infty$. Then, the corresponding ESM is the function ϕ such that for $t \in [t_k, t_{k+1}), k = 0, 1, \ldots, m$,

$$\phi(t) = \phi(t_k) = \pi_{G_{t_k}}\big(\phi(t_{k-1}) + \psi(t_k) - \psi(t_{k-1})\big). \tag{2.13}$$

It is well known that the ESM of a simple function satisfies equation (2.13) in the case of a traditional fixed restraining set. It was used in [7], [9], and [17] for instance. In the case of an evolving constraining system it can be shown directly that ϕ defined by (2.13) satisfies the conditions of Definition 2.4. Alternatively, it can be shown that the ESM described by Theorem 2.1 satisfies (2.13).

3. LIPSCHITZ PROPERTIES OF THE ESM WITH AN ORTHOGONAL EVOLVING BLOCK CONSTRAINING SYSTEM

The more regular the ESM is the more useful it is in applications. The most important regularity feature of the ESM is the Lipschitz property. Some interesting geometric conditions were shown to be sufficient for Lipschitz continuity in [7] and [8]. Here we obtain the best Lipschitz constant for the ESM with an orthogonal evolving block constraining system.

PROPOSITION 3.1. *Let* $\{(G_t, d_t(\cdot)) : t \geqslant 0\}$ *be an orthogonal evolving block constraining system with* $G_t = S([\alpha_t^1, \beta_t^1] \times \ldots \times [\alpha_t^n, \beta_t^n])$ *and let* ψ_1, ψ_2 *be two functions from* $D([0,\infty), \mathbb{R}^n)$. *Then*

$$\|(\Gamma_G(\psi_1) - \Gamma_G(\psi_2)) - (\psi_1 - \psi_2)\| \leqslant \sqrt{n}\|\psi_1 - \psi_2\|, \tag{3.1}$$

where $\|\psi\| = \sup_{t \geqslant t} \|\psi(t)\|$.

P r o o f. We will first establish (3.1) in the one-dimensional case. Let (ϕ, η) be the solution of the ESP for ψ on the interval $[\alpha, \beta]$. We will show that

$$\|(\phi_1 - \phi_2) - (\psi_1 - \psi_2)\| \leqslant \|\psi_1 - \psi_2\|. \tag{3.2}$$

In the case when $\psi_1, \psi_2, \alpha, \beta \in S([0,\infty), \mathbb{R})$ this is equation (4.40) from [20] and it is proven there by induction. In the general case, as in the proof of Remark 4.5 or Proposition 4.6 in [20], we can find sequences $\psi_1^n, \psi_2^n, \alpha^n, \beta^n \in S([0,\infty), \mathbb{R})$ converging uniformly on compact sets to ψ_1, ψ_2, α, and β, respectively. Since (3.2) holds for $\psi_1^n, \psi_2^n, \alpha^n, \beta^n$, taking limits and using (4.4) and Remark 4.3 in [20], we conclude (3.2) in the general case when $\psi_1, \psi_2, \alpha, \beta$ are from $D([0,\infty), \mathbb{R})$. Inequality (3.2) can also be concluded from (1.3) in [22].

For each $j = 1, 2$, let $\psi_j = (\psi_j^1, \psi_j^2, \ldots, \psi_j^n) \in D([0,\infty), \mathbb{R}^n)$. By Theorem 2.1, $\Gamma_G(\psi_j) = \phi_j = (\phi_j^1, \phi_j^2, \ldots, \phi_j^n)$, where ϕ_j^i is the ESM for ψ_j^i on the interval $[\alpha_t^i, \beta_t^i]$. Therefore, by (3.2), for each $i = 1, 2, \ldots, n$,

$$\|(\phi_1^i - \phi_2^i) - (\psi_1^i - \psi_2^i)\| \leqslant \|\psi_1^i - \psi_2^i\|. \tag{3.3}$$

Applying (3.3) to all components, we get

$$\begin{aligned}
&\|(\phi_1(t) - \phi_2(t)) - (\psi_1(t) - \psi_2(t))\|^2 \\
&\qquad = \sum_{i=1}^n |(\phi_1^i(t) - \phi_2^i(t)) - (\psi_1^i(t) - \psi_2^i(t))|^2 \\
&\qquad \leqslant \sum_{i=1}^n \|\psi_1^i - \psi_2^i\|^2 \leqslant n\|\psi_1 - \psi_2\|^2,
\end{aligned}$$

which implies (3.1). ∎

The following example will show that the Lipschitz constant in Proposition 3.1 is tight.

EXAMPLE 3.1. Let $G = [0,2] \times [0,2] \times \ldots \times [0,2]$, let $\psi_1 = -\mathbf{e}_1 I_{[0,1)} - \mathbf{e}_2 I_{[1,2)} - \ldots - \mathbf{e}_n I_{[n-1,n)} + \mathbf{e}_n I_{[n,\infty)}$, and $\psi_2 = \mathbf{0}$. Then $\phi_2 = \mathbf{0}$ and, by (2.13),

$$\begin{aligned}
&\phi_1(0) = \pi_{G(0)}\big(\psi_1(0)\big) = \mathbf{0},\\
&\phi_1(1) = \pi_{G(1)}\big(\phi_1(0) + \psi_1(1) - \psi_1(0)\big) = \pi_G\,(\mathbf{e}_1 - \mathbf{e}_2) = \mathbf{e}_1,\\
&\phi_1(2) = \pi_{G(2)}\big(\phi_1(1) + \psi_1(2) - \psi_1(1)\big) = \pi_G\,(\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3) = \mathbf{e}_1 + \mathbf{e}_2,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\phi_1(n-1) = \pi_G\,(\mathbf{e}_1 + \mathbf{e}_2 + \ldots + \mathbf{e}_{n-1} - \mathbf{e}_n) = \mathbf{e}_1 + \mathbf{e}_2 + \ldots + \mathbf{e}_{n-1},\\
&\phi_1(n) = \pi_G\,(\mathbf{e}_1 + \mathbf{e}_2 + \ldots + \mathbf{e}_{n-1} + 2\mathbf{e}_n) = \mathbf{e}_1 + \mathbf{e}_2 + \ldots + \mathbf{e}_{n-1} + 2\mathbf{e}_n.
\end{aligned}$$

Note that

$$\big\|\big(\phi_1(n) - \phi_2(n)\big) - \big(\psi_1(n) - \psi_2(n)\big)\big\| = \|\mathbf{e}_1 + \mathbf{e}_2 + \ldots + \mathbf{e}_n\| = \sqrt{n},$$

while $\|\psi_1(t) - \psi_2(t)\| = 1$ for every $t \geqslant 0$. Thus

$$\|(\phi_1 - \phi_2) - (\psi_1 - \psi_2)\| \geqslant \sqrt{n}\|\psi_1 - \psi_2\|,$$

and so the Lipschitz constant in Proposition 3.1 is tight.

THEOREM 3.1. *Let* $\big\{\big(G_t, d_t(\cdot)\big) : t \geqslant 0\big\}$ *be an orthogonal evolving block constraining system with* $G_t = S([\alpha_t^1, \beta_t^1] \times \ldots \times [\alpha_t^{n-1}, \beta_t^{n-1}], [\alpha_t^n, \beta_t^n])$. *Then the ESM for the evolving ESP of Theorem* 2.1 *is Lipschitz continuous with constant* $1 + \sqrt{n}$, *i.e.*

$$\|\Gamma_G(\psi_1) - \Gamma_G(\psi_2)\| \leqslant \big(1 + \sqrt{n}\big)\,\|\psi_1 - \psi_2\|. \tag{3.4}$$

P r o o f. By Proposition 3.1,

$$\begin{aligned}
\|\Gamma_G(\psi_1) - \Gamma_G(\psi_2)\| &\leqslant \big\|\big(\Gamma_G(\psi_1) - \Gamma_G(\psi_2)\big) - (\psi_1 - \psi_2)\big\| + \|\psi_1 - \psi_2\|\\
&\leqslant \sqrt{n}\|\psi_1 - \psi_2\| + \|\psi_1 - \psi_2\| \leqslant \big(\sqrt{n} + 1\big)\,\|\psi_1 - \psi_2\|,
\end{aligned}$$

which completes the proof. ■

The Lipschitz constant in Theorem 3.1 is tight as the following modification of Example 3.1 will clearly establish.

EXAMPLE 3.2. We will use Example 3.1 with an added extra jump. Let $G = [0,2] \times [0,2] \times \ldots \times [0,2]$, $\psi_1 = -\mathbf{e}_1 I_{[0,1)} - \mathbf{e}_2 I_{[1,2)} - \ldots - \mathbf{e}_n I_{[n-1,n)} + \mathbf{e}_n I_{[n,n+1)} + (1/\sqrt{n})\,(\mathbf{e}_1 + \mathbf{e}_2 + \ldots + \mathbf{e}_n)\,I_{[n+1,\infty)}$ and $\psi_2 = \mathbf{0}$. Then

$$\begin{aligned}
&\phi_1(n+1) = \pi_{G(n+1)}\big(\phi_1(n) + \psi_1(n+1) - \psi_1(n)\big)\\
&= \pi_{G(n+1)}\big((\mathbf{e}_1 + \mathbf{e}_2 + \ldots + \mathbf{e}_{n-1} + 2\mathbf{e}_n) + (1/\sqrt{n})(\mathbf{e}_1 + \mathbf{e}_2 + \ldots + \mathbf{e}_n) - \mathbf{e}_n\big)\\
&= \pi_G\big((1 + 1/\sqrt{n})(\mathbf{e}_1 + \mathbf{e}_2 + \ldots + \mathbf{e}_n)\big) = (1 + 1/\sqrt{n})(\mathbf{e}_1 + \mathbf{e}_2 + \ldots + \mathbf{e}_n).
\end{aligned}$$

As in Example 3.1, $\|\psi_1 - \psi_2\| = 1$, however

$$\begin{aligned}\|\phi_1 - \phi_2\| \geqslant \|\phi_1(n+1) - \phi_2(n+1)\| &= \left(1 + 1/\sqrt{n}\right) \|\mathbf{e}_1 + \mathbf{e}_2 + \ldots + \mathbf{e}_n\| \\ &= \left(1 + 1/\sqrt{n}\right) \sqrt{n} = \sqrt{n} + 1.\end{aligned}$$

Thus the Lipschitz constant $\sqrt{n} + 1$ in Theorem 3.1 is tight.

4. LIPSCHITZ PROPERTIES OF THE ESM WITH AN ORTHOGONAL EVOLVING STRATUM CONSTRAINING SYSTEM

Our final goal is to find the best Lipschitz constant for the extended Skorokhod map with an orthogonal evolving stratum constraining system. Consider such a system $\left\{\left(G_t, d_t(\cdot)\right) : t \geqslant 0\right\}$ as defined in Definition 2.3. For each $t \geqslant 0$, let C_t^A and C_t^B be the best Lipschitz constants for A_t and B_t, where A and B are as in (2.7). We define the best Lipschitz constant for G as follows:

$$K_G = \sup_{t \geqslant 0} \{C_t^A \vee C_t^B\}. \tag{4.1}$$

As it turns out the best Lipschitz constant for Γ_G depends on K_G. However, before we derive the best Lipschitz constant, it will be useful to establish any Lipschitz condition. We can obtain one by applying the real-valued results coordinatewise.

LEMMA 4.1. *Let* $\left\{\left(G_t, d_t(\cdot)\right) : t \geqslant 0\right\}$ *be an orthogonal evolving stratum constraining system with* $G_t = S\left([\alpha_t^1, \beta_t^1] \times \ldots \times [\alpha_t^d, \beta_t^d] \times [A_t, B_t]\right) \subset \mathbb{R}^{d+1}$. *Then the ESM for the evolving ESP of Theorem* 2.1 *is Lipschitz continuous with constant* $\left(4 + 3K_G\sqrt{d+1}\right)$, *i.e.*

$$\|\Gamma_G(\psi_1) - \Gamma_G(\psi_2)\| \leqslant \left(4 + 3K_G\sqrt{d+1}\right) \cdot \|\psi_1 - \psi_2\|. \tag{4.2}$$

Proof. By Theorem 3.6 of [21], for every $i = 1, 2, \ldots, d$,

$$\|\Gamma_G^i(\psi_1) - \Gamma_G^i(\psi_2)\| = \|\Gamma_{\alpha_i,\beta_i}(\psi_1^i) - \Gamma_{\alpha_i,\beta_i}(\psi_2^i)\| \leqslant 2\|\psi_1 - \psi_2\|.$$

By Theorem 2.1 and by Theorem 3.5 of [21],

$$\begin{aligned}&\|\Gamma_G^{d+1}(\psi_1) - \Gamma_G^{d+1}(\psi_2)\| \\ &= \|\Gamma_{\alpha^{d+1}(\psi_1),\beta^{d+1}(\psi_1)}(\psi_1^{d+1}) - \Gamma_{\alpha^{d+1}(\psi_2),\beta^{d+1}(\psi_2)}(\psi_2^{d+1})\| \\ &\leqslant 4\|\psi_1 - \psi_2\| + 3[\|\alpha^{d+1}(\psi_1) - \alpha^{d+1}(\psi_2)\| \vee \|\beta^{d+1}(\psi_1) - \beta^{d+1}(\psi_2)\|],\end{aligned}$$

where

$$\begin{aligned}
&\|\alpha^{d+1}(\psi_1) - \alpha^{d+1}(\psi_2)\| \\
&= \sup_{0\leqslant t\leqslant T} \big\| A_t\big(\psi_1^1(t) - \Xi_{\alpha_t^1,\beta_t^1}, \psi_1^2(t) - \Xi_{\alpha_t^2,\beta_t^2}, \dots, \psi_1^d(t) - \Xi_{\alpha_t^d,\beta_t^d}\big) \\
&\qquad\qquad - A_t\big(\psi_2^1(t) - \Xi_{\alpha_t^1,\beta_t^1}, \psi_2^2(t) - \Xi_{\alpha_t^2,\beta_t^2}, \dots, \psi_2^d(t) - \Xi_{\alpha_t^d,\beta_t^d}\big)\big\| \\
&\leqslant K_G \cdot \sup_{0\leqslant t\leqslant T} \big\| \big(\psi_1^1(t) - \psi_2^1(t) - \Xi_{\alpha_t^1,\beta_t^1}(\psi_1^1)(t) + \Xi_{\alpha_t^1,\beta_t^1}(\psi_2^1)(t), \\
&\qquad\qquad \psi_1^2(t) - \psi_2^2(t) - \Xi_{\alpha_t^2,\beta_t^2}(\psi_1^2)(t) + \Xi_{\alpha_t^2,\beta_t^2}(\psi_2^2)(t), \\
&\qquad\qquad \dots, \psi_1^d(t) - \psi_2^d(t) - \Xi_{\alpha_t^d,\beta_t^d}(\psi_1^d)(t) + \Xi_{\alpha_t^d,\beta_t^d}(\psi_2^d)(t)\big)\big\| \\
&\leqslant K_G\big(\|\psi_1 - \psi_2\| + \big\|\Xi_{\alpha^1,\beta^1}(\psi_1^1) - \Xi_{\alpha_t^1,\beta^1}(\psi_2^1), \Xi_{\alpha^2,\beta^2}(\psi_1^2) - \Xi_{\alpha^2,\beta^2}(\psi_2^2), \\
&\qquad\qquad \dots, \Xi_{\alpha^d,\beta^d}(\psi_1^d) - \Xi_{\alpha^d,\beta^d}(\psi_2^d)\big\|\big) \\
&\leqslant K_G\big(\|\psi_1 - \psi_2\| + \sqrt{d}\|\psi_1 - \psi_2\|\big) = K_G\big(1 + \sqrt{d}\big)\|\psi_1 - \psi_2\|,
\end{aligned}$$

where the last inequality follows from inequality (3.2) applied coordinatewise.

Similarly we can show that

$$\|\beta^{d+1}(\psi_1) - \beta^{d+1}(\psi_2)\| \leqslant K_G\big(1 + \sqrt{d}\big)\|\psi_1 - \psi_2\|.$$

Therefore

$$\begin{aligned}
\|\Gamma_G(\psi_1) - \Gamma_G(\psi_2)\| &\leqslant 4\|\psi_1 - \psi_2\| + 3K_G\big(1 + \sqrt{d}\big)\|\psi_1 - \psi_2\| \\
&= \big(4 + 3K_G\sqrt{d}\big)\|\psi_1 - \psi_2\|. \quad \blacksquare
\end{aligned}$$

It is well known that every function in $D([0,T],\mathbb{R}^n)$ can be uniformly approximated by functions taking a finite number of values. In fact, it is true in more general spaces.

REMARK 4.1. *If* (X,d) *is a metric space, then* $S([0,T],X)$ *is dense in* $D([0,T],X)$.

P r o o f. In the case when (X,d) is a complete separable metric space the above result can be surmised from Lemma 1 in Section 12 of [2] and from the remarks following its proof. Let (X,d) be any metric space and let $\psi \in D([0,T],X)$. For any $\epsilon > 0$ there is a finite partition $0 = t_0 < t_1 < \ldots < t_n < t_{n+1} = T$ such that $d\big(\psi(t), \psi(t_k)\big) \leqslant \epsilon$ for $t_k \leqslant t < t_{k+1}$ and $0 \leqslant k \leqslant n$. It is enough to define $t_k = \inf\big\{t > t_{k-1} : d\big(\psi(t), \psi(t_{k-1})\big) > \epsilon\big\}$. Using this partition we can define

$$\psi_\epsilon(t) = \sum_{k=0}^{n-1} \psi(t_k)\, I_{[t_k,t_{k+1})} + \psi(t_n)\, I_{[t_n,T]},$$

where $\sup_{0\leqslant t\leqslant T} d\big(\psi(t), \psi_\epsilon(t)\big) \leqslant \epsilon$ and $\psi_\epsilon(t) \in S([0,T],X)$. $\blacksquare$

Consider the space $\mathcal{G}$ equipped with the Hausdorff metric d_H. It can be shown that $(\mathcal{G}, d_H)$ is a complete separable space. We define first a more suitable metric on $\mathcal{G}$.

DEFINITION 4.1. Let G_1 and G_2 be two strata in $\mathcal{G}$ with representations $G_i = S([a_i^1, b_i^1] \times \ldots \times [a_i^{n-1}, b_i^{n-1}], [A_i, B_i])$ for $i = 1, 2$. We define

$$d_M(G_1, G_2) = \max_{1 \leqslant k \leqslant d} [|a_1^k - a_2^k| \vee |b_1^k - b_2^k|] \vee [\|\bar{A}_1 - \bar{A}_2\| \vee \|\bar{B}_1 - \bar{B}_2\|]. \tag{4.3}$$

It is easy to verify that d_M is a *metric* on $\mathcal{G}$.

PROPOSITION 4.1. *The metrics d_M and d_H are equivalent on $\mathcal{G}$. In fact, for any $G_1, G_2 \in \mathcal{G}$,*

$$d_H(G_1, G_2) \leqslant \sqrt{d+1} \cdot d_M(G_1, G_2), \tag{4.4}$$

$$d_M(G_1, G_2) \leqslant \left(K + \sqrt{K^2 + 1}\right) \cdot d_H(G_1, G_2), \tag{4.5}$$

where $K = \max\{K_{G_1}, K_{G_2}\}$.

P r o o f. We begin with a simple observation: given any two intervals $[a_1, b_1]$ and $[a_2, b_2]$

$$|x - \pi_{a_2, b_2}(x)| \leqslant |a_2 - a_1| \vee |b_2 - b_1| \quad \text{for every } x \in [a_1, b_1]. \tag{4.6}$$

Let $D_i = \left[a_i^1, b_i^1\right] \times \left[a_i^2, b_i^2\right] \times \ldots \times \left[a_i^d, b_i^d\right]$ for $i = 1, 2$. Using (2.4) and applying (4.6) coordinatewise we get a multidimensional version of this inequality:

$$\|x - \pi_{D_2}(x)\|^2 \leqslant \sum_{k=1}^{d} (|a_1^k - a_2^k| \vee |b_1^k - b_2^k|)^2 \quad \text{for every } x \in D_1. \tag{4.7}$$

We consider (4.4) first. If $G_1 \neq G_2$ then $d_H(G_1, G_2) = d(x_1, G_2)$ for some $x_1 \in \partial G_1 \setminus G_2$ or $d_H(G_1, G_2) = d(x_2, G_1)$ for some $x_2 \in \partial G_2 \setminus G_1$. We can assume without loss of generality that it is the former case. Then $x_1^{d+1} = A(x_1^1, x_1^2, \ldots, x_1^d)$ or $x_1^{d+1} = B(x_1^1, x_1^2, \ldots, x_1^d)$, or $x_1^k = a_1^k$, or $x_1^k = b_1^k$ for some $k = 1, 2, \ldots, d$. Let $D_2 = [a_2^1, b_2^1] \times [a_2^2, b_2^2] \times \ldots \times [a_2^d, b_2^d]$ and let π_{G_2} be as defined in (2.5) or (2.6). Then we can write

$$\pi_{G_2}(x_1) = \left(\pi_{D_2}(x_1), \pi_{A_2(\pi_{D_2}(x_1)), B_2(\pi_{D_2}(x_1))}(x_1^{d+1})\right).$$

Now, by (4.6) and (4.7),

$$
\begin{aligned}
&\big(d\,(x_1, G_2)\big)^2 \leqslant \|x_1 - \pi_{G_2}(x_1)\|^2\\
&= \big\|\big((x_1^1, x_1^2, \dots, x_1^d) - \pi_{D_2}(x_1), x_1^{d+1} - \pi_{A_2(\pi_{D_2}(x_1)),B_2(\pi_{D_2}(x_1))}(x_1^{d+1})\big)\big\|^2\\
&= \|(x_1^1, x_1^2, \dots, x_1^d) - \pi_{D_2}(x_1)\|^2 + \|x_1^{d+1} - \pi_{A_2(\pi_{D_2}(x_1)),B_2(\pi_{D_2}(x_1))}(x_1^{d+1})\|^2\\
&\leqslant \sum_{k=1}^{d} (|a_1^k - a_2^k| \vee |b_1^k - b_2^k|)^2\\
&\quad + \Big(\big|A_1(x_1^1, \dots, x_1^d) - A_2\big(\pi_{D_2}(x_1)\big)\big| \vee \big|B_1(x_1^1, \dots, x_1^d) - B_2\big(\pi_{D_2}(x_1)\big)\big|\Big)^2\\
&\leqslant d \cdot \max_{1\leqslant k\leqslant d} (|a_1^k - a_2^k| \vee |b_1^k - b_2^k|)^2\\
&\quad + \big(|\bar{A}_1(x_1^1, \dots, x_1^d) - \bar{A}_2(x_1^1, \dots, x_1^d)| \vee |\bar{B}_1(x_1^1, \dots, x_1^d) - \bar{B}_2(x_1^1, \dots, x_1^d)|\big)^2\\
&\leqslant (d+1)\big(d_M(G_1, G_2)\big)^2.
\end{aligned}
$$

This completes the proof of (4.4).

We now move on to (4.5). Let $1 \leqslant k \leqslant d$ and suppose that $a_1^k \leqslant a_2^k$. Let $a_1 = (a_1^1, \dots, a_1^d)$. Since $a_2^k \leqslant x^k$ for every $x \in G_2$, we have

$$
\begin{aligned}
|a_2^k - a_1^k| = a_2^k - a_1^k &\leqslant \inf_{x\in G_2} |x^k - a_1^k| \leqslant \inf_{x\in G_2} \big\|x - \big(a_1, A_1(a_1)\big)\big\|\\
&\leqslant d_H\Big(\big(a_1, A_1(a_1)\big), G_2\Big) \leqslant \sup_{x\in G_1} d\,(x, G_2) \leqslant d_H(G_1, G_2).
\end{aligned}
$$

If $a_1^k \geqslant a_2^k$ we proceed analogously. We can show that $|b_2^k - b_1^k| \leqslant d_H\,(G_1, G_2)$ in the same way. Thus we have $\max_{1\leqslant k\leqslant d}[|a_1^k - a_2^k| \vee |b_1^k - b_2^k|] \leqslant d_H\,(G_1, G_2)$.

Next we will show that

$$\|\bar{A}_1 - \bar{A}_2\| \leqslant \big(K + \sqrt{K^2+1}\big) \cdot d_H(G_1, G_2).$$

Note first that

$$\|\bar{A}_1 - \bar{A}_2\| = \inf_{x\in D_1\cup D_2} |\bar{A}_1(x) - \bar{A}_2(x)|.$$

Thus we need to show that for every $\big(x_0^1, x_0^2, \dots, x_0^d\big) \in D_1 \cup D_2$ we have

(4.8)
$$|\bar{A}_1(x_0^1, x_0^2, \dots, x_0^d) - \bar{A}_2(x_0^1, x_0^2, \dots, x_0^d)| \leqslant \big(K + \sqrt{K^2+1}\big) \cdot d_H(G_1, G_2).$$

Since (4.8) holds trivially when $\bar{A}_2(x_0^1, x_0^2, \dots, x_0^d) = \bar{A}_1(x_0^1, x_0^2, \dots, x_0^d)$, by a standard symmetry argument, it suffices to prove (4.8) when

$$\bar{A}_2(x_0^1, x_0^2, \dots, x_0^d) < \bar{A}_1(x_0^1, x_0^2, \dots, x_0^d). \tag{4.9}$$

We assume first that $(x_0^1,\dots,x_0^d)\in D_2$. Then, $\big((x_0^1,\dots,x_0^d),\bar A_2(x_0^1,\dots,x_0^d)\big)\in G_2\diagdown G_1$, and so we can find $x_1\in G_1$ such that

$$\begin{aligned} d\Big(\big((x_0^1,x_0^2,\dots,x_0^d),\bar A_2(x_0^1,x_0^2,\dots,x_0^d)\big),G_1\Big)\\ &= \inf_{x\in G_1}\big\|x-\big((x_0^1,x_0^2,\dots,x_0^d),A_2(x_0^1,x_0^2,\dots,x_0^d)\big)\big\|\\ &= \big\|x_1-\big((x_0^1,x_0^2,\dots,x_0^d),A_2(x_0^1,x_0^2,\dots,x_0^d)\big)\big\|. \end{aligned}$$

Then $x_1^{d+1}=A_1(x_1^1,x_1^2,\dots,x_1^d)$, and using the Cauchy–Schwarz inequality, we get

$$\begin{aligned} &|\bar A_1(x_0^1,\dots,x_0^d)-\bar A_2(x_0^1,\dots,x_0^d)|\\ &\leqslant |\bar A_1(x_0^1,\dots,x_0^d)-A_1(x_1^1,\dots,x_1^d)|+|A_1(x_1^1,\dots,x_1^d)-A_2(x_0^1,\dots,x_0^d)|\\ &\leqslant K\|(x_0^1,\dots,x_0^d)-(x_1^1,\dots,x_1^d)\|+|A_1(x_1^1,\dots,x_1^d)-A_2(x_0^1,\dots,x_0^d)|\\ &=\big\langle (K,1),\big(\|(x_0^1,\dots,x_0^d)-(x_1^1,\dots,x_1^d)\|,|A_1(x_1^1,\dots,x_1^d)-A_2(x_0^1,\dots,x_0^d)|\big)\big\rangle\\ &\leqslant \sqrt{K^2+1}\cdot\big\|\big((x_0^1,\dots,x_0^d)-(x_1^1,\dots,x_1^d),A_1(x_1^1,\dots,x_1^d)-A_2(x_0^1,\dots,x_0^d)\big)\big\|\\ &=\sqrt{K^2+1}\cdot\big\|\big((x_0^1,\dots,x_0^d,A_2(x_0^1,\dots,x_0^d))-(x_1^1,\dots,x_1^d,A_1(x_1^1,\dots,x_1^d)\big)\big)\big\|\\ &=\sqrt{K^2+1}\cdot d\Big(\big(x_0^1,\dots,x_0^d,A_2(x_0^1,\dots,x_0^d)\big),G_1\Big)\leqslant\sqrt{K^2+1}\cdot d_H(G_1,G_2). \end{aligned}$$

In particular, since $\pi_{D_2}(x_0^1,x_0^2,\dots,x_0^d)\in D_2$, we have

$$\begin{aligned} \big|\bar A_1\big(\pi_{D_2}(x_0^1,x_0^2,\dots,x_0^d)\big)-\bar A_2\big(\pi_{D_2}(x_0^1,x_0^2,\dots,x_0^d)\big)\big|\\ \leqslant\sqrt{K^2+1}\cdot d_H(G_1,G_2). \end{aligned}$$

Suppose now that $(x_0^1,x_0^2,\dots,x_0^d)\in D_1\diagdown D_2$. Then

$$\begin{aligned} &|\bar A_1(x_0^1,x_0^2,\dots,x_0^d)-\bar A_2(x_0^1,x_0^2,\dots,x_0^d)|\\ &\leqslant \big|A_1(x_0^1,x_0^2,\dots,x_0^d)-\bar A_1\big(\pi_{D_2}(x_0^1,x_0^2,\dots,x_0^d)\big)\big|\\ &\quad+\big|\bar A_1\big(\pi_{D_2}(x_0^1,x_0^2,\dots,x_0^d)\big)-\bar A_2(x_0^1,x_0^2,\dots,x_0^d)\big|\\ &\leqslant K\|(x_0^1,x_0^2,\dots,x_0^d)-\pi_{D_2}(x_0^1,x_0^2,\dots,x_0^d)\|\\ &\quad+\big|\bar A_1\big(\pi_{D_2}(x_0^1,x_0^2,\dots,x_0^d)\big)-A_2\big(\pi_{D_2}(x_0^1,x_0^2,\dots,x_0^d)\big)\big|\\ &\leqslant K\Big\|\big(x_0^1,\dots,x_0^d,A_1(x_0^1,\dots,x_0^d)\big)-\pi_{G_2}\Big(\big(x_0^1,\dots,x_0^d,A_1(x_0^1,\dots,x_0^d)\big)\Big)\Big\|\\ &\quad+\big|\bar A_1\big(\pi_{D_2}(x_0^1,\dots,x_0^d)\big)-A_2\big(\pi_{D_2}(x_0^1,\dots,x_0^d)\big)\big|\\ &\leqslant K\cdot d_H(G_1,G_2)+\sqrt{K^2+1}\cdot d_H(G_1,G_2)\\ &\leqslant\big(K+\sqrt{K^2+1}\big)\cdot d_H(G_1,G_2). \end{aligned}$$

Since x_0 is arbitrary, we conclude that

$$\|\bar{A}_1 - \bar{A}_2\| \leqslant \left(K + \sqrt{K^2+1}\right) \cdot d_H(G_1, G_2).$$

If $\bar{A}_1(x_0) \leqslant \bar{A}_2(x_0)$, the proof proceeds analogously. Similarly, we can prove that

$$\|\bar{B}_1 - \bar{B}_2\| \leqslant \left(K + \sqrt{K^2+1}\right) \cdot d_H(G_1, G_2),$$

and so the proof of (4.5) is complete. ■

PROPOSITION 4.2. $(\mathcal{G}, d_H)$ *is a complete separable metric space.*

Proof. It is well known that the space $\mathcal{H}(\mathbb{R}^n)$ of all non-empty compact subsets of $\mathbb{R}^n$ equipped with the Hausdorff metric d_H is a complete separable metric space. For the completeness see Theorem 2.4.4 in [10] or Theorem 7.1 of Chapter II in [1]. Since $\mathcal{G}$ is a closed subset of $\mathcal{H}(\mathbb{R}^n)$, it follows that $(\mathcal{G}, d_H)$ is also complete.

Let $\mathcal{R}$ be a countable dense subset of the space of real-valued continuous functions of $n-1$ variables, $C\left(\mathbb{R}^{n-1}\right)$, and let $\mathcal{S}$ be a subset of $\mathcal{G}$ consisting of all strata $S([a^1, b^1] \times \ldots \times [a^{n-1}, b^{n-1}], [A, B])$ such that $a^1, a^2, \ldots, a^d$ and $b^1, b^2, \ldots, b^d$ are rational numbers and A, B are restrictions of functions from $\mathcal{R}$ to $[a^1, b^1] \times \ldots \times [a^{n-1}, b^{n-1}]$. Then $\mathcal{S}$ is countable. It is also easy to verify that $\mathcal{S}$ is dense in $(\mathcal{G}, d_M)$. Since d_H and d_M are equivalent on $\mathcal{G}$, $\mathcal{S}$ is also dense in $(\mathcal{G}, d_H)$. ■

COROLLARY 4.1. *For every* $K \geqslant 0$, $(\mathcal{G}_K, d_H)$ *is a complete separable metric space.*

Proof. It is easy to see that $\mathcal{G}_K$ is a closed subset of $(\mathcal{G}, d_M)$. Thus $(\mathcal{G}_K, d_M)$ and therefore also $(\mathcal{G}_K, d_H)$ is complete. The intersection of $\mathcal{S}$ from the proof of Proposition 4.2 with $\mathcal{G}_K$ is a countable dense set in $(\mathcal{G}_K, d_H)$. ■

Consider the space $\mathcal{G}_K^T$ of $\mathcal{G}_K$-valued càdlàg functions defined on $[0, T]$. We define two metrics d_M^T and d_H^T on $\mathcal{G}_K^T$ by

$$d_M^T(G_1, G_2) = \sup_{0 \leqslant t \leqslant T} d_M\big(G_1(t), G_2(t)\big),$$
$$d_H^T(G_1, G_2) = \sup_{0 \leqslant t \leqslant T} d_H\big(G_1(t), G_2(t)\big).$$

The following statement is a direct result of combining Corollary 4.1 with Remark 4.1. It will play a significant role in obtaining the best Lipschitz constant in the proof of Theorem 4.1.

COROLLARY 4.2. *For every* $\epsilon > 0$ *and every function* $G \in \mathcal{G}_K^T$ *there is* $G' \in \mathcal{G}_K^T \cap \mathcal{S}$ *such that* $d_M^T(G, G') < \epsilon$ *or, equivalently, such that* $d_H^T(G, G') < \epsilon$.

The next result will show that for any $\psi \in D([0,T],\mathbb{R}^n)$ the Skorokhod map $\Gamma_{(\cdot)}(\psi)$ is a Lipschitz continuous function from $\mathcal{G}_K^T$ to $D([0,T],\mathbb{R}^n)$.

LEMMA 4.2. *Let $T>0$ and let $\psi \in D([0,T],\mathbb{R}^{d+1})$. For any $G_1, G_2 \in \mathcal{G}_K^T$,*

$$\|\Gamma_{G_1}(\psi)-\Gamma_{G_2}(\psi)\|_T \leqslant \sqrt{9d+9K^2d+6K\sqrt{d}+1}\cdot d_T(G_1,G_2). \tag{4.10}$$

Proof. We can assume $G_{1,t}=S([\alpha^1_{1,t},\beta^1_{1,t}]\times\ldots\times[\alpha^d_{1,t},\beta^d_{1,t}],[A_{1,t},B_{1,t}])$ and $G_{2,t}=S([\alpha^1_{2,t},\beta^1_{2,t}]\times\ldots\times[\alpha^d_{2,t},\beta^d_{2,t}],[A_{2,t},B_{2,t}])$ for any $0\leqslant t\leqslant T$. Then

$$\begin{aligned}\|\Gamma_{G_1}(\psi)-\Gamma_{G_2}(\psi)\|_T^2 = \sup_{0\leqslant t\leqslant T}\Big(\sum_{i=1}^d |\Xi_{\alpha_1^i,\beta_1^i}(\psi^i)(t)-\Xi_{\alpha_2^i,\beta_2^i}(\psi^i)(t)|^2 \\ + |\Xi_{\alpha_1^{d+1},\beta_1^{d+1}}(\psi^{d+1})(t)-\Xi_{\alpha_2^{d+1},\beta_2^{d+1}}(\psi^{d+1})(t)|^2\Big),\end{aligned}$$

where $\alpha_1^{d+1},\beta_1^{d+1},\alpha_2^{d+1},\beta_2^{d+1}$ are as described in equations (2.9) and (2.10).

By Theorem 3.5 of [21], for each $i=1,2,\ldots,d$ and for every $0\leqslant t\leqslant T$,

$$\begin{aligned}|\Xi_{\alpha_1^i,\beta_1^i}(\psi^i)(t)-\Xi_{\alpha_2^i,\beta_2^i}(\psi^i)(t)| &= |\Gamma_{\alpha_1^i,\beta_1^i}(\psi^i)(t)-\Gamma_{\alpha_2^i,\beta_2^i}(\psi^i)(t)| \\ &\leqslant 3\cdot(\|\alpha_1^i-\alpha_2^i\|\vee\|\beta_1^i-\beta_2^i\|),\end{aligned}\tag{4.11}$$

and

$$\begin{aligned}&|\Xi_{\alpha_1^{d+1},\beta_1^{d+1}}(\psi^{d+1})(t)-\Xi_{\alpha_2^{d+1},\beta_2^{d+1}}(\psi^{d+1})(t)| \\ &\qquad = |\Gamma_{\alpha_1^{d+1},\beta_1^{d+1}}(\psi^{d+1})(t)-\Gamma_{\alpha_2^{d+1},\beta_2^{d+1}}(\psi^{d+1})(t)| \\ &\qquad \leqslant 3\cdot(\|\alpha_1^{d+1}-\alpha_2^{d+1}\|\vee\|\beta_1^{d+1}-\beta_2^{d+1}\|),\end{aligned}\tag{4.12}$$

where, by (2.9),

$$\begin{aligned}&\|\alpha_1^{d+1}-\alpha_2^{d+1}\| = \|A_1(\psi^1-\Xi_{\alpha_1^1,\beta_1^1},\psi^2-\Xi_{\alpha_1^2,\beta_1^2},\ldots,\psi^d-\Xi_{\alpha_1^d,\beta_1^d}) \\ &\qquad - A_2(\psi^1-\Xi_{\alpha_2^1,\beta_2^1},\psi^2-\Xi_{\alpha_2^2,\beta_2^2},\ldots,\psi^d-\Xi_{\alpha_2^d,\beta_2^d})\| \\ &= \sup_{1\leqslant t\leqslant T}\big|A_{1,t}\big(\psi^1(t)-\Xi_{\alpha_1^1,\beta_1^1}(t),\psi^2(t)-\Xi_{\alpha_1^2,\beta_1^2}(t),\ldots,\psi^d(t)-\Xi_{\alpha_1^d,\beta_1^d}(t)\big) \\ &\qquad - A_{2,t}\big(\psi^1(t)-\Xi_{\alpha_2^1,\beta_2^1}(t),\psi^2(t)-\Xi_{\alpha_2^2,\beta_2^2}(t),\ldots,\psi^d(t)-\Xi_{\alpha_2^d,\beta_2^d}(t)\big)\big| \\ &\leqslant \sup_{1\leqslant t\leqslant T}\big|A_{1,t}\big(\psi^1(t)-\Xi_{\alpha_1^1,\beta_1^1}(t),\psi^2(t)-\Xi_{\alpha_1^2,\beta_1^2}(t),\ldots,\psi^d(t)-\Xi_{\alpha_1^d,\beta_1^d}(t)\big) \\ &\qquad - \bar{A}_{1,t}\big(\psi^1(t)-\Xi_{\alpha_2^1,\beta_2^1}(t),\psi^2(t)-\Xi_{\alpha_2^2,\beta_2^2}(t),\ldots,\psi^d(t)-\Xi_{\alpha_2^d,\beta_2^d}(t)\big)\big| \\ &+ \sup_{1\leqslant t\leqslant T}\big|\bar{A}_{1,t}\big(\psi^1(t)-\Xi_{\alpha_2^1,\beta_2^1}(t),\psi^2(t)-\Xi_{\alpha_2^2,\beta_2^2}(t),\ldots,\psi^d(t)-\Xi_{\alpha_2^d,\beta_2^d}(t)\big) \\ &\qquad - A_{2,t}\big(\psi^1(t)-\Xi_{\alpha_2^1,\beta_2^1}(t),\psi^2(t)-\Xi_{\alpha_2^2,\beta_2^2}(t),\ldots,\psi^d(t)-\Xi_{\alpha_2^d,\beta_2^d}(t)\big)\big|\end{aligned}$$

and hence

$$
\begin{aligned}
&\|\alpha_1^{d+1}-\alpha_2^{d+1}\| \\
&\leqslant \sup_{1\leqslant t\leqslant T} C_t^{A_1} \sup_{1\leqslant t\leqslant T} \left\|\left(\Xi_{\alpha_1^1,\beta_1^1}(t)-\Xi_{\alpha_2^1,\beta_2^1}(t),\ldots,\Xi_{\alpha_1^d,\beta_1^d}(t)-\Xi_{\alpha_2^d,\beta_2^d}(t)\right)\right\| \\
&\quad + \sup_{1\leqslant t\leqslant T}\|A_{1,t}-A_{2,t}\|.
\end{aligned}
$$

Thus, by (4.11),

$$
\begin{aligned}
\text{(4.13)}\quad &\|\alpha_1^{d+1}-\alpha_2^{d+1}\| \\
&\leqslant K_{G_1}\cdot 3\cdot\sqrt{\sum_{i=1}^{d}(\|\alpha_1^i-\alpha_2^i\|\vee\|\beta_1^i-\beta_2^i\|)^2}+\|A_1-A_2\| \\
&\leqslant 3K\sqrt{d}\cdot d_M^T(G_1,G_2)+d_M^T(G_1,G_2) \\
&\leqslant \left(3K\sqrt{d}+1\right)\cdot d_M^T(G_1,G_2).
\end{aligned}
$$

Similarly, using (4.12), we obtain

$$
\text{(4.14)}\qquad \|\beta_1^{d+1}-\beta_2^{d+1}\|\leqslant\left(3K\sqrt{d}+1\right)\cdot d_M^T(G_1,G_2).
$$

Therefore,

$$
\begin{aligned}
\|\Gamma_{G_1}(\psi)-\Gamma_{G_2}(\psi)\|_T^2 &\leqslant d\cdot\left(3d_M^T(G_1,G_2)\right)^2+\left(\left(3K\sqrt{d}+1\right)\cdot d_M^T(G_1,G_2)\right)^2 \\
&=\left(9d+9K^2d+6K\sqrt{d}+1\right)\left(d_M^T(G_1,G_2)\right)^2. \quad \blacksquare
\end{aligned}
$$

We are ready now to state the main result of this section showing the best Lipschitz constant for the type of the ESM under consideration in this study.

THEOREM 4.1. *Let $\{(G_t,d_t(\cdot)) : t\geqslant 0\}$ be an orthogonal evolving stratum constraining system with $G_t=S([\alpha_t^1,\beta_t^1]\times\ldots\times[\alpha_t^d,\beta_t^d]\times[A_t,B_t])\in\mathcal{G}_K$ for every $t\geqslant 0$. Then the ESM for the evolving ESP on $\{(G_t,d_t(\cdot)) : t\geqslant 0\}$ is Lipschitz continuous with constant*

$$
1+\sqrt{(d+1)(K^2+1)+2K\sqrt{d(K^2+1)}},
$$

i.e. for any $\psi_1,\psi_1\in D\big([0,\infty),\mathbb{R}^{d+1}\big)$,

$$
\begin{aligned}
\text{(4.15)}\quad &\|\Gamma_G(\psi_1)-\Gamma_G(\psi_2)\| \\
&\leqslant\left(1+\sqrt{(d+1)(K^2+1)+2K\sqrt{d(K^2+1)}}\right)\|\psi_1-\psi_2\|.
\end{aligned}
$$

This result is an extension of Theorem 3.6 in [21] into the vector-valued case. It should be noted that applying Theorem 3.6 in [21] coordinatewise would produce the Lipschitz property but with a constant that increases in direct proportion to the dimension of the space. Instead, through a significant effort and several intermediate results we will produce the best constant. Similarly to the one-dimensional case, the Lipschitz properties of the ESM follow from the Lipschitz properties of the constraining term.

PROPOSITION 4.3. *Let* $\{(G_t, d_t(\cdot)) : t \geqslant 0\}$ *be an orthogonal evolving stratum constraining system with* $G_t = S([\alpha_t^1, \beta_t^1] \times \ldots \times [\alpha_t^d, \beta_t^d] \times [A_t, B_t]) \in \mathcal{G}_K$ *for every* $t \geqslant 0$*. Then*

$$\begin{aligned}(4.16)\quad &\big\|\big(\Gamma_G(\psi_1) - \Gamma_G(\psi_2)\big) - (\psi_1 - \psi_2)\big\| \\ &\qquad\leqslant \Big(\sqrt{(d+1)(K^2+1) + 2K\sqrt{d(K^2+1)}}\,\Big)\|\psi_1 - \psi_2\|.\end{aligned}$$

The one-dimensional version of this result is (4.40) of [20]. In fact, we are going to use the same approach as in the proof of Proposition 4.6 in [20]. Namely, we shall first prove that the inequality holds for a dense family of functions taking a finite number of values, and then complete the proof by taking a limit. For the sake of convenience let $\Lambda = (\phi_2 - \phi_1) - (\psi_2 - \psi_1)$.

REMARK 4.2. *For any real numbers* a, b *and* K, *by the Cauchy–Schwarz inequality*, *we have*

$$(Ka + b)^2 \leqslant (K^2 + 1)(a^2 + b^2). \tag{4.17}$$

LEMMA 4.3. *Let* $G = ([\alpha^1, \beta^1] \times \ldots \times [\alpha^d, \beta^d] \times [A, B])$ *be a fixed stratum in* $\mathbb{R}^{d+1}$*. Then, for every* $x, y \in \mathbb{R}^{d+1}$,

$$\big|\big(\pi_G^{d+1}(x) - \pi_G^{d+1}(y)\big) - (x^{d+1} - y^{d+1})\big| \leqslant \sqrt{K_G^2 + 1} \cdot \|x - y\|. \tag{4.18}$$

P r o o f. We consider several cases with different positions of x and y relative to G.

C a s e A. If x and y lie between the graphs of $\bar{A}$ and $\bar{B}$, i.e.

$$\bar{A}(x^1, x^2, \ldots, x^d) \leqslant x^{d+1} \leqslant \bar{B}(x^1, x^2, \ldots, x^d)$$

and

$$\bar{A}(y^1, y^2, \ldots, y^d) \leqslant y^{d+1} \leqslant \bar{B}(y^1, y^2, \ldots, y^d),$$

then $\pi_G^{d+1}(x) = x^{d+1}$ and $\pi_G^{d+1}(y) = y^{d+1}$, and therefore

$$\pi_G^{d+1}(x^{d+1}) - \pi_G^{d+1}(y^{d+1}) - (x^{d+1} - y^{d+1}) = 0.$$

C a s e B. Both points lie below the graph of $\bar{A}$ or both points lie above the graph of $\bar{B}$.

If $x^{d+1} > \bar{B}(x^1, x^2, \ldots, x^d)$ and $y^{d+1} > \bar{B}(y^1, y^2, \ldots, y^d)$, then, using Remark 4.2, we have

$$
\begin{aligned}
&\left|\left(\pi_G^{d+1}(x) - \pi_G^{d+1}(y)\right) - (x^{d+1} - y^{d+1})\right| \\
&\qquad = \left|\left(\bar{B}(x^1, x^2, \ldots, x^d) - \bar{B}(y^1, y^2, \ldots, y^d)\right) - (x^{d+1} - y^{d+1})\right| \\
&\qquad \leqslant |\bar{B}(x^1, x^2, \ldots, x^d) - \bar{B}(y^1, y^2, \ldots, y^d)| + |x^{d+1} - y^{d+1}| \\
&\qquad \leqslant K_G \|(x^1, x^2, \ldots, x^d) - (y^1, y^2, \ldots, y^d)\| + |x^{d+1} - y^{d+1}| \\
&\qquad \leqslant \sqrt{K_G^2 + 1} \cdot \|x - y\|.
\end{aligned}
$$

C a s e C. One of the points lies below the graph of $\bar{A}$ and the other lies above the graph of $\bar{B}$.

If $y^{d+1} > \bar{B}(y^1, y^2, \ldots, y^d)$ and $x^{d+1} < \bar{A}(x^1, x^2, \ldots, x^d)$, then

$$
\begin{aligned}
&\left|\left(\pi_G^{d+1}(y) - \pi_G^{d+1}(x)\right) - (y^{d+1} - x^{d+1})\right| \\
&\qquad = \left|\left(\bar{B}(y^1, y^2, \ldots, y^d) - \bar{A}(x^1, x^2, \ldots, x^d)\right) - (y^{d+1} - x^{d+1})\right| \\
&\qquad = y^{d+1} - \bar{B}(y^1, y^2, \ldots, y^d) + \bar{A}(x^1, x^2, \ldots, x^d) - x^{d+1} \\
&\qquad \leqslant y^{d+1} - \bar{B}(y^1, y^2, \ldots, y^d) + \bar{B}(x^1, x^2, \ldots, x^d) - x^{d+1} \\
&\qquad \leqslant |\bar{B}(y^1, y^2, \ldots, y^d) - \bar{B}(x^1, x^2, \ldots, x^d)| + |y^{d+1} - x^{d+1}| \\
&\qquad \leqslant K_G \|(x^1, x^2, \ldots, x^d) - (y^1, y^2, \ldots, y^d)\| + |x^{d+1} - y^{d+1}| \\
&\qquad \leqslant \sqrt{K_G^2 + 1} \cdot \|x - y\|.
\end{aligned}
$$

C a s e D. Exactly one of the points lies between the graphs of $\bar{A}$ and $\bar{B}$. If $y^{d+1} > \bar{B}(y^1, y^2, \ldots, y^d)$ and $\bar{A}(x^1, x^2, \ldots, x^d) \leqslant x^{d+1} \leqslant \bar{B}(x^1, x^2, \ldots, x^d)$, then

$$
\begin{aligned}
&\quad \left|\left(\pi_G^{d+1}(y) - \pi_G^{d+1}(x)\right) - (y^{d+1} - x^{d+1})\right| \\
&= \left|\left(\bar{B}(y^1, \ldots, y^d) - x^{d+1}\right) - (y^{d+1} - x^{d+1})\right| = |\bar{B}(y^1, \ldots, y^d) - y^{d+1}| \\
&= y^{d+1} - \bar{B}(y^1, \ldots, y^d) \leqslant y^{d+1} - \bar{B}(y^1, \ldots, y^d) + \bar{B}(x^1, \ldots, x^d) - x^{d+1} \\
&\leqslant |\bar{B}(y^1, \ldots, y^d) - \bar{B}(x^1, \ldots, x^d)| + |y^{d+1} - x^{d+1}| \\
&\leqslant K_G \|(x^1, \ldots, x^d) - (y^1, \ldots, y^d)\| + |x^{d+1} - y^{d+1}| \leqslant \sqrt{K_G^2 + 1} \cdot \|x - y\|.
\end{aligned}
$$

If $y^{d+1} < \bar{A}(y^1, y^2, \ldots, y^d)$ and $\bar{A}(x^1, x^2, \ldots, x^d) \leqslant x^{d+1} \leqslant \bar{B}(x^1, x^2, \ldots, x^d)$ we complete the proof with analogous arguments. ∎

LEMMA 4.4. *Under the assumptions of Proposition* 4.3, *if* $\Gamma_G(\psi_1) = \phi_1$, $\Gamma_G(\psi_2) = \phi_2$, *and*

$$\frac{|\phi_2^{d+1}(t) - \phi_1^{d+1}(t)|}{\big\|\big(\phi_2^1(t), \phi_2^2(t), \ldots, \phi_2^d(t)\big) - \big(\phi_1^1(t), \phi_1^2(t), \ldots, \phi_1^d(t)\big)\big\|} \leqslant m,$$

then

$$|\Lambda^{d+1}(t)| \leqslant \big(m\sqrt{d} + \sqrt{m^2+1}\big)\|\psi_2 - \psi_1\|.$$

P r o o f. By Proposition 3.1,

$$\big\|\big(\Lambda^1(t), \Lambda^2(t), \ldots, \Lambda^d(t)\big)\big\| \leqslant \sqrt{d}\|\psi_2 - \psi_1\|,$$

hence

$$\begin{aligned}
&\big\|\big(\phi_2^1(t), \phi_2^2(t), \ldots, \phi_2^d(t)\big) - \big(\phi_1^1(t), \phi_1^2(t), \ldots, \phi_1^d(t)\big)\big\| \\
&\leqslant \sqrt{d}\|\psi_2 - \psi_1\| + \big\|\big(\psi_2^1(t), \psi_2^2(t), \ldots, \psi_2^d(t)\big) - \big(\psi_1^1(t), \psi_1^2(t), \ldots, \psi_1^d(t)\big)\big\|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|\Lambda^{d+1}(t)| &= \big|\big(\phi_2^{d+1}(t) - \phi_1^{d+1}(t)\big) - \big(\psi_2^{d+1}(t) - \psi_1^{d+1}(t)\big)\big| \\
&\leqslant |\phi_2^{d+1}(t) - \phi_1^{d+1}(t)| + |\psi_2^{d+1}(t) - \psi_1^{d+1}(t)| \\
&\leqslant m\big\|\big(\phi_2^1(t), \ldots, \phi_2^d(t)\big) - \big(\phi_1^1(t), \ldots, \phi_1^d(t)\big)\big\| + |\psi_2^{d+1}(t) - \psi_1^{d+1}(t)| \\
&\leqslant m\sqrt{d}\|\psi_2 - \psi_1\| + m\big\|\big(\psi_2^1(t), \ldots, \psi_2^d(t)\big) - \big(\psi_1^1(t), \ldots, \psi_1^d(t)\big)\big\| \\
&\quad + |\psi_2^{d+1}(t) - \psi_1^{d+1}(t)| \\
&\leqslant m\sqrt{d}\|\psi_2 - \psi_1\| + \sqrt{m^2+1}\|\psi_2(t) - \psi_1(t)\| \\
&= \big(m\sqrt{d} + \sqrt{m^2+1}\big)\|\psi_2 - \psi_1\|,
\end{aligned}$$

where the last inequality follows from Remark 4.2. ■

LEMMA 4.5. *Let* $\big\{\big(G_t, d_t(\cdot)\big) : 0 \leqslant t \leqslant T\big\}$ *be an orthogonal evolving stratum constraining system with* $G_t = S([\alpha_t^1, \beta_t^1] \times \ldots \times [\alpha_t^d, \beta_t^d] \times [A_t, B_t]) \in \mathcal{G}_K$ *for every* $0 \leqslant t \leqslant T$. *Then, for any* $\psi_1, \psi_2 \in D\big([0,T], \mathbb{R}^n\big)$ *and for every* $0 \leqslant t \leqslant T$, *we have*

$$\begin{aligned}
(4.19)\quad &\big|\big(\phi_2^{d+1}(t) - \phi_1^{d+1}(t)\big) - \big(\psi_2^{d+1}(t) - \psi_1^{d+1}(t)\big)\big| \\
&\qquad\qquad \leqslant \big(K\sqrt{d} + \sqrt{K^2+1}\big)\|\psi_1 - \psi_2\|,
\end{aligned}$$

where $\phi_i = \Gamma_G(\psi_i)$ *for* $i = 1, 2$.

P r o o f. We assume first that ψ_1, ψ_2 and G take only a finite number of values. Then we can find numbers $0 = t_0 < t_1 < t_2 < \ldots < t_m \leqslant T$ such that

$$\begin{aligned}
\psi_1 &= \sum_{i=0}^{m-1} \psi_1(t_i)\, I_{[t_i,t_{i+1})} + \psi_1(t_m)\, I_{[t_m,T]},\\
\psi_2 &= \sum_{i=0}^{m-1} \psi_2(t_i)\, I_{[t_i,t_{i+1})} + \psi_2(t_m)\, I_{[t_m,T]},\\
G &= \bigcup_{i=0}^{m-1} G_{t_i} I_{[t_i,t_{i+1})} \cup G_{t_m} I_{[t_m,T]},
\end{aligned}$$

where I_E is the indicator function of the interval E. We shall prove by induction that, for every $k = 0, 1, \ldots, m$,

$$\begin{aligned}
(4.20)\quad &\left|\left(\phi_2^{d+1}(t_k) - \phi_1^{d+1}(t_k)\right) - \left(\psi_2^{d+1}(t_k) - \psi_1^{d+1}(t_k)\right)\right| \\
&\qquad\leqslant \left(K\sqrt{d} + \sqrt{K^2+1}\right)\|\psi_1 - \psi_2\|.
\end{aligned}$$

The initial step for $k = 0$ follows immediately from Lemma 4.3, since

$$\begin{aligned}
&\left|\left(\phi_2^{d+1}(t_0) - \phi_1^{d+1}(t_0)\right) - \left(\psi_2^{d+1}(t_0) - \psi_1^{d+1}(t_0)\right)\right| \\
&\qquad = \left|\left(\pi_{G_0}^{d+1}\big(\psi_2(0)\big) - \pi_{G_0}^{d+1}\big(\psi_2(0)\big)\right) - \left(\psi_2^{d+1}(0) - \psi_1^{d+1}(0)\right)\right|.
\end{aligned}$$

To prove the inductive step, first observe that for any $i = 1$ or 2 and $k = 0, 1, \ldots, m$ we have the following:

If $\bar{A}(t_{k+1}) \leqslant \phi_i^{d+1}(t_k) + \psi_i^{d+1}(t_{k+1}) - \psi_i^{d+1}(t_k) \leqslant \bar{B}(t_{k+1})$, then

$$(4.21)\qquad \phi_i^{d+1}(t_{k+1}) = \phi_i^{d+1}(t_k) + \psi_i^{d+1}(t_{k+1}) - \psi_i^{d+1}(t_k).$$

For every $t \geqslant 0$ we can partition $\mathbb{R}^{d+1}$ into three regions: S_t, N_t, C_t, where

$$\begin{aligned}
S_t &= \{(x^1,\ldots,x^d,x^{d+1}) :\ x^{d+1} < \bar{A}(x^1,\ldots,x^d)\},\\
N_t &= \{(x^1,\ldots,x^d,x^{d+1}) :\ x^{d+1} > \bar{B}(x^1,\ldots,x^d)\},\\
C_t &= \{(x^1,\ldots,x^d,x^{d+1}) :\ \bar{A}(x^1,\ldots,x^d) \leqslant x^{d+1} \leqslant \bar{B}(x^1,\ldots,x^d)\}.
\end{aligned}$$

We consider several cases as in the proof of Lemma 4.3.

C a s e C C. If $\phi_i(t_k) + \psi_i(t_{k+1}) - \psi_i(t_k) \in C_{t_{k+1}}$ for $i = 1, 2$, then, by (4.21) and the inductive assumption, we have

$$|\Lambda^{d+1}(t_{k+1})| = |\Lambda^{d+1}(t_k)| \leqslant \left(K\sqrt{d} + \sqrt{K^2+1}\right)\|\psi_1 - \psi_2\|.$$

C a s e C C$'$. Suppose now that $\phi_1(t_k) + \psi_1(t_{k+1}) - \psi_1(t_k) \in C_{t_{k+1}}$ and $\phi_2(t_k) + \psi_2(t_{k+1}) - \psi_2(t_k) \in C^c_{t_{k+1}} = S_{t_{k+1}} \cup N_{t_{k+1}}$ or *vice versa*. We will show the proof only when $\phi_1(t_k)+\psi_1(t_{k+1})-\psi_1(t_k)\in C_{t_{k+1}}$ and $\phi_2(t_k)+\psi_2(t_{k+1})-\psi_2(t_k)\in N_{t_{k+1}}$. If

$$
\begin{aligned}
(4.22)\quad |\phi_2^{d+1}(t_{k+1})-\phi_1^{d+1}(t_{k+1})| \leqslant K\big\|&\big(\phi_2^1(t_{k+1}),\phi_2^2(t_{k+1}),\ldots,\phi_2^d(t_{k+1})\big)\\
&-\big(\phi_1^1(t_{k+1}),\phi_1^2(t_{k+1}),\ldots,\phi_1^d(t_{k+1})\big)\big\|,
\end{aligned}
$$

then (4.19) holds for $t=t_{k+1}$ by Lemma 4.4. If (4.22) does not hold, then we must have

$$
(4.23)\qquad \phi_2^{d+1}(t_{k+1})-\phi_1^{d+1}(t_{k+1})>0.
$$

Indeed, suppose that $\phi_2^{d+1}(t_{k+1})-\phi_1^{d+1}(t_{k+1})\leqslant 0$. Then

$$
\begin{aligned}
0&\geqslant \phi_2^{d+1}(t_{k+1})-\phi_1^{d+1}(t_{k+1})\geqslant \phi_2^{d+1}(t_{k+1})-\bar{B}\big(\phi_1^1(t_{k+1}),\ldots,\phi_1^d(t_{k+1})\big)\\
&=\bar{B}\big(\phi_2^1(t_{k+1}),\ldots,\phi_2^d(t_{k+1})\big)-\bar{B}\big(\phi_1^1(t_{k+1}),\ldots,\phi_1^d(t_{k+1})\big)\\
&>-K\big\|\big(\phi_2^1(t_{k+1}),\ldots,\phi_2^d(t_{k+1})\big)-\big(\phi_1^1(t_{k+1}),\ldots,\phi_1^d(t_{k+1})\big)\big\|,
\end{aligned}
$$

and so (4.22) does hold for $t=t_{k+1}$ and we have a contradiction.

First suppose that $\psi_2^{d+1}(t_{k+1})-\psi_1^{d+1}(t_{k+1})>\phi_2^{d+1}(t_{k+1})-\phi_1^{d+1}(t_{k+1})$. Then

$$
\begin{aligned}
|\Lambda^{d+1}(t_{k+1})|&=-\Lambda^{d+1}(t_{k+1})\\
&=\big(\psi_2^{d+1}(t_{k+1})-\psi_1^{d+1}(t_{k+1})\big)-\big(\phi_2^{d+1}(t_{k+1})-\phi_1^{d+1}(t_{k+1})\big)\\
&\leqslant \psi_2^{d+1}(t_{k+1})-\psi_1^{d+1}(t_{k+1})\leqslant\|\psi_1-\psi_2\|,
\end{aligned}
$$

and so (4.19) holds.

Suppose now that $\Lambda^{d+1}(t_{k+1})\geqslant 0$. Then

$$
\begin{aligned}
0&\leqslant\Lambda^{d+1}(t_{k+1})=\big(\phi_2^{d+1}(t_{k+1})-\phi_1^{d+1}(t_{k+1})\big)-\big(\psi_2^{d+1}(t_{k+1})-\psi_1^{d+1}(t_{k+1})\big)\\
&\leqslant\big(\phi_2^{d+1}(t_k)+\psi_2^{d+1}(t_{k+1})-\psi_2^{d+1}(t_k)\big)\\
&\quad-\big(\phi_1^{d+1}(t_k)+\psi_1^{d+1}(t_{k+1})-\psi_1^{d+1}(t_k)\big)-\big(\psi_2^{d+1}(t_{k+1})-\psi_1^{d+1}(t_{k+1})\big)\\
&=\big(\phi_2^{d+1}(t_k)-\phi_1^{d+1}(t_k)\big)-\big(\psi_2^{d+1}(t_k)-\psi_1^{d+1}(t_k)\big)=\Lambda^{d+1}(t_k),
\end{aligned}
$$

and so (4.19) holds by the inductive assumption.

Case C′C′. Suppose now that $\phi_i(t_k)+\psi_i(t_{k+1})-\psi_i(t_k)\notin C_{t_{k+1}}$ for $i=1,2$. That means that either both points are in $S_{t_{k+1}}$ or both are in $N_{t_{k+1}}$, or one is in $S_{t_{k+1}}$ and the other is in $N_{t_{k+1}}$.

Suppose first that both points are in $S_{t_{k+1}}$ or both are in $N_{t_{k+1}}$. Then both $\phi_1(t_{k+1})$ and $\phi_2(t_{k+1})$ lie on the graph of $\bar{A}$ or both lie on the graph of $\bar{B}$. In either case (4.22) holds, and so (4.19) follows by Lemma 4.4.

Suppose that $\phi_1^{d+1}(t_k) + \psi_1^{d+1}(t_{k+1}) - \psi_1^{d+1}(t_k) \in S_{t_{k+1}}$ and $\phi_2^{d+1}(t_k) + \psi_2^{d+1}(t_{k+1}) - \psi_2^{d+1}(t_k) \in N_{t_{k+1}}$. If (4.22) holds, then (4.19) holds for $t = t_{k+1}$ by Lemma 4.4. If (4.22) does not hold, then, as in case CC′, (4.23) holds and we consider two situations: $\Lambda^{d+1}(t_{k+1}) < 0$ and $\Lambda^{d+1}(t_{k+1}) \geqslant 0$.

Suppose first that $\Lambda^{d+1}(t_{k+1}) < 0$. Then

$$\begin{aligned}|\Lambda^{d+1}(t_{k+1})| &= -\Lambda^{d+1}(t_{k+1})\\ &= \big(\psi_2^{d+1}(t_{k+1}) - \psi_1^{d+1}(t_{k+1})\big) - \big(\phi_2^{d+1}(t_{k+1}) - \phi_1^{d+1}(t_{k+1})\big)\\ &\leqslant \psi_2^{d+1}(t_{k+1}) - \psi_1^{d+1}(t_{k+1}) \leqslant \|\psi_1 - \psi_2\|.\end{aligned}$$

If $\Lambda^{d+1}(t_{k+1}) \geqslant 0$, then

$$\begin{aligned}0 &\leqslant \Lambda^{d+1}(t_{k+1}) = \big(\phi_2^{d+1}(t_{k+1}) - \phi_1^{d+1}(t_{k+1})\big) - \big(\psi_2^{d+1}(t_{k+1}) - \psi_1^{d+1}(t_{k+1})\big)\\ &\leqslant \big(\phi_2^{d+1}(t_k) + \psi_2^{d+1}(t_{k+1}) - \psi_2^{d+1}(t_k)\big)\\ &\quad - \big(\phi_1^{d+1}(t_k) + \psi_1^{d+1}(t_{k+1}) - \psi_1^{d+1}(t_k)\big) - \big(\psi_2^{d+1}(t_{k+1}) - \psi_1^{d+1}(t_{k+1})\big)\\ &= \big(\phi_2^{d+1}(t_k) - \phi_1^{d+1}(t_k)\big) - \big(\psi_2^{d+1}(t_k) - \psi_1^{d+1}(t_k)\big) = \Lambda^{d+1}(t_k),\end{aligned}$$

and so (4.19) holds by the inductive assumption.

At this point we have proven (4.19) when ψ_1, ψ_2 and G take only a finite number of values. Suppose now that $G \in \mathcal{G}_K^T$ is arbitrary and let $\epsilon > 0$. By Corollary 4.2 there is $G' \in \mathcal{G}_K^T \cap \mathcal{S}$ such that $d_H^T(G, G') < \epsilon$. Therefore, by (4.12), (4.13), (4.14), and by (4.19) applied to G', for every $0 \leqslant t \leqslant T$,

$$\begin{aligned}&\big|\big(\Gamma_G^{d+1}(\psi_2)(t) - \Gamma_G^{d+1}(\psi_1)(t)\big) - \big(\psi_2^{d+1}(t) - \psi_1^{d+1}(t)\big)\big|\\ &\quad\leqslant |\Gamma_G^{d+1}(\psi_2)(t) - \Gamma_{G'}^{d+1}(\psi_2)(t)| + |\Gamma_G^{d+1}(\psi_1)(t) - \Gamma_{G'}^{d+1}(\psi_1)(t)|\\ &\qquad + \big|\big(\Gamma_{G'}^{d+1}(\psi_2)(t) - \Gamma_{G'}^{d+1}(\psi_1)(t)\big) - \big(\psi_2^{d+1}(t) - \psi_1^{d+1}(t)\big)\big|\\ &\quad\leqslant 2 \cdot 3 \cdot \big(3K\sqrt{d} + 1\big) \cdot d_M^T(G, G') + \big(K\sqrt{d} + \sqrt{K^2+1}\big)\|\psi_1 - \psi_2\|\\ &\quad= 6 \cdot \big(3K\sqrt{d} + 1\big) \cdot \epsilon + \big(K\sqrt{d} + \sqrt{K^2+1}\big)\|\psi_1 - \psi_2\|.\end{aligned}$$

Since ϵ is arbitrary, it follows that (4.19) holds for any G.

Finally, we can extend (4.19) to arbitrary $\psi_1, \psi_2 \in D([0,T], \mathbb{R}^n)$. By Remark 4.1, we can find sequences $\psi_{1,n}, \psi_{2,n} \in S([0,T], \mathbb{R}^n)$ such that $\lim_{n\to\infty} \psi_{1,n} = \psi_1$ and $\lim_{n\to\infty} \psi_{2,n} = \psi_2$ uniformly. We have already proven that (4.19) holds for $\psi_{1,n}, \psi_{2,n}$ for every n. By Lemma 4.1, $\lim_{n\to\infty} \Gamma_G(\psi_{1,n}) = \Gamma_G(\psi_1)$ and $\lim_{n\to\infty} \Gamma_G(\psi_{2,n}) = \Gamma_G(\psi_2)$ uniformly. Therefore,

$$\begin{aligned}&\big|\big(\phi_2^{d+1}(t) - \phi_1^{d+1}(t)\big) - \big(\psi_2^{d+1}(t) - \psi_1^{d+1}(t)\big)\big|\\ &= \lim_{n\to\infty} \big|\big(\Gamma_G^{d+1}(\psi_{2,n})(t) - \Gamma_G^{d+1}(\psi_{1,n})(t)\big) - \big(\psi_{2,n}^{d+1}(t) - \psi_{1,n}^{d+1}(t)\big)\big|\\ &\leqslant \big(K\sqrt{d} + \sqrt{K^2+1}\big) \lim_{n\to\infty} \|\psi_{1,n} - \psi_{2,n}\| = \big(K\sqrt{d} + \sqrt{K^2+1}\big)\|\psi_1 - \psi_2\|,\end{aligned}$$

which completes the proof of the lemma. ■

P r o o f o f P r o p o s i t i o n 4.3. Since (4.19) in Lemma 4.5 holds for every T, it also holds for $\psi_1, \psi_2 \in D\big([0,\infty), \mathbb{R}^n\big)$ and $G \in \mathcal{G}_K$. Therefore, by Theorem 2.1, applying (3.1) coordinatewise we obtain

$$\begin{aligned}
&\big\|\big(\phi_2(t) - \phi_1(t)\big) - \big(\psi_2(t) - \psi_1(t)\big)\big\|^2 = \|\Lambda(t)\|^2 \\
&= \big\|\big(\Lambda^1(t), \ldots, \Lambda^d(t)\big)\big\|^2 + |\Lambda^{d+1}(t)|^2 \leqslant d\|(\psi_2^1 - \psi_1^1, \ldots, \psi_2^d - \psi_1^d)\|^2 \\
&\quad + \big(K\sqrt{d} + \sqrt{K^2+1}\big)^2 \|\psi_2 - \psi_1\|^2 \\
&= \big(d + K^2 d + K^2 + 1 + 2K\sqrt{d(K^2+1)}\big)\|\psi_2 - \psi_1\|^2 \\
&= \big((d+1)(K^2+1) + 2K\sqrt{d(K^2+1)}\big)\|\psi_2 - \psi_1\|^2. \quad \blacksquare
\end{aligned}$$

P r o o f o f T h e o r e m 4.1. By Proposition 4.3,

$$\begin{aligned}
\big\|\big(\phi_2(t) - \phi_1(t)\big)\big\| &\leqslant \big\|\big(\psi_2(t) - \psi_1(t)\big)\big\| + \big\|\big(\phi_2(t) - \phi_1(t)\big) - \big(\psi_2(t) - \psi_1(t)\big)\big\| \\
&\leqslant \Big(1 + \sqrt{(d+1)(K^2+1) + 2K\sqrt{d(K^2+1)}}\Big)\|\psi_2 - \psi_1\|,
\end{aligned}$$

which proves the assertion. $\blacksquare$

Finally, we construct an example showing that the Lipschitz constant of Theorem 4.1 is tight.

EXAMPLE 4.1. Let b and h be positive real numbers and let d be a positive integer. Define functions $A(x_1, x_2, \ldots, x_d) = c(x_1 + x_2 + \ldots + x_d)/\sqrt{d}$ and $B(x_1, x_2, \ldots, x_d) = c(x_1 + x_2 + \ldots + x_d)/\sqrt{d} + h$ and consider the orthogonal constant stratum constraining system $\big\{\big(G_t, d_t(\cdot)\big) : t \geqslant 0\big\}$ with $G_t = S([0,b] \times \ldots \times [0,b] \times [A,B])$. Let $\psi_1 = \mathbf{0}$ and let

$$\begin{aligned}
\psi_2 = &-\sum_{i=1}^{d-1} \mathbf{e}_i I_{[i-1,i)} + \frac{1}{\sqrt{c^2+1}}\Big(\frac{c}{\sqrt{d}}\sum_{i=1}^{d} \mathbf{e}_i - \mathbf{e}_{d+1}\Big) I_{[d,d+1)} + \\
&\frac{1}{\sqrt{(d+1)(c^2+1) + 2c\sqrt{d(c^2+1)}}}\big(\sum_{i=1}^{d} \mathbf{e}_i + (c\sqrt{d} + \sqrt{c^2+1})\mathbf{e}_{d+1}\big) I_{[d+1,\infty)}.
\end{aligned}$$

Then the best Lipschitz constants for A and B are $C_t^A = c$ and $C_t^B = c$, respectively, and therefore $K_G = c$ as well. Furthermore,

$$\|\phi_2 - \phi_1\| = \Big(\sqrt{(d+1)(c^2+1) + 2c\sqrt{d(c^2+1)}} + 1\Big)\|\psi_2 - \psi_1\|, \tag{4.24}$$

where $\phi_1 = \Gamma_G(\psi_1)$ and $\phi_2 = \Gamma_G(\psi_2)$, i.e. the Lipschitz constant of the ESM in Theorem 4.1 is tight.

P r o o f. Since $\psi_1(t) = \mathbf{0} \in G_t$ for every $t \geqslant 0$, we have $\phi_1(t) = \psi_1(t) = \mathbf{0}$. As for $\phi_2 = \Gamma_G(\psi_2)$ we will have to determine its values at each jump point of ψ. It follows that

$$
\begin{aligned}
&\psi_2(0) = -\mathbf{e}_1 = (-1,0,0,\ldots,0), \text{ hence } \phi_2(0) = \pi_{G_0}\big(\psi_2(0)\big) = 0,\\
&\psi_2(1) = -\mathbf{e}_2 = (0,-1,0,\ldots,0), \text{ hence } \phi_2(1) = \pi_{G_1}\big(\phi_2(0)+\psi_2(1)-\psi_2(0)\big)\\
&\qquad\qquad = \pi_G(1,-1,0,\ldots,0) = (1,0,0,\ldots,0),\\
&\psi_2(2) = -\mathbf{e}_3 = (0,0,-1,0,\ldots,0), \text{ hence}\\
&\qquad\qquad \phi_2(2) = \pi_{G_2}\big(\phi_2(1)+\psi_2(2)-\psi_2(1)\big)\\
&\qquad\qquad = \pi_G(1,1,-1,0,\ldots,0) = (1,1,0,\ldots,0),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\psi_2(d-1) = -\mathbf{e}_d = (0,\ldots,0,-1,0), \text{ hence}\\
&\qquad\qquad \phi_2(d-1) = \pi_{G_{d-1}}\big(\phi_2(d-2)+\psi_2(d-1)-\psi_2(d-2)\big)\\
&\qquad\qquad = \pi_G(1,\ldots,1,-1,0) = (1,\ldots,1,0,0).
\end{aligned}
$$

Since

$$
\begin{aligned}
\psi_2(d) &= \left(\frac{c}{\sqrt{d(c^2+1)}},\ldots,\frac{c}{\sqrt{d(c^2+1)}},\frac{-1}{\sqrt{c^2+1}}\right)\\
&= \frac{c}{\sqrt{d(c^2+1)}}\left(1,\ldots,1,-\frac{\sqrt{d}}{c}\right),
\end{aligned}
$$

we get

$$
\begin{aligned}
\phi_2(d) &= \pi_{G_d}\big(\phi_2(d-1)+\psi_2(d)-\psi_2(d-1)\big)\\
&= \pi_G\left(\left(1+\frac{c}{\sqrt{d(c^2+1)}}\right)(1,1,\ldots,1,0) - \frac{1}{\sqrt{c^2+1}}(0,0,\ldots,0,1)\right)\\
&= \left(1+\frac{c}{\sqrt{d(c^2+1)}}\right)\big(1,1,\ldots,1,c\sqrt{d}\big).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&\psi_2(d+1) =\\
&= \frac{1}{\sqrt{(d+1)(c^2+1)+2c\sqrt{d(c^2+1)}}}\Big(\sum_{i=1}^{d}\mathbf{e}_i + \big(c\sqrt{d}+\sqrt{c^2+1}\big)\mathbf{e}_{d+1}\Big)\\
&= \frac{\big(1,\ldots,1,c\sqrt{d}+\sqrt{c^2+1}\big)}{\sqrt{(d+1)(c^2+1)+2c\sqrt{d(c^2+1)}}} = \frac{\big(1,\ldots,1,c\sqrt{d}+\sqrt{c^2+1}\big)}{\big\|\big(1,\ldots,1,c\sqrt{d}+\sqrt{c^2+1}\big)\big\|},
\end{aligned}
$$

hence

$$\phi_2(d)+\psi_2(d+1)-\psi_2(d)=\Big(1+\frac{c}{\sqrt{d(c^2+1)}}\Big)(1,1,\dots,1,c\sqrt{d})$$
$$+\frac{\big(1,1,\dots,1,c\sqrt{d}+\sqrt{c^2+1}\big)}{\big\|\big(1,1,\dots,1,c\sqrt{d}+\sqrt{c^2+1}\big)\big\|}-\frac{c}{\sqrt{d(c^2+1)}}\Big(1,1,\dots,1,-\frac{\sqrt{d}}{c}\Big)$$
$$=\big(1,1,\dots,1,c\sqrt{d}+\sqrt{c^2+1}\big)+\frac{\big(1,1,\dots,1,c\sqrt{d}+\sqrt{c^2+1}\big)}{\big\|\big(1,1,\dots,1,c\sqrt{d}+\sqrt{c^2+1}\big)\big\|}$$
$$=\frac{\sqrt{(d+1)(c^2+1)+2c\sqrt{d(c^2+1)}}+1}{\sqrt{(d+1)(c^2+1)+2c\sqrt{d(c^2+1)}}}\big(1,1,\dots,1,c\sqrt{d}+\sqrt{c^2+1}\big).$$

When b and h are large enough, $\phi_2(d)+\psi_2(d+1)-\psi_2(d)\in G$, and so

$$\phi_2(d+1)=\pi_{G_{d+1}}\big(\phi_2(d)+\psi_2(d+1)-\psi_2(d)\big)=\phi_2(d)+\psi_2(d+1)-\psi_2(d)$$
$$=\Big(\sqrt{(d+1)(c^2+1)+2c\sqrt{d(c^2+1)}}+1\Big)\frac{\big(1,1,\dots,1,c\sqrt{d}+\sqrt{c^2+1}\big)}{\big\|\big(1,1,\dots,1,c\sqrt{d}+\sqrt{c^2+1}\big)\big\|}.$$

Therefore,

$$\|\phi_2(d+1)-\phi_1(d+1)\|=\|\phi_2(d+1)\|$$
$$=\sqrt{(d+1)(c^2+1)+2c\sqrt{d(c^2+1)}}+1.$$

On the other hand,

$$\|\psi_2-\psi_1\|=\|\psi_2\|=\max_{0\leqslant k\leqslant d+1}\|\psi_2(k)\|=1,$$

and so (4.24) holds. ■

REMARK 4.3. *The Lipschitz constant for the constraining term given in* (4.16) *is tight.*

P r o o f. In Example 4.1 we have

$$\Lambda(d)=\big(\phi_2(d)-\phi_1(d)\big)-\big(\psi_2(d)-\psi_1(d)\big)=\phi_2(d)-\psi_2(d)$$
$$=\Big(1+\frac{c}{\sqrt{d(c^2+1)}}\Big)(1,1,\dots,1,c\sqrt{d})-\frac{c}{\sqrt{d(c^2+1)}}\Big(1,\dots,1,-\frac{\sqrt{d}}{c}\Big)$$
$$=\big(1,1,\dots,1,c\sqrt{d}+\sqrt{c^2+1}\big).$$

Therefore,

$$\|\Lambda(d)\| = \sqrt{(d+1)(c^2+1) + 2c\sqrt{d(c^2+1)}} \cdot \|\psi_2 - \psi_1\|,$$

which completes the assertion. ■

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