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FOCK SPACE REALIZATIONS OF SOME CLASSICAL MARKOV PROCESSES*

BY

WŁODZIMIERZ BRYC (CINCINNATI) AND WIKTOR EJSMONT (WROCŁAW)

Abstract. We define a pair of non-commutative processes on a perturbed Fock space. Both processes have the same univariate distributions and satisfy a weak form of the polynomial martingale property. The processes give two non-equivalent Fock-space realizations of the same classical Markov process: the two-parameter bi-Poisson processes introduced in [12], and constructed in [13].

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1. INTRODUCTION

It is known that any probability measure with finite moments has a Fock space representation, see [1]. On the other hand, while there are many examples, there is no general construction for the Fock space representation of a classical processes $(X_t)_{t\geqslant 0}$. The difficulty here consists in selecting a convenient Fock space $\mathbb{H}$ and a suitable mapping $(0,\infty)\to\mathbb{H}$ such that all ordered non-commutative moments would agree with the corresponding moments of the classical process, see Definition 3.3. It is natural to expect, and known to experts, that when such a Fock space representation exists then it is not unique. In this paper we exhibit two non-equivalent processes on the same Fock space that represent the same classical bi-Poisson process [13]; for particular values of the parameters these constructions coincide with two non-equivalent Fock space representations of the classical Markov q-Poisson processes that already appeared in the literature in [2] and [17], and were in fact suggested to us by these two papers. We were also guided by the perturbations of the free Fock space in [10] and [11], and by the observation in

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[14], Proposition 4.1, that for $q = 0$ the univariate laws of the bi-Poisson processes form a semigroup with respect to the c-free convolution of [9]. Our construction does not enjoy the symmetry of the bi-Poisson processes in classical (commutative) probability, where $(X_t)_{t>0}$ and $(tX_{1/t})_{t>0}$ represent the same Markov process.

2. NON-COMMUTATIVE PROCESSES

Throughout this paper, parameters $|q| < 1$ and $1 + \eta\theta > q^+$ are fixed (where $q^+ = \max\{q, 0\}$). For this choice of the parameters, $1 + \eta\theta[n]_q \geqslant 0$ for all n. (Recall that $[n]_q = \sum_{j=1}^{n} q^{j-1}$.)

The following three-step recurrence

$$
\begin{aligned}
(2.1)\quad & xp_n(x;t) \\
&= p_{n+1}(x;t) + (\theta + t\eta)[n]_q p_n(x;t) + t(1 + \eta\theta[n-1]_q)[n]_q p_{n-1}(x;t),
\end{aligned}
$$

$n \geqslant 0$, with $p_{-1} = 0, p_0 = 1$ was introduced in [12], Example 4.9, in connection with a regression problem. When $\eta\theta = 0$ and t is fixed this recurrence determines polynomials which are orthogonal with respect to the q-Poisson distribution that already appeared in non-commutative probability, see [2] and [16]–[18].

Our goal is to construct two different non-commutative processes that correspond to this recurrence. We will then show that these processes fail to be equivalent, but are still classically equivalent (see Definitions 3.1 and 3.3) and they both have the classical bi-Poisson Markov process of [13] as the classical version.

2.1. Perturbed Fock space. We will be working with the perturbation of the q-Fock space from [7] which coincides with the perturbation in [10], Section 7, when $q = 0$. For a real Hilbert space $\mathbb{H} = L_2\big((0,\infty), ds\big)$ with the complexification $\mathbb{H}_c := \mathbb{H} \oplus i\mathbb{H}$, the associated perturbed q-Fock space $\mathcal{F}_{q,\eta\theta}(\mathbb{H})$ is the closure of $\bigoplus_{n=0}^{\infty} \mathbb{H}_c^{\otimes n}$ with respect to the scalar product obtained as the sesquilinear extension of

$$
\begin{aligned}
&\langle g_1 \otimes \ldots \otimes g_n | h_1 \otimes \ldots \otimes h_m \rangle_q \\
&\qquad = \delta_{m,n} \prod_{k=1}^{n-1} (1 + \eta\theta[k-1]_q) \sum_{\sigma \in S_n} q^{|\sigma|} \prod_{j=1}^{n} \langle g_j, h_{\sigma(j)} \rangle.
\end{aligned}
$$

Here, $\mathbb{H}_c^{\otimes 0} := \mathbb{C}\mathbf{1}$, where $\mathbf{1}$ is called the *vacuum vector*, i.e. it is a distinguished vector of norm one, S_n is the set of all the permutations of $\{1, \ldots, n\}$, and $|\sigma| := \mathrm{card}\{(i,j) : i < j, \sigma(i) > \sigma(j)\}$ is the number of inversions of $\sigma \in S_n$. The explicit structure of $\mathbb{H}$ is used in (2.7). (This is just a “weighted” version of the q-Gaussian Fock space of [7].)

We denote by $\|\cdot\|_{\mathcal{F}_{q,\eta\theta}(\mathbb{H})}$ the corresponding norm. We denote by $\overline{\mathbb{H}^{\otimes n}}$ the $\|\cdot\|_{\mathcal{F}_{q,\eta\theta}(\mathbb{H})}$-closure of the algebraic tensor product $\mathbb{H}_c^{\otimes n}$ so that

$$
\mathcal{F}_{q,\eta\theta}(\mathbb{H}) = \bigoplus_{n=0}^{\infty} \overline{\mathbb{H}^{\otimes n}}.
$$

Let $T : \mathbb{H} \to \mathbb{H}$ be a bounded operator. Define its gauge operator $\mathbf{p}(T)$ on $\bigoplus_{n=0}^{\infty} \mathbb{H}_c^{\otimes n}$ by

$$\mathbf{p}(T)\mathbf{1} = 0,$$

$$\mathbf{p}(T) g_1 \otimes \ldots \otimes g_n := \sum_{j=1}^{n} q^{j-1} (Tg_j) \otimes g_1 \otimes \ldots \otimes g_{j-1} \otimes g_{j+1} \otimes \ldots \otimes g_n.$$

We will need the perturbed version of Lemma 1 in [3].

LEMMA 2.1. *If T is self-adjoint, then $\mathbf{p}(T)$ is self-adjoint and extends to a bounded operator on $\mathcal{F}_{q,\eta\theta}(\mathbb{H})$.*

The proof relies on an auxiliary operator

$$P_{q,\eta\theta} = \bigoplus_{n=0}^{\infty} P_{q,\eta\theta}^{(n)} \qquad \text{with } P_{q,\eta\theta}^{(n)} \colon \mathbb{H}_c^{\otimes n} \to \mathbb{H}_c^{\otimes n}.$$

We define the operator $P_{q,\eta\theta}^{(n)}$ by

$$P_{q,\eta\theta}^{(n)} = \prod_{k=1}^{n-1} (1 + \eta\theta [k-1]_q) \sum_{\sigma \in S_n} q^{|\sigma|} U_\sigma, \tag{2.2}$$

where

$$U_\sigma g_1 \otimes \ldots \otimes g_n = g_{\sigma(1)} \otimes \ldots \otimes g_{\sigma(n)}, \qquad \sigma \in S_n.$$

We can now write the scalar product in terms of $P_{q,\eta\theta}^{(n)}$ and the usual scalar product on the full Fock space as

$$\langle \zeta | \zeta' \rangle_{q,\eta\theta} = \langle \zeta | P_{q,\eta\theta}^{(n)} (\zeta') \rangle_{0,0}\,.$$

REMARK 2.1. *The operator $P_{q,\eta\theta}$ is positive for all $q \in (-1, 1)$ with respect to the scalar product $\langle \cdot | \cdot \rangle_{0,0}$. That is, for all $\zeta \in \mathbb{H}_c^{\otimes n}$, $\langle \zeta | P_{q,\eta,\theta}^{(n)} \zeta \rangle_{0,0} \geqslant 0$.*

Indeed, recalling the fact that

$$P_q^{(n)} = P_{q,0}^{(n)} = \sum_{\sigma \in S_n} q^{|\sigma|} U_\sigma$$

is positive definite ([8], Proposition 1) we see that

$$\langle \zeta | P_{q,\eta\theta}^{(n)} \zeta \rangle_{0,0} = \prod_{k=1}^{n-1} (1 + \eta\theta [k-1]_q) \big\langle \zeta | \sum_{\sigma \in S_n} q^{|\sigma|} U_\sigma \zeta \big\rangle_{0,0} \geqslant 0. \tag{2.3}$$

Proof of Lemma 2.1. We first prove that if T is self-adjoint then $\mathbf{p}(T)$ is self-adjoint on a dense set $\bigoplus_{n=0}^{\infty} \mathbb{H}_c^{\otimes n}$. Take any $\zeta, \zeta' \in \mathbb{H}_c^n$. Operator $P_{q,\eta\theta}^{(n)}$, restricted to $\mathbb{H}_c^n$, can be written as $P_{q,\eta\theta}^{(n)} = \prod_{k=1}^{n-1} (1 + \eta\theta [k-1]_q) P_q^{(n)}$. The main idea of the proof is to apply Proposition 2.2 of [2] which tells us that the adjoint of $\mathbf{p}(T)$ with respect to the $\langle\cdot|\cdot\rangle_{q,0}$-inner product is $\mathbf{p}(T^*)$. This gives us

$$\begin{aligned}
\langle\mathbf{p}(T)(\zeta)|\zeta'\rangle_{q,\eta\theta} &= \langle\mathbf{p}(T)(\zeta)|P^{(n)}_{q,\eta\theta}(\zeta')\rangle_{0,0} \qquad (2.4)\\
&= \prod_{k=1}^{n-1}(1+\eta\theta[k-1]_q)\langle\mathbf{p}(T)(\zeta)|P^{(n)}_{q}(\zeta')\rangle_{0,0}\\
&= \prod_{k=1}^{n-1}(1+\eta\theta[k-1]_q)\langle\mathbf{p}(T)(\zeta)|\zeta'\rangle_{q,0}\\
&= \prod_{k=1}^{n-1}(1+\eta\theta[k-1]_q)\langle\zeta|\mathbf{p}(T^*)(\zeta')\rangle_{q,0}\\
&= \prod_{k=1}^{n-1}(1+\eta\theta[k-1]_q)\langle\zeta|P^{(n)}_{q}\mathbf{p}(T^*)(\zeta')\rangle_{0,0}\\
&= \langle\zeta|\mathbf{p}(T^*)(\zeta')\rangle_{q,\eta\theta}. \qquad (2.5)
\end{aligned}$$

Next we show that $\mathbf{p}(T)$ is bounded. We begin by showing that $\mathbf{p}(T)$ is bounded in $\mathcal{F}_{0,0}(\mathbb{H})$. The proof is similar to that of Lemma 1 in Anshelevich [3]. We express the $\mathbf{p}(T)$ as $\mathbf{p}(T)=p_0(T)p(\mathbf{1})$, where $p_0(T)(g_1\otimes\ldots\otimes g_n)=T(g_1)\otimes\ldots\otimes T(g_n)$ and $\mathbf{1}$ is the vacuum vector. Of course, we have $\|p_0\|\leqslant\|T\|$. Operator $p(\mathbf{1})$ is bounded and has norm $\|p(\mathbf{1})\|_{0,0}\leqslant\max\big(1,1/(1-q)\big)$. This follows by the same method as in [3] (the first part of the proof of Lemma 1). From the equality stated in (2.4)–(2.5) it is clear that $P_{q,\eta\theta}\mathbf{p}(T^*)=\mathbf{p}(T)^*P_{q,\eta\theta}$, where $\mathbf{p}(T)^*$ is taken with respect to the zero-inner product ($\langle\cdot|\cdot\rangle_{0,0}$). This gives us $P_{q,\eta\theta}\mathbf{p}(T^*)\mathbf{p}(T)=\mathbf{p}(T)^*P_{q,\eta\theta}\mathbf{p}(T)\geqslant 0$. In particular,

$$P_{q,\eta\theta}\mathbf{p}(T^*)\mathbf{p}(T)[\mathbf{p}(T^*)\mathbf{p}(T)]^*P_{q,\eta\theta}\leqslant\|\mathbf{p}(T^*)\mathbf{p}(T)[\mathbf{p}(T^*)\mathbf{p}(T)]^*\|_{0,0}P^2_{q,\eta\theta}$$

or

$$\begin{aligned}
P_{q,\eta\theta}\mathbf{p}(T^*)\mathbf{p}(T) &\leqslant \sqrt{\|\mathbf{p}(T^*)\mathbf{p}(T)[\mathbf{p}(T^*)\mathbf{p}(T)]^*\|_{0,0}}P_{q,\eta\theta} \qquad (2.6)\\
&\leqslant \|\mathbf{p}(T^*)\|_{0,0}\|\mathbf{p}(T)\|_{0,0}P_{q,\eta\theta}.
\end{aligned}$$

If we take $\zeta\in\mathcal{F}_{q,\eta\theta}(\mathbb{H})$, we get

$$\begin{aligned}
&\langle\mathbf{p}(T)(\zeta)|\mathbf{p}(T)(\zeta)\rangle_{q,\eta\theta}=\langle\zeta|\mathbf{p}(T^*)\mathbf{p}(T)(\zeta)\rangle_{q,\eta\theta}\\
&=\langle\zeta|P_{q,\eta\theta}\mathbf{p}(T^*)\mathbf{p}(T)(\zeta)\rangle_{0,0}\overset{(2.6)}{\leqslant}\|\mathbf{p}(T^*)\|_{0,0}\|\mathbf{p}(T)\|_{0,0}\langle\zeta|P_{q,\eta\theta}(\zeta)\rangle_{0,0}\\
&=\|\mathbf{p}(T^*)\|_{0,0}\|\mathbf{p}(T)\|_{0,0}\langle\zeta|\zeta\rangle_{q,\eta\theta}.
\end{aligned}$$

Since $\|T^*\|=\|T\|$, we conclude that

$$\|\mathbf{p}(T)\|_{q,\eta\theta}\leqslant\sqrt{\|\mathbf{p}(T^*)\|_{0,0}\|\mathbf{p}(T)\|_{0,0}}\leqslant\max\big(1,1/(1-q)\big)\|T\|. \quad \blacksquare$$

2.2. Operators. Our process will be a linear combination of the usual objects: the annihilator, creator and two gauge operators.

2.2.1. *Annihilators and creators.* For $h \in \mathbb{H}$, the annihilation operator

$$\mathbf{a}_h : \mathcal{F}_{q,\eta\theta}(\mathbb{H}) \to \mathcal{F}_{q,\eta\theta}(\mathbb{H})$$

and its adjoint, the creation operator

$$\mathbf{a}_h^* : \mathcal{F}_{q,\eta\theta}(\mathbb{H}) \to \mathcal{F}_{q,\eta\theta}(\mathbb{H}),$$

are the bounded linear extensions of

$$\mathbf{a}_h \mathbf{1} := 0,$$

$$\begin{aligned}&\mathbf{a}_h g_1 \otimes \ldots \otimes g_n \\ &\quad := (1 + \eta\theta[n-1]_q) \sum_{j=1}^{n} q^{j-1} \langle h, g_j \rangle g_1 \otimes \ldots \otimes g_{j-1} \otimes g_{j+1} \otimes \ldots \otimes g_n\end{aligned}$$

and

$$\mathbf{a}_h^* \mathbf{1} = h, \qquad \mathbf{a}_h^* g_1 \otimes \ldots \otimes g_n := h \otimes g_1 \otimes \ldots \otimes g_n,$$

where $g_1, g_2, \ldots, g_n \in \mathbb{H}_c$.

2.2.2. *Gauge operator* $\mathbf{p}$. We follow [2]. For $h \in L_\infty\big((0,\infty)\big) \cap L_2\big((0,\infty)\big) \subset \mathbb{H}$ we define the gauge operator $\mathbf{p}_h$ as the bounded linear extension of

$$\mathbf{p}_h g_1 \otimes \ldots \otimes g_n := \sum_{j=1}^{n} q^{j-1} (h g_j) \otimes g_1 \otimes \ldots \otimes g_{j-1} \otimes g_{j+1} \otimes \ldots \otimes g_n, \tag{2.7}$$

$\mathbf{p}_h \mathbf{1} = 0.$

In the terminology of Lemma 2.1, this operator corresponds to the self-adjoint operator $T : \mathbb{H} \to \mathbb{H}$ of multiplication by a bounded function h.

2.2.3. *Gauge operator* $\mathbf{q}$. For $h \in \mathbb{H}$, we define the gauge operator $\mathbf{q}_h$ as the bounded linear extension of

$$\mathbf{q}_h g_1 \otimes \ldots \otimes g_n := \sum_{j=1}^{n} q^{j-1} \langle h, g_j \rangle h \otimes g_1 \otimes \ldots \otimes g_{j-1} \otimes g_{j+1} \otimes \ldots \otimes g_n,$$

$\mathbf{q}_h \mathbf{1} = 0.$

This is just the operator that in the unperturbed theory is written as $a_h^* a_h$. In the terminology of Lemma 2.1, this operator corresponds to the self-adjoint operator $T : \mathbb{H} \to \mathbb{H}$ defined by $T(f) = \langle h, f \rangle h$.

3. NON-COMMUTATIVE BI-POISSON PROCESSES

In this section we analyze simple but somewhat paradoxical properties of a pair of two-parameter processes defined on the same Fock space $\mathcal{F}_{q,\eta\theta}(\mathbb{H})$, and indexed by $t \in (0,\infty)$. Both processes at time t have the same univariate bi-Poisson distribution determined by (2.1), and satisfy a weak form of the polynomial martingale property. Both processes give (non-equivalent) Fock-space realizations of the same classical Markov process from [13].

We will define our processes in terms of the auxiliary self-adjoint bounded linear operator $\mathbf{Z}_{\eta,\theta}(f)$ on $\mathcal{F}_{q,\eta\theta}(\mathbb{H})$, which for $f \in L_\infty\big((0,\infty)\big) \cap L_2\big((0,\infty)\big)$ is given as

$$\mathbf{Z}_{\eta,\theta}(f) = \mathbf{a}_f + \mathbf{a}_f^* + \eta\mathbf{q}_f + \theta\mathbf{p}_f. \tag{3.1}$$

PROPOSITION 3.1. *$\mathbf{Z}_{\eta,\theta}(f)$ is a bounded self-adjoint operator, and vacuum $\mathbf{1}$ is a cyclic and separating vector for the algebra generated by $\big\{\mathbf{Z}(f) : f \in L_\infty\big((0,\infty)\big) \cap L_2\big((0,\infty)\big)\big\}$.*

Proof. (Compare [3], Propositions 2 and 4.) By Lemma 2.1, it is clear that $\mathbf{Z}_{\eta,\theta}(f)$ is a bounded self-adjoint operator.

Let $\mathcal{A}_{q,\eta,\theta}$ be the algebra of operators on $\mathcal{F}_{q,\eta\theta}(\mathbb{H})$ generated by $\big\{\mathbf{Z}(f) : f \in L_\infty\big((0,\infty)\big) \cap L_2\big((0,\infty)\big)\big\}$ (no closure!). Recall that $\mathbf{1}$ is a separating vector if for any $A \in \mathcal{A}_{q,\eta,\theta}$ we have the implication $A\mathbf{1} = 0 \Rightarrow A = 0$. Let $\mathbb{H}' = L_\infty\big((0,\infty)\big) \cap L_2\big((0,\infty)\big)$ and we define a Wick map

$$W : \bigoplus_{n=0}^{\infty} \mathbb{H}'^{\otimes n} \to \mathcal{A}_{q,\eta,\theta}$$

as

$$\begin{aligned}
&W(f_0 \otimes f_1 \otimes \ldots \otimes f_n) = Z_{\eta,\theta}(f_0) W(f_1 \otimes \ldots \otimes f_n)\\
&- (1 + \eta\theta[k-1]_q) \sum_{i=1}^{n} q^{i-1} \langle f_0 | f_i \rangle W(f_1 \otimes \ldots \otimes f_{i-1} \otimes f_{i+1} \otimes \ldots \otimes f_n)\\
&- \eta \sum_{i=1}^{n} q^{i-1} \langle f_0 | f_i \rangle W(f_0 \otimes f_1 \otimes \ldots \otimes f_{i-1} \otimes f_{i+1} \otimes \ldots \otimes f_n)\\
&- \theta \sum_{i=1}^{n} q^{i-1} W\big((f_0 f_i) \otimes f_1 \otimes \ldots \otimes f_{i-1} \otimes f_{i+1} \otimes \ldots \otimes f_n\big)
\end{aligned}$$

with the extra condition $W(f) = Z_{\eta,\theta}(f)$ and $W(\mathbf{1}) = \mathbf{1}$. It is obvious that

$$W(f_1 \otimes \ldots \otimes f_n)\mathbf{1} = f_1 \otimes \ldots \otimes f_n.$$

Since $\mathbb{H}'$ is a dense subspace of $\mathbb{H}$, we conclude that the Wick map W extends linearly to $\mathcal{F}_{q,\eta\theta}(\mathbb{H})$. It is clear that the Wick map $W(f_1 \otimes \ldots \otimes f_n)$ is a polynomial in variables $Z_{\eta,\theta}(f), f \in \mathbb{H}'$. Induction on n shows that

$$Z_{\eta,\theta}(f_1)\dots Z_{\eta,\theta}(f_n) = W(f_1 \otimes \dots \otimes f_n) + W(\zeta), \tag{3.2}$$

where $\zeta \in \bigoplus_{i=0}^{n-1} \mathbb{H}'^{\otimes i}$. So we have $A = W(\phi)$, where $\phi \in \bigoplus_{n=0}^{\infty} \mathbb{H}^{\otimes n}$ for every $A \in \mathcal{A}_{q,\eta,\theta}$. If $A\mathbf{1} = 0$, then $W(\phi)\mathbf{1} = \phi = 0$, and so $A = 0$. Thus $\mathbf{1}$ is a separating vector for $\mathcal{A}_{q,\eta,\theta}$.

Recall that the vacuum vector $\mathbf{1}$ is cyclic for an algebra of bounded operators $\mathcal{A}_{q,\eta,\theta}$ on a Hilbert space $\mathcal{F}_{q,\eta\theta}(\mathbb{H})$, if $\{T\mathbf{1} : T \in \mathcal{A}_{q,\eta,\theta}\}$ is a dense subspace of $\mathcal{F}_{q,\eta\theta}(\mathbb{H})$. By definition,

$$Z_{\eta,\theta}(f_1)\dots Z_{\eta,\theta}(f_n)(\mathbf{1}) = f_1 \otimes \dots \otimes f_n + \zeta, \tag{3.3}$$

where $\zeta \in \bigoplus_{i=0}^{n-1} \mathbb{H}'^{\otimes i}$. $\mathcal{A}_{q,\eta,\theta}$ is the algebra generated by $Z_{\eta,\theta}(f)$, so it follows by induction that the set $B = \{T\mathbf{1} : T \text{ is a polynomial of } Z_{\eta,\theta}(f), f \in \mathbb{H}'\}$ is equal to $\bigoplus_{i=0}^{\infty} \mathbb{H}'^{\otimes i}$. The set B is dense in $\mathcal{F}_{q,\eta\theta}(\mathbb{H})$ because $\mathbb{H}'$ is dense in $\mathbb{H}$ with respect to $\|\cdot\|_2$ (obviously, $B \subset \{T\mathbf{1} : T \in \mathcal{A}_{q,\eta,\theta}\}$). ■

Let $\mathcal{A}_{q,\eta,\theta}$ be the algebra generated by $\mathbf{Z}_{\eta,\theta}(f)$ from the above proof. We define the expectation state $\tau : \mathcal{A}_{q,\eta,\theta} \to \mathbb{C}$ on this algebra by setting

$$\tau(\mathbf{U}) := \langle \mathbf{U1}, \mathbf{1} \rangle$$

for any polynomial $\mathbf{U} \in \mathcal{A}_{q,\eta,\theta}$.

For fixed $\{0,1\}$-valued f, the random variable $\mathbf{Z}_{\eta,\theta}(f)$ has the bi-Poisson distribution in the following sense.

PROPOSITION 3.2. *If $f^2 = f$, then the distribution of $\mathbf{Z}_{\eta,\theta}(f)$ is the orthogonality measure of polynomials $\{Q_n : n \geqslant 0\}$ defined by the three-step recurrence*

$$\begin{aligned} &xQ_n(x) - Q_{n+1}(x) \\ &\quad = (\theta + \eta\|f\|^2)[n]_q Q_n(x) + (1 + \eta\theta[n-1]_q)\|f\|^2[n]_q Q_{n-1}(x), \end{aligned} \tag{3.4}$$

$n \geqslant 0$, $Q_{-1} = 0$, $Q_0 = 1$.

P r o o f. Since $\mathbf{Z}_{\eta,\theta}(f)\mathbf{1} = f$, we see that $\tau\big(\mathbf{Z}_{\eta,\theta}(f)\big) = 0$ and the covariance is of the form

$$\tau\big(\mathbf{Z}_{\eta,\theta}(f)\mathbf{Z}_{\eta,\theta}(g)\big) = \langle g, f \rangle.$$

We need to verify that for $n \neq m$ we have

$$\tau\Big(Q_m\big(\mathbf{Z}_{\eta,\theta}(f)\big)Q_n\big(\mathbf{Z}_{\eta,\theta}(f)\big)\Big) = 0.$$

To see this, we verify by induction that

$$Q_n\big(\mathbf{Z}_{\eta,\theta}(f)\big)\mathbf{1} = f^{\otimes n}, \quad n \geqslant 0, \tag{3.5}$$

with the interpretation $f^{\otimes 0} = \mathbf{1}$. Indeed, since $\mathbf{Z}_{\eta,\theta}(f)\mathbf{1} = f$, formula (3.5) holds for $n = 0, 1$. Suppose that (3.5) holds for some $n \geqslant 1$. From (3.4) and the induction assumption we see that

$$Q_{n+1}\big(\mathbf{Z}_{\eta,\theta}(f)\big)\mathbf{1}$$
$$= \mathbf{Z}_{\eta,\theta}(f)f^{\otimes n} - (\theta + \eta\|f\|^2)[n]_q f^{\otimes n} - (1 + \eta\theta[n-1]_q)\|f\|^2[n]_q f^{\otimes n-1}.$$

Using (3.1) it is easy to see that

$$\mathbf{Z}_{\eta,\theta}(f)f^{\otimes n} = \mathbf{a}_f^* f^{\otimes n} + \eta\mathbf{q}_f f^{\otimes n} + \theta\mathbf{p}_f f^{\otimes n} + \mathbf{a}_f f^{\otimes n}$$
$$= f^{\otimes n+1} + \eta\|f\|^2[n]_q f^{\otimes n} + \theta[n]_q f^{\otimes n} + (1 + \eta\theta[n-1]_q)\|f\|^2[n]_q f^{\otimes n-1}.$$

Thus $Q_{n+1}\big(\mathbf{Z}_{\eta,\theta}(f)\big)\mathbf{1} = f^{\otimes n+1}$ and (3.5) follows. ∎

We will be interested in two closely related non-commutative stochastic processes, indexed by $t > 0$ and derived from $\mathbf{Z}$ by taking appropriate $\{0, 1\}$-valued functions f. Let

$$\mathbf{X}(t) = \mathbf{Z}_{\eta,\theta}(1_{(0,t)}), \tag{3.6}$$

$$\mathbf{Y}(t) = t\mathbf{Z}_{\theta,\eta}(1_{(0,1/t)}). \tag{3.7}$$

Since the parameters q, η, θ are fixed throughout, in our notation we suppress the dependence of $\mathbf{X}, \mathbf{Y}$ on these parameters. Note that in (3.7) the parameters η, θ are switched.

We remark that when $\eta = 0$, process $\mathbf{X}(t)$ becomes the centered version of the q-Poisson process as defined in [2], Definition 6.15, and $\mathbf{Y}(t)$ is the time-transform of the centered q-Poisson process introduced in [17]. (That is, $\big(t\mathbf{Y}(1/t)\big)_{t>0}$ is the centered version of the q-Poisson process from [17].) Similarly, when $\theta = 0$, $\mathbf{Y}(t)$ is the time-transform of the centered q-Poisson process from [2], Definition 6.15, and process $\mathbf{X}(t)$ becomes the centered q-Poisson process as defined in [17], and is closely related to the construction in [18].

It is clear that both processes have the same covariance structure: $\tau\big(\mathbf{X}(t)\big) = \tau\big(\mathbf{Y}(t)\big) = 0$, and $\tau\big(\mathbf{X}(t)\mathbf{X}(s)\big) = \tau\big(\mathbf{Y}(t)\mathbf{Y}(s)\big) = \min\{t, s\}$. From Proposition 3.2 it follows that both processes also have the same orthogonal polynomials given by (2.1). Indeed, $p_m\big(\mathbf{Y}(t), t\big)\mathbf{1} = t^m 1_{(0,1/t)}^{\otimes m}$. Thus their one-dimensional distributions are equal.

We are interested in the question of equivalence of processes $\mathbf{X}$ and $\mathbf{Y}$. We introduce several related moments-based notions of equivalence, and we show that processes $\mathbf{X}$ and $\mathbf{Y}$ are equivalent only in the weakest sense: they both have the same classical versions.

DEFINITION 3.1. Processes $\mathbf{X}_1$, $\mathbf{X}_2$ are *equivalent* if for every finite choice $t_1, \ldots, t_k \in (0, \infty)$

$$\tau\big(\mathbf{X}_1(t_1)\ldots\mathbf{X}_1(t_k)\big) = \tau\big(\mathbf{X}_2(t_1)\ldots\mathbf{X}_2(t_k)\big), \quad k = 1, 2, \ldots$$

For an example of equivalent processes, see [6], Theorem 6.2. Belavkin [5] advocates the following concept of weak equivalence.

DEFINITION 3.2. Processes $\mathbf{X}_1$, $\mathbf{X}_2$ are *weakly equivalent* if for every finite choice of ordered numbers $s_1 \geqslant \ldots \geqslant s_m > 0$ and $s_m < t_1 \leqslant \ldots \leqslant t_k$

$$\begin{aligned}\tau\big(\mathbf{X}_1(s_1)\ldots\mathbf{X}_1(s_m)\mathbf{X}_1(t_1)\ldots\mathbf{X}_1(t_k)\big)\\ = \tau\big(\mathbf{X}_2(s_1)\ldots\mathbf{X}_2(s_m)\mathbf{X}_2(t_1)\ldots\mathbf{X}_2(t_k)\big), \quad m,k = 1,2,\ldots\end{aligned}$$

The weakest version of the concept of equivalence of non-commutative processes relies on time-ordered moments, compare [7], Definition 4.1.

DEFINITION 3.3. Processes $\mathbf{X}_1$, $\mathbf{X}_2$ are *classically equivalent* if for every finite choice of real numbers $0 < t_1 \leqslant t_2 \leqslant \ldots \leqslant t_k$

$$\tau\big(\mathbf{X}_1(t_1)\ldots\mathbf{X}_1(t_k)\big) = \tau\big(\mathbf{X}_2(t_1)\ldots\mathbf{X}_2(t_k)\big), \quad k = 1,2,\ldots \tag{3.8}$$

Recall that a classical version of the process $\mathbf{X}(t)$ is a classical stochastic process (X_t) that is classically equivalent to $\mathbf{X}(t)$, i.e.

$$\tau\big(\mathbf{X}(t_1)\ldots\mathbf{X}(t_k)\big) = E(X_{t_1}\ldots X_{t_k}) \quad \text{for all } t_1 \leqslant \ldots \leqslant t_k \in (0,\infty),$$

cf. [7], Theorem 4.4, and [4], Corollary A1 (c).

First we show that, generically, processes $\mathbf{X}$ and $\mathbf{Y}$ are not (weakly) equivalent.

PROPOSITION 3.3. *Processes $\mathbf{X}(t)$ and $\mathbf{Y}(t)$ are equivalent if and only if $\eta = \theta = 0$.*

Proof. A calculation shows that

$$\mathbf{X}(t)\mathbf{X}(s)\mathbf{1} = (t\wedge s)\mathbf{1} + \theta 1_{(0,t\wedge s)} + (t\wedge s)\eta 1_{(0,t)} + 1_{(0,t)} \otimes 1_{(0,s)},$$

and, recalling that η, θ are swapped in (3.7), we have

$$\mathbf{Y}(t)\mathbf{Y}(s)\mathbf{1} = (t\wedge s)\mathbf{1} + ts\eta 1_{(0,1/t\wedge 1/s)} + (t\wedge s)\theta 1_{(0,1/t)} + st 1_{(0,1/t)} \otimes 1_{(0,1/s)}.$$

Since $\mathbf{X}(t), \mathbf{Y}(t)$ are self-adjoint and different tensor powers are orthogonal, we have

$$\tau\big(\mathbf{X}(t)\mathbf{X}(s)\mathbf{X}(t)\big) = \langle 1_{(0,s)}, \theta 1_{(0,t\wedge s)} + (t\wedge s)\eta 1_{(0,t)}\rangle$$

and

$$\tau\big(\mathbf{Y}(t)\mathbf{Y}(s)\mathbf{Y}(t)\big) = \langle t1_{(0,1/t)}, ts\eta 1_{(0,1/t\wedge 1/s)} + (t\wedge s)\theta 1_{(0,1/t)}\rangle.$$

Fix $0 < t_1 \leqslant t_2$. The above formulas give

$$\tau\big(\mathbf{X}(t_2)\mathbf{X}(t_1)\mathbf{X}(t_2)\big) = \theta t_1 + \eta t_1^2 \tag{3.9}$$

and

$$\tau\big(\mathbf{Y}(t_2)\mathbf{Y}(t_1)\mathbf{Y}(t_2)\big) = \eta t_1 t_2 + \theta t_2. \tag{3.10}$$

Thus the moments cannot be equal for all $0 < t_1 \leqslant t_2$ unless $\eta = \theta = 0$.

On the other hand, if $\eta = \theta = 0$, then $\tau\big(\mathbf{X}(t)\mathbf{X}(s)\big) = \tau\big(\mathbf{Y}(t)\mathbf{Y}(s)\big)$, so the joint moments $\tau\big(\mathbf{X}(t_1)\dots\mathbf{X}(t_k)\big)$ and $\tau\big(\mathbf{Y}(t_1)\dots\mathbf{Y}(t_k)\big)$ are given by the same combinatorial expression for all $t_1, \dots, t_k \geqslant 0$, see [15], Corollary 2.1. Thus in this case the processes are equivalent. ■

PROPOSITION 3.4. *Processes $\mathbf{X}(t)$ and $\mathbf{Y}(t)$ are classically equivalent. Moreover, the recurrence* (2.1) *defines the classical martingale polynomials: if $t_1 \leqslant t_2 \leqslant \dots \leqslant t_k \leqslant u$, then*

$$\tau\Big(\mathbf{X}(t_1)\dots\mathbf{X}(t_k)p_m\big(\mathbf{X}(u);u\big)\Big) = \tau\Big(\mathbf{X}(t_1)\dots\mathbf{X}(t_k)p_m\big(\mathbf{X}(t_k);t_k\big)\Big) \tag{3.11}$$

and

$$\tau\Big(\mathbf{Y}(t_1)\dots\mathbf{Y}(t_k)p_m\big(\mathbf{Y}(u);u\big)\Big) = \tau\Big(\mathbf{Y}(t_1)\dots\mathbf{Y}(t_k)p_m\big(\mathbf{Y}(t_k);t_k\big)\Big). \tag{3.12}$$

Proof. Write $f_t = 1_{(0,t)}, g_t = 1_{(0,1/t)}$. Fix $t_1, t_2, \dots, t_k \in (0,u]$. Induction on k shows that $\mathbf{X}(t_k)\mathbf{X}(t_{k-1})\dots\mathbf{X}(t_1)\mathbf{1}$ is given by a unique linear combination

$$\sum_{S=\{s_1,\dots,s_r\}\subset\{t_1,\dots,t_k\}} \alpha_S f_{s_1}\otimes f_{s_2}\otimes\dots\otimes f_{s_r}.$$

Indeed, each of the operators $\mathbf{a}_f, \mathbf{a}_f^*, \mathbf{p}_f, \mathbf{q}_f$ preserves this form.

If $u \geqslant t \geqslant \max\{t_1,\dots,t_k\}$ then $\langle f_u, f_s\rangle = \langle f_t, f_s\rangle$ for all $s \in \{t_1,\dots,t_k\}$. Therefore,

$$\langle f_u^{\otimes m}, f_{s_1}\otimes f_{s_2}\otimes\dots\otimes f_{s_r}\rangle = \langle f_t^{\otimes m}, f_{s_1}\otimes f_{s_2}\otimes\dots\otimes f_{s_r}\rangle,$$

which implies (3.11).

Similarly, $\mathbf{Y}(t_k)\mathbf{Y}(t_{k-1})\dots\mathbf{Y}(t_1)\mathbf{1}$ is given by a unique linear combination

$$\sum_{S=\{s_1,\dots,s_r\}\subset\{t_1,\dots,t_k\}} \alpha_S g_{s_1}\otimes g_{s_2}\otimes\dots\otimes g_{s_r}.$$

If $u \geqslant t \geqslant \max\{t_1,\dots,t_k\}$ then the identity $u\langle g_u, g_s\rangle = 1 = t\langle g_t, g_s\rangle$ for all $s \in \{t_1,\dots,t_k\}$ now gives

$$u^m\langle g_u^{\otimes m}, g_{s_1}\otimes g_{s_2}\otimes\dots\otimes g_{s_r}\rangle = t^m\langle g_t^{\otimes m}, g_{s_1}\otimes g_{s_2}\otimes\dots\otimes g_{s_r}\rangle,$$

which implies (3.12).

Since for each fixed $t > 0$ random variables $\mathbf{X}(t)$ and $\mathbf{Y}(t)$ are bounded and their one-dimensional distributions coincide, the classical equivalence of $\mathbf{X}(t)$ and $\mathbf{Y}(t)$ follows from (3.11) and (3.12) by induction as follows. Suppose that we have m different values $t_1 < t_2 < \ldots < t_m$ each repeated $n_1, \ldots, n_m$ times, so that the definition (3.8) takes the form

$$\tau\big(\mathbf{X}(t_1)^{n_1} \ldots \mathbf{X}(t_m)^{n_m}\big) = \tau\big(\mathbf{Y}(t_1)^{n_1} \ldots \mathbf{Y}(t_m)^{n_m}\big).$$

From (3.11) and (3.12) we see that for a polynomials $p_r(x; , t_{m+1})$ determined from (2.1) it follows that

$$\tau\Big(\mathbf{X}(t_1)^{n_1} \ldots \mathbf{X}(t_m)^{n_m} p_r\big(\mathbf{X}(t_{m+1}); t_{m+1}\big)\Big)$$

reduces to the expectation of the polynomial in $\mathbf{X}(t_1), \ldots, \mathbf{X}(t_m)$. Thus

$$\begin{aligned}\tau\Big(\mathbf{X}(t_1)^{n_1} \ldots \mathbf{X}(t_m)^{n_m} p_r\big(\mathbf{X}(t_{m+1}); t_{m+1}\big)\Big)& \\ = \tau\Big(\mathbf{Y}(t_1)^{n_1} \ldots \mathbf{Y}(t_m)^{n_m} p_r\big(\mathbf{Y}(t_{m+1}); t_{m+1}\big)\Big)&,\end{aligned}$$

which by linearity implies

$$\tau\big(\mathbf{X}(t_1)^{n_1} \ldots \mathbf{X}(t_m)^{n_m} \mathbf{X}(t_{m+1})^r\big) = \tau\big(\mathbf{Y}(t_1)^{n_1} \ldots \mathbf{Y}(t_m)^{n_m} \mathbf{Y}(t_{m+1})^r\big).$$

This completes the induction step. ■

Anshelevich [3], Proposition 25, points out that, generically, q-Lévy processes do not have normal tracial states. Since our process generalizes the q-Poisson process, it is not surprising that τ is not tracial.

COROLLARY 3.1. *If* $\eta^2 + \theta^2 > 0$, *then* $\tau : \mathcal{A}_{q,\eta,\theta} \to \mathbb{C}$ *is not tracial.*

P r o o f. The tracial property and Proposition 3.4 imply that for $t_1 < t_2$

$$\begin{aligned}&\tau\big(\mathbf{X}(t_2)\mathbf{X}(t_1)\mathbf{X}(t_2)\big) = \tau\big(\mathbf{X}(t_1)\mathbf{X}(t_2)^2\big) \\ &= \tau\Big(\mathbf{X}(t_1)\Big(p_2\big(\mathbf{X}(t_2); t_2\big) + (\theta + t\eta)p_1\big(\mathbf{X}(t_2); t_2\big) + tp_0\big(\mathbf{X}(t_2); t_2\big)\Big)\Big) \\ &= \tau\Big(\mathbf{X}(t_1)\Big(p_2\big(\mathbf{X}(t_1); t_1\big) + (\theta + t\eta)p_1\big(\mathbf{X}(t_1); t_1\big) + tp_0\big(\mathbf{X}(t_1); t_1\big)\Big)\Big) \\ &= \tau\Big(\mathbf{Y}(t_1)\Big(p_2\big(\mathbf{Y}(t_1); t_1\big) + (\theta + t\eta)p_1\big(\mathbf{Y}(t_1); t_1\big) + tp_0\big(\mathbf{Y}(t_1); t_1\big)\Big)\Big) \\ &= \tau\big(\mathbf{Y}(t_2)\mathbf{Y}(t_1)\mathbf{Y}(t_2)\big).\end{aligned}$$

From the proof of Proposition 3.3 (see (3.9) and (3.10)), we know that this cannot hold for all $t_1 < t_2$ unless $\eta = \theta = 0$. ■

In [13] the authors construct a (classical) Markov process $(X_t)_{t>0}$, which they call the *bi-Poisson process*, such that polynomials $\{p_n(x;t) : n \geqslant 0\}$ are the orthogonal martingale polynomials for (X_t). Proposition 3.4 implies the following.

COROLLARY 3.2. *The classical bi-Poisson process $(X_t)_{t>0}$ is the classical version of $\mathbf{X}$ and of $\mathbf{Y}$.*

Proof. The univariate distributions and moments of X_t, $\mathbf{X}(t)$ and $\mathbf{Y}(t)$ agree since $p_n(x;t)$ are orthogonal polynomials for all three processes.

By the polynomial martingale property and the Markov property the classical process satisfies

$$\begin{aligned} E\big(X_{t_1}\dots X_{t_k}p_m(X_u;u)\big) &= E\Big(X_{t_1}\dots X_{t_k}E\big(p_m(X_u;u)|X_{t_k}\big)\Big) \\ &= E\big(X_{t_1}\dots X_{t_k}p_m(X_{t_k};t_k)\big). \end{aligned}$$

Thus the multivariate distributions agree. ■

We remark that part of Corollary 3.2 that refers to process $\mathbf{X}$ is not new, as this result follows from a more general statement (cf. [4], Corollary A.3 (c)).

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Department of Mathematics
University of Cincinnati
Cincinnati, OH 45221–0025, USA
E-mail: Wlodzimierz.Bryc@UC.edu

Mathematical Institute
University of Wrocław
pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland
and
Department of Mathematics and Cybernetics
University of Economics
ul. Komandorska 118/120
53-345 Wrocław, Poland
E-mail: wiktor.ejsmont@math.uni.wroc.pl

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