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ESTIMATION BASED ON SEQUENTIAL ORDER STATISTICS WITH RANDOM REMOVALS

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Abstract. Suppose that n individuals are scrutinized in an experiment. Each failure is accompanied by a fixed number of removals. The experiment terminates after r $(\leqslant n)$ failures. An explicit expression for the likelihood function of the available progressive sequential order statistics (PSOS) data is proposed. Under the conditional proportional hazard rate (CPHR) model, the maximum likelihood (ML) estimates of parameters are derived. Under the CPHR model and the assumption that the baseline distribution belongs to the Weibull family of distributions, the existence and uniqueness of the ML estimates are investigated. Moreover, two general classes of lifetime distributions, as an extension of the Weibull distribution, are studied in more detail. An algorithm for generating PSOS data under the CPHR model is proposed. Finally, some concluding remarks are given.

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1. INTRODUCTION

Suppose that n individuals with lifetimes $X_1, \ldots, X_n$, respectively, are scrutinized in an experiment and $X_{1:n} \leqslant \ldots \leqslant X_{n:n}$ denote arranged lifetimes in order of magnitude. It is usually assumed that $X_1, \ldots, X_n$ are independent and identically distributed (iid) random variables, and so the theory of ordinary order statistics (OS) may be applied for modeling purposes. Kamps [14] introduced the concept of sequential order statistics (SOS) as an extension of the OS for modeling experiments in which failing an individual may change the distribution of the surviving individuals. For more details, see [14], [15], [8]–[11], and [5].

In this paper, we extend the concept of SOS and assume that each failure is accompanied by a fixed number of removals and the experiment terminates after r $(\leqslant n)$ failures. Therefore, the rest of the paper is organized as follows. In Section 2, the concept of progressive sequential order statistics (PSOS) and the associated likelihood function (LF) of a given PSOS data are presented. The PSOS data is investigated in Subsection 2.3 under a special model known as a conditional proportional hazard rate (CPHR) model. In Section 3, the problem of a maximum likelihood (ML) estimation of parameters on the basis of the available PSOS data under the CPHR model is considered. In Section 4, we obtain the ML estimates of parameters for some parametric families of lifetime distributions based on PSOS data under the CPHR model. Section 5 points out an algorithm generating a PSOS sample. Some proofs are given in the Appendix.

2. PSOS SAMPLE

2.1. Scheme of PSOS. Suppose that n individuals are scrutinized in an experiment. Each failure is accompanied by a fixed number of removals. The experiment terminates after r $(\leqslant n)$ failures. We assume that the common lifetime distribution of the surviving individuals in the experiment changes at the moment of failure. More precisely, let F_j be the common lifetime cdf of the units when $n-j+1-\sum_{i=1}^{j-1} R_i$ units are tested. Thus, all n units put under test with the common cdf F_1 at time zero. Following immediately the first failure at time x_1, R_1 units are randomly selected and removed. The remaining $n-1-R_1$ units continue to work with the common cdf F_2. Following the second failure at time x_2, R_2 units are randomly selected and removed, and then the remaining $n-2-R_1-R_2$ units continue to work with the common cdf F_3. This process continues until the r-th failure observed at time x_r, the remaining $R_r = n-r-R_1-\ldots-R_{r-1}$ units are removed, and then the experiment terminates. Figure 1 shows a perspective for a PSOS censored Type-II sample.

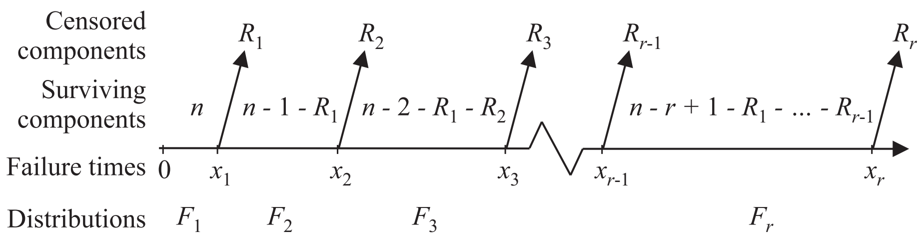

FIGURE 1. A perspective for a PSOS data

2.2. The LF of the PSOS sample. Let $\mathbf{x} = (x_1, \ldots, x_r)$ be a PSOS sample of size r. The associated LF is

$$
\begin{aligned}
(2.1)\quad & L(F_1, \ldots, F_r; \mathbf{x}) \\
&= \binom{n}{1} f_1^*(x_1)[\bar{F}_1^*(x_1)]^{n-1} \binom{n-1-R_1}{1} f_2^*(x_2)[\bar{F}_2^*(x_2)]^{n-2-R_1} \times \ldots \\
&\quad \times \binom{n-r+1-R_1-\ldots-R_{r-1}}{1} f_r^*(x_r)[\bar{F}_r^*(x_r)]^{n-r-R_1-\ldots-R_{r-1}} \\
&= c_p \prod_{j=1}^{r} \{ f_j^*(x_j)[\bar{F}_j^*(x_j)]^{m_j} \}, \quad x_1 < \ldots < x_r,
\end{aligned}
$$

where, by convention, $R_0 \equiv 0$, and

$$m_j = n - j - R_0 - \ldots - R_{j-1}, \tag{2.2}$$

$$F_j^*(x_j) = \frac{F_j(x_j) - F_j(x_{j-1})}{1 - F_j(x_{j-1})} \quad \text{and} \quad f_j^*(x_j) = \frac{f_j(x_j)}{1 - F_j(x_{j-1})}$$

for $1 \leqslant j \leqslant r$. If $F_1(x_0) = 0$, the LF in (2.1) takes the form

$$\begin{aligned} &L(F_1, \ldots, F_r; \mathbf{x}) = \\ &\quad = c_p \prod_{j=1}^{r-1} \left\{ f_j(x_j) \frac{[1 - F_j(x_j)]^{m_j}}{[1 - F_{j+1}(x_j)]^{1+m_{j+1}}} \right\} f_r(x_r) \, [1 - F_r(x_r)]^{m_r} . \end{aligned} \tag{2.3}$$

REMARK 2.1. *If $R_j = 0, j = 1, \ldots, r-1$, and $R_r = n - r$, then $m_j = n - j$ and $1 + m_{j+1} = n - j$ for $j = 1, \ldots, r$. In this case, the PSOS reduces to the SOS.*

REMARK 2.2. *If $F_1 = \ldots = F_n = F$, the PSOS data is reduced to the progressive censored Type-II data* (*POS*). (For more details see [1].)

2.3. Conditional proportional hazard rate model. In the sequel, we assume that the cdfs F_j in equation (2.3) satisfy

$$F_j(t) = 1 - [1 - F_0(t)]^{\alpha_j}, \quad j = 1, \ldots, r, \tag{2.4}$$

where F_0 is the baseline cdf. This model is known as the *conditional proportional hazard rate* (CPHR) model. Under the CPHR model, the hazard rate function of the cdf F_j is proportional to the hazard rate function of the baseline cdf F_0. More precisely, let $h_j(t) = f_j(t)/\bar{F}_j(t)$ for $j = 0, \ldots, n$. Then, for $t > 0$, we have

$$h_j(t) = \alpha_j h_0(t), \quad j = 1, \ldots, n. \tag{2.5}$$

For more details, see [13].

Under the CPHR model, the LF is

$$L(\boldsymbol{\alpha}, F_0; \mathbf{x}) = c_p \prod_{j=1}^{r} \{\alpha_j f_0(x_j) [\bar{F}_0(x_j)]^{\alpha_j(1+m_j) - \alpha_{j+1}(1+m_{j+1}) - 1}\}, \tag{2.6}$$

where, by convention, $\alpha_{r+1} \equiv 0$, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r)$ and m_j is given by (2.2).

The joint pdf (2.6) may be written as an *exponential family* structure, i.e.,

$$L(\boldsymbol{\alpha}, F_0; \mathbf{x}) = g(\boldsymbol{\alpha}) q_1(\mathbf{x}) \exp\big[\sum_{j=1}^{r} \alpha_j T_j(\mathbf{x})\big],$$

where $q_1(\mathbf{x}) = \prod_{j=1}^{r} f_0(x_j)/\bar{F}_0(x_j)$, $T_j(\mathbf{x}) = (m_j + 1) \ln[\bar{F}_0(x_j)/\bar{F}_0(x_{j-1})]$, and $g(\boldsymbol{\alpha}) = c_p\big(\prod_{j=1}^{r} \alpha_j\big)$. If the baseline cdf $F_0(t)$ is free of any unknown parameters, then the statistic $\mathbf{T} = (T_1, \ldots, T_r)$ is complete and sufficient for the vector parameter $\boldsymbol{\alpha}$. If $F_0(t)$ depends on some unknown parameters, say θ, one cannot estimate both parameters θ and $\boldsymbol{\alpha}$ based on only one PSOS sample. For example, Cramer and Kamps [8] proved that the ML estimate of the shape parameter of the two-parameter Weibull distribution on the basis of a single SOS sample does not exist. However, with at least two independent SOS data, the ML estimates exist. Similarly, we show that this statement holds for PSOS data.

3. ML ESTIMATION BASED ON THE PSOS DATA UNDER THE CPHR MODEL

Suppose that we observe s independent PSOS samples each of size r. The available data may be represented as

$$\mathbf{x} = \begin{bmatrix} x_{11}^* & x_{12}^* & \dots & x_{1r}^* \\ \dots & \dots & \dots & \dots \\ x_{s1}^* & x_{s2}^* & \dots & x_{sr}^* \end{bmatrix}, \tag{3.1}$$

where each row of $\mathbf{x}$ is a PSOS sample. The LF of $\mathbf{x}$, as an extension of (2.3), is

$$\text{(3.2)} \quad L(F_1, \dots, F_r; \mathbf{x}) =$$
$$= c_p^s \Big(\prod_{i=1}^{s} \prod_{j=1}^{r-1} \Big\{ f_j(x_{ij}) \frac{[1 - F_j(x_{ij})]^{m_j}}{[1 - F_{j+1}(x_{ij})]^{1+m_{j+1}}} \Big\} \Big) \big(\prod_{i=1}^{s} f_r(x_{ir}) \, [1 - F_r(x_{ir})]^{m_r} \big),$$

and under the CPHR model, the LF (3.2) reduces to

$$L(\boldsymbol{\alpha}, F_0; \mathbf{x}) = c_p^s \prod_{i=1}^{s} \prod_{j=1}^{r} \{ \alpha_j f_0(x_{ij}) [\bar{F}_0(x_{ij})]^{\alpha_j(1+m_j) - \alpha_{j+1}(1+m_{j+1}) - 1} \}. \tag{3.3}$$

The LF (3.3) may be written as an *exponential family* structure, i.e.

$$L(\boldsymbol{\alpha}, F_0; \mathbf{x}) = g^s(\boldsymbol{\alpha}) q_2(\mathbf{x}) \exp \big[\sum_{j=1}^{r} \alpha_j T_j^*(\mathbf{x}) \big],$$

where

$$q_2(\mathbf{x}) = \prod_{i=1}^{s} \prod_{j=1}^{r} \frac{f_0(x_{ij})}{\bar{F}_0(x_{ij})}, \quad T_j^*(\mathbf{x}) = \sum_{i=1}^{s} (m_j + 1) \ln \left[\frac{\bar{F}_0(x_{ij})}{\bar{F}_0(x_{i,j-1})} \right],$$

and m_j is given by (2.2). If the baseline cdf $F_0(t)$ is free of any unknown parameters, then the statistic $\mathbf{T}^* = (T_1^*, \dots, T_r^*)$ is complete and sufficient for the vector parameter $\boldsymbol{\alpha}$.

In the sequel, we consider the problem of estimating unknown parameters based on PSOS data from the baseline cdf F_0 under the CPHR model in two cases separately.

Case I. The baseline cdf $F_0(t)$ is known.

PROPOSITION 3.1. *If the baseline cdf $F_0(t)$ is known, the ML estimates of α_j for $1 \leqslant j \leqslant r$, based on PSOS data, exist and are unique.*

P r o o f. By (3.3), the log-likelihood function (LLF), denoted by ℓ, reads

$$\begin{aligned}(3.4)\qquad \ell &= s\ln c_p + s\sum_{j=1}^{r}\ln\alpha_j + \sum_{j=1}^{r}\sum_{i=1}^{s}\ln\frac{f_0(x_{ij})}{\bar{F}_0(x_{ij})}\\ &+\sum_{j=1}^{r}\alpha_j(1+m_j)\sum_{i=1}^{s}[\ln\bar{F}_0(x_{ij})-\ln\bar{F}_0(x_{i,j-1})],\end{aligned}$$

where, by convention, $x_{i,0}\equiv 0$ for $1\leqslant i\leqslant s$. By setting $H_0(t)=-\ln\bar{F}_0(t)$, the LLF (3.4) is summarized to

$$(3.5)\qquad \ell = s\ln c_p + s\sum_{j=1}^{r}\ln\alpha_j + \sum_{j=1}^{r}\sum_{i=1}^{s}\ln h_0(x_{ij}) - \sum_{j=1}^{r}\alpha_j(1+m_j)A_j,$$

where $A_j=\sum_{i=1}^{s}[H_0(x_{ij})-H_0(x_{i,j-1})]$, $1\leqslant j\leqslant r$. Hence, the likelihood equations for the vector parameter $\boldsymbol{\alpha}$ yield

$$(3.6)\qquad \alpha_j^* = \frac{s}{(1+m_j)A_j} = -\frac{s}{1+m_j}\left(\ln\prod_{i=1}^{s}\frac{\bar{F}_0(x_{ij})}{\bar{F}_0(x_{i,j-1})}\right)^{-1},\qquad 1\leqslant j\leqslant r.$$

The ML estimates of α_j $(1\leqslant j\leqslant r)$ in (3.6) are unique since the corresponding Hessian matrix $[\partial^2 l/\partial\alpha_i\alpha_j]=\mathrm{diag}[-s/\alpha_i^2]$ is strictly negative definite, which means that $(\alpha_1^*,\ldots,\alpha_r^*)$ is the unique local and global maximum point of the LLF (3.5). ■

C a s e II. The baseline cdf $F_0(t)$ is unknown.

In this case, α_j^* for $1\leqslant j\leqslant r$ in the equation (3.6) depend on the unknown baseline cdf $F_0(t)$. If the cdf $F_0(t)$ belongs to a parametric family of distributions, say $\mathcal{P}=\{F_{\underline{\theta}}:\underline{\theta}\in\Theta\}$, then we must maximize the function ℓ^* with respect to $\underline{\theta}$, where ℓ^* is obtained by replacing α_j with α_j^* in (3.5), i.e.,

$$(3.7)\qquad \ell^* = s\ln c_p + s\sum_{j=1}^{r}\ln\alpha_j^* + \sum_{j=1}^{r}\sum_{i=1}^{s}\ln h_0(x_{ij}) - \sum_{j=1}^{r}\alpha_j^*(1+m_j)A_j.$$

Hence, the ML estimate of $\underline{\theta}$ is obtained by the *plug-in* method. We illustrate the proposed procedure for some parametric families of distributions in the next section.

4. SOME PARAMETRIC FAMILIES OF DISTRIBUTIONS

In this section, we study some parametric families of distributions in more detail.

4.1. The Weibull model. The Weibull distribution is one of the most commonly lifetime distributions in the reliability engineering studies. Because of its flexible shape and ability to model a wide range of failure rates, it has been used successfully in many applications as a purely empirical model. It is appropriate to describe the relationship between failure times under the accelerated conditions and normal operating conditions. The two-parameter Weibull model has the cdf

$$F_0(t) = 1 - \exp\left[- \left(\frac{t}{\sigma} \right)^{\beta} \right], \quad t \geqslant 0,\ \sigma > 0,\ \beta > 0, \tag{4.1}$$

where σ and β are called the *scale* and *shape parameters*, respectively. For more details, see [16]. The problem of estimation of the parameters of the Weibull distribution for a random sample has been extensively studied in the literature. See, e.g., [7], [12], [17]–[22], [6], [4], and [2]. For a general class of distributions including the Weibull distribution, Cramer and Kamps [8] considered the problem of estimation for the parameters based on the SOS data. Here, we consider this problem on the basis of the PSOS data. Substituting (4.1) into (3.6), we obtain

$$\alpha_j^* = \frac{s\sigma^{\beta}}{1+m_j} \Big(\sum_{i=1}^{s} [x_{ij}^{\beta} - x_{i,j-1}^{\beta}] \Big)^{-1}, \quad 1 \leqslant j \leqslant r. \tag{4.2}$$

Substituting (4.1) and (4.2) into (3.7), we have

$$\ell^*(\beta, \sigma) = rs[\ln(s\beta) - 1] - s \sum_{j=1}^{r} \ln \sum_{i=1}^{s} (x_{ij}^{\beta} - x_{i,j-1}^{\beta}) + (\beta - 1) \sum_{j=1}^{r} \sum_{i=1}^{s} \ln x_{ij},$$

which depends only on β. Thus, the ML estimate for σ is not available.

Another form for the two-parameter Weibull distribution has the cdf

$$F_0(t) = 1 - \exp[-\lambda t^{\beta}], \quad t \geqslant 0,\ \lambda > 0,\ \beta > 0. \tag{4.3}$$

Similarly, the upper bound ℓ^* in (3.7) does not depend on the parameter λ, and hence the ML estimate of λ is not available. To see this, by substituting (4.3) into (3.6), we obtain

$$\alpha_j^* = \frac{s}{\lambda(1+m_j)} \Big(\sum_{i=1}^{s} [x_{ij}^{\beta} - x_{i,j-1}^{\beta}] \Big)^{-1}, \quad 1 \leqslant j \leqslant r. \tag{4.4}$$

Then, putting (4.4) into (3.7), we conclude that

$$\ell^*(\beta, \lambda) = rs[\ln(s\beta) - 1] - s \sum_{j=1}^{r} \ln \sum_{i=1}^{s} (x_{ij}^{\beta} - x_{i,j-1}^{\beta}) + (\beta - 1) \sum_{j=1}^{r} \sum_{i=1}^{s} \ln x_{ij},$$

which depends only on β. Thus, the ML estimate for λ is not available.

If either σ or λ is unknown, we showed that there exist no ML estimates of α_j ($1 \leqslant j \leqslant r$). We shall prove that for the new reparameterizing $\tilde{\alpha}_j = \lambda\alpha_j,\ 1 \leqslant j \leqslant r$, there exist ML estimates of $\tilde{\alpha}_j$ provided that $s > 1$. To show this, we first give some lemmas.

LEMMA 4.1. *Let* $0 < b < a$, $0 < d < c$, $a < c$, $0 < b_i < a_i$, $1 \leqslant i \leqslant s$, *and* $a_{(s)} = \max_{1 \leqslant i \leqslant s} a_i$. *Then*

$$\text{(i)} \qquad \lim_{x \to \infty} \frac{a^x \ln a - b^x \ln b}{a^x - b^x} = \ln a,$$

$$\text{(ii)} \qquad \lim_{x \to \infty} \frac{a^x \ln a - b^x \ln b}{c^x - d^x} = 0,$$

$$\text{(iii)} \qquad \lim_{x \to \infty} \frac{\sum_{i=1}^{s} (a_i^x \ln a_i - b_i^x \ln b_i)}{\sum_{i=1}^{s} (a_i^x - b_i^x)} = \ln a_{(s)}.$$

LEMMA 4.2 (Cramer and Kamps [8]). *Let* $a_1, \ldots, a_s,\ b_1, \ldots, b_s \in \mathbb{R}$ *with* $a_i \geqslant b_i,\ 1 \leqslant i \leqslant s$, *and let*

$$S(k) = \sum_{i=1}^{s} (a_i^k e^{a_i} - b_i^k e^{b_i}), \qquad k = 0, 1, 2.$$

Then, we have $S(2)\ S(0) - S^2(1) + S^2(0) \geqslant 0$ *with equality if and only if* $a_i = b_i$ *for all* $1 \leqslant i \leqslant s$.

PROPOSITION 4.1. *Let* $F(t) = 1 - \exp[-\lambda t^{\beta}],\ t \geqslant 0,\ \lambda > 0,\ \beta > 0$, *and* $\tilde{\alpha}_j = \lambda \alpha_j,\ 1 \leqslant j \leqslant r$. *Then, based on the PSOS data,*

(a) *for* $s = 1$, *there exist no ML estimates for the parameters*;

(b) *for* $s > 1$, *the ML estimates for* $\tilde{\alpha}_j$ *are*

$$\tilde{\alpha}_j^* = \frac{s}{1 + m_j} \Big(\sum_{i=1}^{s} [x_{ij}^{\beta} - x_{i,j-1}^{\beta}] \Big)^{-1}, \qquad 1 \leqslant j \leqslant r, \tag{4.5}$$

and the ML estimate of the shape parameter β *is obtained numerically from the equation*

$$-s \sum_{j=1}^{r} \frac{\sum\limits_{i=1}^{s} (x_{ij}^{\beta} \ln x_{ij} - x_{i,j-1}^{\beta} \ln x_{i,j-1})}{\sum\limits_{i=1}^{s} (x_{ij}^{\beta} - x_{i,j-1}^{\beta})} + \frac{rs}{\beta} + \sum_{j=1}^{r} \sum_{i=1}^{s} \ln x_{ij} = 0.$$

REMARK 4.1. *We showed that the ML estimates of the parameters* $\lambda \alpha_j$ $(1 \leqslant j \leqslant r)$ *exist whereas* λ *and* α_j $(1 \leqslant j \leqslant r)$ *themselves are not ML-estimable. As mentioned by an anonymous referee, this problem lies in identifiability of parameters under the CPHR PSOS model. More precisely, in the LF* (3.3), *if we take the baseline cdf as* $\tilde{F}_0 = 1 - (1 - F_0)^{\gamma}$, *and* $\tilde{\alpha}_j = \alpha_j / \gamma$ $(1 \leqslant j \leqslant r)$ *for every* $\gamma > 0$, *then the same LF in* (3.3) *is obtained, i.e.*,

$$L(\boldsymbol{\alpha}, F_0; \mathbf{x}) = L\big(\boldsymbol{\alpha}/\gamma, 1 - (1 - F_0)^{\gamma}; \mathbf{x}\big) \quad \textit{for all } \gamma > 0, \tag{4.6}$$

where $\boldsymbol{\alpha}/\gamma = (\alpha_1/\gamma, \dots, \alpha_r/\gamma)$. *Therefore the ML estimates* $\tilde{\alpha}_j^*$ $(1 \leqslant j \leqslant r)$ *in* (4.5) *are useless when the parameter* λ *is unknown. Hence, we need that the parameter* λ *in the baseline cdf* (4.3) *is known for practical applications of the obtained results in this subsection and its generalizations in Subsections* 4.2 *and* 4.3.

REMARK 4.2. *For* $\beta = 1$, *the Weibull distribution in* (4.3) *reduces to the one-parameter exponential distribution with mean* $1/\lambda$ *and the ML estimates of* α_j *and* $\tilde{\alpha}_j$ *are obtained from* (4.4) *and* (4.5), *respectively, by setting* $\beta = 1$.

4.2. A general family of scale distributions. Cramer and Kamps [8] considered a general family of distributions of the form

$$F_0(t) = 1 - e^{-\lambda g(t)}, \quad t \geqslant 0,\ \lambda > 0, \tag{4.7}$$

where $g(t)$ is a known, increasing and differentiating function on $[0, \infty)$, satisfying $g(0) = 0$ and $\lim_{t\to\infty} g(t) = \infty$. For $g(t) = t^\beta$ and $g(t) = \ln(t/\sigma)$, the Weibull and the Pareto distributions are obtained, respectively. They proved that the ML estimates of α_j for $1 \leqslant j \leqslant r$ based on the SOS data do exist, while for λ do not exist. Then, they derived ML estimates of $\tilde{\alpha}_j = \lambda\alpha_j$ for $1 \leqslant j \leqslant r$.

Here, we show that the ML estimate for λ in (4.7) based on a single PSOS data does not exist as we showed for the Weibull distribution in Subsection 4.1. Substituting (4.7) into (3.6) and, by convention, $x_{i,0} \equiv 0$, $1 \leqslant i \leqslant s$, we have

$$\alpha_j^* = \frac{s}{\lambda(1+m_j)}\Big(\sum_{i=1}^{s}[g(x_{ij}) - g(x_{i,j-1})]\Big)^{-1}, \quad 1 \leqslant j \leqslant r. \tag{4.8}$$

From (4.7) it is clear that $H_0(t) = \lambda g(t)$ and $h_0(t) = \lambda g'(t)$, where $g'(t) = \frac{d}{dt}g(t)$. Then, by substituting (4.8) into (3.7), we see that the associated LLF is simplified to the following:

$$\begin{aligned}\ell^*(\lambda) = rs\ln s - s\sum_{j=1}^{r}\ln\Big(\sum_{i=1}^{s}[g(x_{ij}) - g(x_{i,j-1})]\Big)\\ + \sum_{j=1}^{r}\sum_{i=1}^{s}\ln[g'(x_{ij})] - rs.\end{aligned} \tag{4.9}$$

The LLF in (4.9) does not depend on the parameter λ. If λ is unknown, the ML estimates of the unknown parameters do not exist.

As for the Weibull distribution in Subsection 4.1, one can obtain the ML estimator for the new parameters $\tilde{\alpha}_j = \lambda\alpha_j$ in the form

$$\tilde{\alpha}_j^* = \frac{s}{1+m_j}\Big(\sum_{i=1}^{s}[g(x_{ij}) - g(x_{i,j-1})]\Big)^{-1}, \quad 1 \leqslant j \leqslant r. \tag{4.10}$$

Therefore, we summarize the above statements in the following theorem.

THEOREM 4.1. *The ML estimates of* $\tilde{\alpha}_j$, $1 \leqslant j \leqslant r$, *based on the PSOS data drawn from the baseline cdf* (4.7), *are unique and given by* (4.10).

4.3. A general family of scale and shift distributions. Another family of distributions, that is an extension of the family (4.7) studied by [8], is

$$F_0(t) = 1 - e^{-\lambda[g(t)-\eta]}, \quad t \geqslant g^{-1}(\eta),\ \lambda > 0,\ \eta \in \mathbb{R}, \tag{4.11}$$

where $g : [a, b] \mapsto \mathbb{R}$, $-\infty \leqslant a < b \leqslant +\infty$, is a known, increasing and differentiable function with $g(a) = -\infty$ and $g(b) = +\infty$. For special cases $g(t) = t$ and $g(t) = \ln t^a,\ a > 0$, the two-parameter exponential and the two-parameter Pareto distributions are obtained, respectively. By convention, $x_{i,0} \equiv g^{-1}(\eta)$ $(1 \leqslant j \leqslant r)$, and substituting (4.11) into (3.6), we obtain

$$\alpha_j^* = \frac{s}{\lambda(1+m_j)}\Big(\sum_{i=1}^{s}[g(x_{ij}) - g(x_{i,j-1})]\Big)^{-1}, \quad 1 \leqslant j \leqslant r,\ s \geqslant 2. \tag{4.12}$$

Substituting (4.12) into (3.4), we see that the upper bound of $\ell^*(\lambda, \eta)$ is reduced to (4.9), which does not depend on the parameter λ. Then, if λ is unknown, the ML estimates of the unknown parameters λ and α_j $(1 \leqslant j \leqslant r)$, based on the PSOS data drawn from the baseline cdf (4.11), do not exist.

On the other hand, $\ell^*(\lambda, \eta)$ increases with respect to η. Then, the ML estimate of η is

$$\eta^* = \min\{g(x_{i1}),\ 1 \leqslant i \leqslant s\}. \tag{4.13}$$

By multiplying two sides of equation (4.12) by λ, the ML estimate for the new parameter $\tilde{\alpha}_j = \lambda\alpha_j$ reads

$$\tilde{\alpha}_j^* = \frac{s}{1+m_j}\Big(\sum_{i=1}^{s}[g(x_{ij}) - g(x_{i,j-1})]\Big)^{-1}, \quad 1 \leqslant j \leqslant r, \tag{4.14}$$

where, by convention, $g(x_{i,0}) = \eta^*$ for $1 \leqslant j \leqslant r$. Therefore, we summarize the above statements in the following theorem.

PROPOSITION 4.2. *The ML estimates of η and $\tilde{\alpha}_j$, $1 \leqslant j \leqslant r$, based on the PSOS data drawn from the baseline cdf* (4.11), *are unique and given by* (4.13) *and* (4.14), *respectively.*

5. CONCLUSIONS

In this paper, we discussed an extension of the SOS and investigated it under a special CPHR model in more detail. The problem of estimating parameters under the CPHR model is considered when individual lifetimes follow some well-known lifetime distributions. The results of this paper may be extended to some other parametric families of lifetime models such as the log-normal and the generalized Pareto distributions. For generating a PSOS sample of size r, we provide a proposition which is an extension of Theorem 1 in [3].

PROPOSITION 5.1. *Let $X^*_{(1)}, \ldots, X^*_{(r)}$ be a PSOS data under the CPHR model with the baseline cdf $F_0(x) = 1 - e^{-x}$, $x \geqslant 0$. Then the random variables $Z_1 = m_1\alpha_1 X^*_{(1)}$ and $Z_j = m_j\alpha_j(X^*_{(j)} - X^*_{(j-1)})$, $1 \leqslant j \leqslant r$, are iid according to $F_0(x)$.*

By Proposition 5.1, a PSOS data drawn from an arbitrary baseline distribution F_0 may be generated. To this end, one can generate r iid random variables from the standard uniform distribution, denoted by $U_1, \ldots, U_r$. Letting, for $1 \leqslant j \leqslant r$,

$$Z_j = -\ln(1 - U_j), \qquad X^*_{(j)} = \sum_{k=1}^{j} \frac{Z_k}{m_k\alpha_k}, \qquad \text{and} \qquad Y^*_{(j)} = F_0^{-1}[1 - \exp(-X^*_{(j)})],$$

it follows that $(Y^*_{(1)}, \ldots, Y^*_{(r)})$ is a PSOS sample of size r coming from the baseline cdf F_0. To assess the performance of the estimations obtained, one may conduct some simulation studies.

6. APPENDIX

Proof of Lemma 4.1. To prove (i) and (ii), after some simple algebraic manipulations we have

$$\lim_{x\to\infty} \frac{a^x \ln a - b^x \ln b}{a^x - b^x} = \lim_{x\to\infty} \frac{\ln a - (b/a)^x \ln b}{1 - (b/a)^x} = \ln a,$$

and

$$\lim_{x\to\infty} \frac{a^x \ln a - b^x \ln b}{c^x - d^x} = \lim_{x\to\infty} \left(\frac{a}{c}\right)^x \frac{\ln a - (b/a)^x \ln b}{1 - (d/c)^x} = 0.$$

For (iii), we have

$$\lim_{x\to\infty} \frac{\sum_{i=1}^{s}(a_i^x \ln a_i - b_i^x \ln b_i)}{\sum_{i=1}^{s}(a_i^x - b_i^x)} = \lim_{x\to\infty} \frac{\sum_{i=1}^{s}[(a_i/a_{(s)})^x \ln a_i - (b_i/a_{(s)})^x \ln b_i]}{\sum_{i=1}^{s}[(a_i/a_{(s)})^x - (b_i/a_{(s)})^x]} = \ln a_{(s)},$$

and the proof is completed. ■

Proof of Proposition 4.1. (a) Obviously, for $s = 1$ the number of observations is less than the number of parameters, so there exist no ML estimates for $\tilde{\alpha}_j$, $1 \leqslant j \leqslant r$, λ and β.

(b) By substituting (4.3) into (3.4), the associated LLF takes the form

$$\ell(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_r, \beta) = s \ln c_p + s \sum_{j=1}^{r} \ln \tilde{\alpha}_j + sr \ln \beta + (\beta - 1) \sum_{j=1}^{r} \sum_{i=1}^{s} \ln x_{ij} - \sum_{j=1}^{r} (1 + m_j)\tilde{\alpha}_j \sum_{i=1}^{s} (x_{ij}^{\beta} - x_{i,j-1}^{\beta}). \tag{6.1}$$

Let $\tilde{\alpha}_j^* = s\big[(1+m_j)\sum_{i=1}^{s}(x_{ij}^{\beta} - x_{i,j-1}^{\beta})\big]^{-1}$ for $1 \leqslant j \leqslant r$. Then

$$\ell(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_r, \beta) \leqslant \ell(\tilde{\alpha}_1^*, \ldots, \tilde{\alpha}_r^*, \beta) \equiv u(\beta). \tag{6.2}$$

Notice that

$$u(\beta) = rs[\ln(s\beta) - 1] - s\sum_{j=1}^{r} \ln\Big[\sum_{i=1}^{s}(x_{ij}^{\beta} - x_{i,j-1}^{\beta})\Big] + (\beta - 1)\sum_{j=1}^{r}\sum_{i=1}^{s} \ln x_{ij},$$

and

$$\frac{\partial u(\beta)}{\partial \beta} = -s\sum_{j=1}^{r} \frac{\sum\limits_{i=1}^{s}(x_{ij}^{\beta}\ln x_{ij} - x_{i,j-1}^{\beta}\ln x_{i,j-1})}{\sum\limits_{i=1}^{s}(x_{ij}^{\beta} - x_{i,j-1}^{\beta})} + \frac{rs}{\beta} + \sum_{j=1}^{r}\sum_{i=1}^{s}\ln x_{ij}, \tag{6.3}$$

where, by convention, $x_{i,0} = 0$ and $\ln x_{i,0} = 0$ for $1 \leqslant i \leqslant s$. For $s = 1$, the equation (6.3) reduces to

$$\frac{\partial u(\beta)}{\partial \beta} = \frac{r}{\beta} - \sum_{j=2}^{r} \frac{\ln(x_{1j}/x_{1,j-1})}{(x_{1j}/x_{1,j-1})^{\beta} - 1}.$$

It is obvious that $x_{1j}/x_{1,j-1} \geqslant 1,\ 2 \leqslant j \leqslant r$. Using the inequality $e^{\beta z} \geqslant \beta z + 1$, $z \geqslant 0$, and letting $z = \ln y$, we have $y^{\beta} \geqslant \beta \ln y + 1$ and $(\ln y)/(y^{\beta} - 1) \leqslant 1/\beta$ for $y \geqslant 1$, which leads immediately to

$$\frac{\partial u(\beta)}{\partial \beta} \geqslant \frac{r}{\beta} - \sum_{j=2}^{r}\frac{1}{\beta} = \frac{1}{\beta} > 0.$$

This means that $u(\beta)$ is an increasing function of β and the ML estimate of β does not exist.

For $s > 1$, we claim that

$$\text{(i)}\ \lim_{\beta\to 0^+}\frac{\partial u(\beta)}{\partial \beta} = \infty, \quad \text{(ii)}\ \lim_{\beta\to\infty}\frac{\partial u(\beta)}{\partial \beta} < 0, \quad \text{and} \quad \text{(iii)}\ \frac{\partial^2 u(\beta)}{\partial \beta^2} \leqslant 0.$$

These statements imply that the MLE of β is unique.

To prove (i), from equation (6.3) we see that

$$\frac{\partial u(\beta)}{\partial \beta} = K(\beta) + \sum_{j=1}^{r}\sum_{i=1}^{s}\ln x_{ij},$$

where

$$K(\beta) = -s\sum_{j=1}^{r} \frac{\sum\limits_{i=1}^{s}(x_{ij}^{\beta}\ln x_{ij} - x_{i,j-1}^{\beta}\ln x_{i,j-1})}{\sum\limits_{i=1}^{s}(x_{ij}^{\beta} - x_{i,j-1}^{\beta})} + \frac{rs}{\beta}.$$

We show that $\lim_{\beta\to 0^+} K(\beta) = +\infty$. Notice that

$$\begin{aligned}
\text{(6.4)}\qquad \lim_{\beta\to 0^+} K(\beta) &= -s \lim_{\beta\to 0^+} \sum_{j=1}^{r} \left[\frac{\sum_{i=1}^{s} (x_{ij}^{\beta} \ln x_{ij} - x_{i,j-1}^{\beta} \ln x_{i,j-1})}{\sum_{i=1}^{s} (x_{ij}^{\beta} - x_{i,j-1}^{\beta})} - \frac{1}{\beta} \right] \\
&= -s \lim_{\beta\to 0^+} \left[\frac{\sum_{i=1}^{s} (x_{i1}^{\beta} \ln x_{i1} - x_{i0}^{\beta} \ln x_{i0})}{\sum_{i=1}^{s} (x_{i1}^{\beta} - x_{i0}^{\beta})} - \frac{1}{\beta} \right] \\
&\quad - s \lim_{\beta\to 0^+} \sum_{j=2}^{r} \left[\frac{\sum_{i=1}^{s} (x_{ij}^{\beta} \ln x_{ij} - x_{i,j-1}^{\beta} \ln x_{i,j-1})}{\sum_{i=1}^{s} (x_{ij}^{\beta} - x_{i,j-1}^{\beta})} - \frac{1}{\beta} \right] \\
&= -s \lim_{\beta\to 0^+} \left[\frac{\sum_{i=1}^{s} x_{i1}^{\beta} \ln x_{i1}}{\sum_{i=1}^{s} x_{i1}^{\beta}} - \frac{1}{\beta} \right] - s \sum_{j=2}^{r} \lim_{\beta\to 0^+} B_j(\beta),
\end{aligned}$$

where

$$B_j(\beta) = \frac{\beta \sum_{i=1}^{s} (x_{ij}^{\beta} \ln x_{ij} - x_{i,j-1}^{\beta} \ln x_{i,j-1}) - \sum_{i=1}^{s} (x_{ij}^{\beta} - x_{i,j-1}^{\beta})}{\beta \sum_{i=1}^{s} (x_{ij}^{\beta} - x_{i,j-1}^{\beta})}.$$

It is clear that the first term in (6.4) goes to infinity as β vanishes. It suffices to show that the last expression in (6.4) is finite. To see this, we consider

$$\lim_{\beta\to 0^+} B_j(\beta) = \lim_{\beta\to 0^+} \frac{\sum_{i=1}^{s} x_{ij}^{\beta}(\beta \ln x_{ij} - 1) - \sum_{i=1}^{s} [x_{i,j-1}^{\beta}(\beta \ln x_{i,j-1} - 1)]}{\beta \sum_{i=1}^{s} (x_{ij}^{\beta} - x_{i,j-1}^{\beta})}.$$

Using the l'Hospital rule, one can write

$$\begin{aligned}
\lim_{\beta\to 0^+} B_j(\beta) &= \lim_{\beta\to 0^+} \frac{\beta \sum_{i=1}^{s} [x_{ij}^{\beta} (\ln x_{ij})^2 - x_{i,j-1}^{\beta} (\ln x_{i,j-1})^2]}{\sum_{i=1}^{s} (x_{i,j-1}^{\beta} - x_{i,j-1}^{\beta}) + \beta \sum_{i=1}^{s} (x_{ij}^{\beta} \ln x_{ij} - x_{i,j-1}^{\beta} \ln x_{i,j-1})} \\
&= \frac{\sum_{i=1}^{s} [(\ln x_{ij})^2 - (\ln x_{i,j-1})^2]}{\sum_{i=1}^{s} (\ln x_{ij} - \ln x_{i,j-1})}.
\end{aligned}$$

To prove (ii), by Lemma 4.1 (iii), one can see that

$$\lim_{\beta\to\infty}\frac{\partial u(\beta)}{\partial\beta}=\sum_{j=1}^{r}\sum_{i=1}^{s}\ln x_{ij}-s\sum_{j=1}^{r}\lim_{\beta\to\infty}\frac{\sum\limits_{i=1}^{s}(x_{ij}^{\beta}\ln x_{ij}-x_{i,j-1}^{\beta}\ln x_{i,j-1})}{\sum\limits_{i=1}^{s}(x_{ij}^{\beta}-x_{i,j-1}^{\beta})}$$
$$=\sum_{j=1}^{r}\sum_{i=1}^{s}\ln x_{ij}-s\sum_{j=1}^{r}\max_{1\leqslant k\leqslant s}\ln x_{kj}$$
$$=\sum_{j=1}^{r}\sum_{i=1}^{s}(\ln x_{ij}-\max_{1\leqslant k\leqslant s}\ln x_{kj})\leqslant 0.$$

To prove (iii), we compute the second derivative of $u(\beta)$ with respect to β from equation (6.3):

$$\frac{\partial^2 u(\beta)}{\partial\beta^2}=-s\sum_{j=1}^{r}\big[\sum_{i=1}^{s}(x_{ij}^{\beta}-x_{i,j-1}^{\beta})\big]^{-2}$$
$$\times\Big\{\sum_{i=1}^{s}\big(x_{ij}^{\beta}(\ln x_{ij})^2-x_{i,j-1}^{\beta}(\ln x_{i,j-1})^2\big)\sum_{i=1}^{s}(x_{ij}^{\beta}-x_{i,j-1}^{\beta})$$
$$-\big[\sum_{i=1}^{s}(x_{ij}^{\beta}\ln x_{ij}-x_{i,j-1}^{\beta}\ln x_{i,j-1})\big]^2\Big\}-\frac{rs}{\beta^2}.$$

Applying Lemma 4.2, we conclude that

$$\frac{\partial^2 u(\beta)}{\partial\beta^2}=\frac{-s}{\beta^2}\sum_{j=1}^{r}\frac{1}{S^2(0)}[S(2)S(0)-S^2(1)+S^2(0)]\leqslant 0.$$

Finally, by substituting the ML estimate of β into (6.1), and then by derivation, we obtain the desired result. ■

P r o o f o f P r o p o s i t i o n 5.1. Putting $F_0(x)=1-e^{-x}$ in equation (2.6), we have

$$L(\boldsymbol{\alpha};\mathbf{x})=c_p\prod_{j=1}^{r}\{\alpha_j\exp[-m_j\alpha_j(x_j-x_{j-1})]\}.$$

Since

$$X_{(1)}^{*}=\frac{Z_1}{m_1\alpha_1},\quad X_{(2)}^{*}=\frac{Z_1}{m_1\alpha_1}+\frac{Z_2}{m_2\alpha_2},\quad\ldots,\quad X_{(r)}^{*}=\frac{Z_1}{m_1\alpha_1}+\ldots+\frac{Z_r}{m_r\alpha_r},$$

the corresponding Jacobian simplifies to $|J|=\prod_{j=1}^{r}\frac{1}{m_j\alpha_j}$. Therefore, the joint pdf of $Z_1,\ldots,Z_r$ is

$$f_{Z_1,\ldots,Z_r}(z_1,\ldots,z_r)=c_p\prod_{j=1}^{r}\{\alpha_j\exp[-z_j]\}\frac{1}{\prod_{j=1}^{r}m_j\alpha_j}=\exp\big[-\sum_{j=1}^{r}z_j\big],$$

and the desired result follows. ■

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