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NOTES ON THE KRUPA–ZAWISZA ULTRAPOWER OF SELF-ADJOINT OPERATORS

BY

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Abstract. Let $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ be a free ultrafilter on $\mathbb{N}$. It is known that there is a difficulty in constructing the ultrapower of unbounded operators. Krupa and Zawisza gave a rigorous definition of the ultrapower A_ω of a self-adjoint operator A. In this note, we give an alternative description of A_ω and the Hilbert space $H(A)$ on which A_ω is densely defined. This provides a criterion to determine a representing sequence $(\xi_n)_n$ of a given vector $\xi \in \mathrm{dom}(A_\omega)$ which has the property that $A_\omega \xi = (A\xi_n)_\omega$ holds. An explicit core for A_ω is also described.

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1. INTRODUCTION

Throughout the paper, we fix a free ultrafilter ω on $\mathbb{N}$ and a separable infinite-dimensional Hilbert space H. We denote by $\mathbb{B}(H)$ the algebra of all bounded operators in H. Let H_ω be the Hilbert space ultraproduct of H. Each bounded sequence $(a_n)_n \subset \mathbb{B}(H)$ of bounded operators in H defines a bounded operator $(a_n)_\omega \in \mathbb{B}(H)$, called the *ultraproduct* of $(a_n)_n$, by the formula

$$(a_n)_\omega (\xi_n)_\omega := (a_n \xi_n)_\omega, \quad (\xi_n)_\omega \in H_\omega.$$

The ultrapower (or, more generally, the ultraproduct) of a sequence of bounded operators has been used as an efficient tool for the analysis on Hilbert spaces. In view of its usefullness, it is natural to consider a corresponding notion of ultrapower A_ω for an unbounded self-adjoint operator A. However, there arise essential difficulties in connection with the following issues:

(1) definition of the domain $\mathrm{dom}(A_\omega)$ of A_ω;
(2) self-adjointness of A_ω;
(3) interpretation of $A_\omega(\xi_n)_\omega = (A\xi_n)_\omega$ for $\xi = (\xi_n)_\omega \in \mathrm{dom}(A_\omega)$.

Regarding (1), it does not make sense to define $\mathrm{dom}(A_\omega)$ to be the subspace $\mathrm{dom}(A)_\omega$ of all $\xi \in H_\omega$ which is represented by a sequence $(\xi_n)_n$, where $\xi_n \in \mathrm{dom}(A)$ for all n, because $\mathrm{dom}(A)_\omega$ is simply the whole H_ω and $A_\omega(\xi_n)_\omega = (A\xi_n)_\omega$ is not well-defined. Importance of the question (2) should be clear. The problem (3) is probably the most delicate. Even if we could manage to define $\mathrm{dom}(A_\omega)$ and suppose $\xi \in \mathrm{dom}(A_\omega)$ is represented by $(\xi_n)_n$ with $\xi_n \in \mathrm{dom}(A)$ for all n, it might be the case where there exists another $(\xi_n')_n$ which also represents ξ (i.e., $\lim_{n\to\omega} \|\xi_n - \xi_n'\| = 0$ holds), and $\xi_n' \in \mathrm{dom}(A)$ for all n holds as well, and yet $(A\xi_n)_\omega \neq (A\xi_n')_\omega$.

EXAMPLE 1.1. Let A be a self-adjoint operator and assume that there is an orthonormal base $\{\eta_n\}_{n=1}^\infty$ of H consisting of eigenvectors of A with $A\eta_n = n\eta_n$, $n \geqslant 1$. Let $\eta \in \mathrm{dom}(A)$, and consider two sequences

$$\xi_n := \eta, \quad \xi_n' := \eta + \frac{1}{n}\eta_n \quad (n \geqslant 1).$$

Then it is clear that $\xi_n, \xi_n' \in \mathrm{dom}(A)$, that $(\xi_n)_n, (\xi_n')_n$ define the same element $\xi = (\xi_n)_\omega = (\xi_n')_\omega \in H_\omega$, but

$$\lim_{n\to\omega} \|A\xi_n - A\xi_n'\| = \lim_{n\to\omega} \|\eta_n\| = 1 \neq 0,$$

whence $(A\xi_n)_\omega \neq (A\xi_n')_\omega$. Should we define $A_\omega\xi = (A\xi_n)_\omega$ or $A_\omega\xi = (A\xi_n')_\omega$?

Despite the above difficulty, Krupa and Zawisza [3], [4] gave a rigorous definition of A_ω, as well as interesting applications to Schrödinger operators. To define $\mathrm{dom}(A_\omega)$ in any sensible way, it is necessary to note that such a domain must be in the subspace of $\mathscr{D}_A$, given as the set of all $\xi \in H_\omega$ which has a representing sequence $(\xi_n)_n$ of vectors from $\mathrm{dom}(A)$ such that $(A\xi_n)_n$ is also norm-bounded. We put $H(A) = \overline{\mathscr{D}_A}$. We recall from [4] the notion of partial ultrapowers.

DEFINITION 1.1. Let $\mathcal{H} \subset H_\omega$ be a closed subspace. A densely defined operator $\mathscr{A}$ in $\mathcal{H}$ is called a *partial ultrapower* (p.u. for short) of A in $\mathcal{H}$ if for any $\xi \in \mathrm{dom}(\mathscr{A})$ there is $(\xi_n)_n \subset \mathrm{dom}(A)$ such that $\xi = (\xi_n)_\omega$ and $\mathscr{A}\xi = (A\xi_n)_\omega$.

One of the fundamental results of Krupa and Zawisza [4] is the following:

THEOREM 1.1. (1) *There is a p.u.* A_ω *of* A *in* $H(A)$ *satisfying* $\mathrm{dom}(A_\omega) = \mathscr{D}_A$, *uniquely determined by the property that for* $\xi \in \mathscr{D}_A$ *and* $\eta \in H(A)$, $A_\omega\xi = \eta$ *if and only if there is a representative* $(\xi_n)_n \subset \mathrm{dom}(A)$ *of* ξ *satisfying* $(A\xi_n)_\omega = \eta$.

(2) A_ω *is the maximal among all p.u.'s of A. That is, if* $\mathscr{A}$ *is a p.u. of* A *in* $\mathcal{H}$, *then* $\mathcal{H} \subset H(A)$ *and* $\mathscr{A} = A_\omega|_{\mathrm{dom}(\mathscr{A})}$.

(3) A_ω *is self-adjoint in* $H(A)$. *Moreover*, $(A_\omega - i)^{-1}$ *is the restriction of* $\left((A-i)^{-1}\right)_\omega$ *to* $H(A)$ *and* $\mathrm{sp}(A_\omega) = \mathrm{sp}(A)$ *holds.*

Note that in (1), the uniqueness of η is guaranteed by the condition $\eta \in H(A)$. Indeed, in Example 1.1, $(A\xi_n)_\omega \in H(A)$, while $(A\xi_n')_\omega \notin H(A)$ (see Remark 4.1

below). Despite their success, what seems to be unsatisfactory is that there is no *a priori* criterion for a given $\xi \in \mathscr{D}_A$ to choose an appropriate representative $(\xi_n)_n$ such that $(A\xi_n)_\omega$ is well-defined and is in $H(A)$. Whether a chosen representative is indeed appropriate or not can be seen only after applying A and knowing that the resulting vector is in the closure of $\mathscr{D}_A$. In this short note, we give an alternative characterization of such an appropriate sequence, which will be called a *proper A-sequence*, and give a new description of A_ω in terms of an auxiliary operator $\widetilde{A}_\omega$ by checking the validity of the equality $A_\omega = \widetilde{A}_\omega$. More precisely, we show that a bounded sequence $(\xi_n)_n$ of vectors from $\mathrm{dom}(A)$ has a property that $A_\omega(\xi_n)_\omega = (A\xi_n)_\omega$ if and only if $(A\xi_n)_n$ is bounded and, for every $\varepsilon > 0$, there is $a > 0$, $(\eta_n)_n \in \ell^\infty(\mathbb{N}, H)$ with $\eta_n \in 1_{[-a,a]}(A)H$ for each $n \in \mathbb{N}$, such that $\lim_{n\to\omega} \|\xi_n - \eta_n\|_A < \varepsilon$. ($\|\cdot\|_A$ is the graph norm.) Moreover, a bounded sequence $(\xi_n)_n$ defines an element in $H(A)$ if and only if the family of maps $\{f_n : \mathbb{R} \to H\}_{n=1}^\infty$ given by $f_n(t) = e^{itA}\xi_n$ is ω-equicontinuous (see Definition 3.1). We believe that this description will make Krupa–Zawisza analyses more accessible and give a new insight into them.

2. PRELIMINARIES

Let $\ell^\infty(\mathbb{N}, H)$ be the space of all bounded sequences in H. The ultrapower H_ω of H is defined by $H_\omega = \ell^\infty(\mathbb{N}, H)/\mathcal{T}_\omega$, where $\mathcal{T}_\omega$ is the subspace of $\ell^\infty(\mathbb{N}, H)$ consisting of sequences tending to zero in norm along ω. The canonical image of $(\xi_n)_n \in \ell^\infty(\mathbb{N}, H)$ is written as $(\xi_n)_\omega$, and H_ω is again a Hilbert space (non-separable in general) by the inner product

$$\langle \xi, \eta \rangle = \lim_{n\to\omega} \langle \xi_n, \eta_n \rangle, \quad \xi = (\xi_n)_\omega,\ \eta = (\eta_n)_\omega \in H_\omega.$$

We identify $\xi \in H$ with its canonical image $(\xi, \xi, \ldots)_\omega \in H_\omega$, so that H is a closed subspace of H_ω. Let $\{a_n\}_{n=1}^\infty$ be a sequence of bounded operators on H. We then define a bounded operator $(a_n)_\omega \in \mathbb{B}(H_\omega)$ by

$$(a_n)_\omega(\xi_n)_\omega := (a_n\xi_n)_\omega, \quad (\xi_n)_\omega \in H_\omega.$$

$(a_n)_\omega$ is well-defined by the above, and $\|(a_n)_\omega\| = \lim_{n\to\omega} \|a_n\|$ holds. For a linear operator T on H, the domain of T is denoted by $\mathrm{dom}(T)$. For $\xi \in \mathrm{dom}(T)$, we denote by $\|\xi\|_T$ the graph norm of T given by $(\|\xi\|^2 + \|T\xi\|^2)^{1/2}$. For details about operator theory, see, e.g., [7].

3. CONSTRUCTION OF $\widetilde{A}_\omega$

Let A be a self-adjoint operator on a separable Hilbert space H, and let $u(t) = e^{itA}$ $(t \in \mathbb{R})$. We introduce several subspaces of H_ω. First, we need to introduce the notion of ω-equicontinuity which has been used in the literature (see [2], [5]).

DEFINITION 3.1. Let $(X_1, d_1),\ (X_2, d_2)$ be metric spaces. A family of maps $\{f_n\colon X_1 \to X_2\}_{n=1}^{\infty}$ is said to be *ω-equicontinuous* if for every $x \in X$ and $\varepsilon > 0$, there exists $\delta = \delta_{x,\varepsilon} > 0$ and $W \in \omega$ such that for every $x' \in X$ with $d_1(x, x') < \delta$ and $n \in W$, we have

$$d_2\big(f_n(x), f_n(x')\big) < \varepsilon.$$

LEMMA 3.1. *Let us assume that $(\xi_n)_n \in \ell^\infty(\mathbb{N}, H)$ is a sequence such that $\{f_n : t \mapsto e^{itA}\xi_n\}_{n=1}^{\infty}$ is ω-equicontinuous. Then $t \mapsto (e^{itA}\xi_n)_\omega$ is continuous. Moreover, if $(\xi'_n)_n \in \ell^\infty(\mathbb{N}, H)$ satisfies $\lim_{n\to\omega} \|\xi_n - \xi'_n\| = 0$, then the sequence $\{f'_n : t \mapsto e^{itA}\xi'_n\}_{n=1}^{\infty}$ is also ω-equicontinuous.*

Proof. Let $t \in \mathbb{R}$, and $\varepsilon > 0$ be given. There exists $\delta > 0$ and $W_1 \in \omega$ such that for any $s \in (t - \delta, t + \delta)$ and $n \in W_1$ we have $\|e^{itA}\xi_n - e^{isA}\xi_n\| < \varepsilon/3$. This means that $\|(e^{itA}\xi_n)_\omega - (e^{isA}\xi_n)_\omega\| < \varepsilon/3$, whence $t \mapsto (e^{itA}\xi_n)_\omega$ is continuous. By $(\xi_n)_\omega = (\xi'_n)_\omega$, it follows that $W_2 := \{n \in \mathbb{N};\, \|\xi_n - \xi'_n\| < \varepsilon/3\} \in \omega$. Then for $s \in (t - \delta, t + \delta)$ and $n \in W := W_1 \cap W_2 \in \omega$, we have

$$\begin{aligned}\|e^{itA}\xi'_n - e^{isA}\xi'_n\| &\leqslant \|e^{itA}(\xi'_n - \xi_n)\| + \|e^{itA}\xi_n - e^{isA}\xi_n\| + \|e^{isA}(\xi_n - \xi'_n)\| \\ &< \varepsilon.\end{aligned}$$

Therefore, $\{t \mapsto e^{itA}\xi'_n\}_{n=1}^{\infty}$ is ω-equicontinuous. ∎

DEFINITION 3.2. A vector $\xi = (\xi_n)_\omega \in H_\omega$ is called *A-regular* if the sequence $\{t \mapsto e^{itA}\xi_n\}_{n=1}^{\infty}$ is ω-equicontinuous. By Lemma 3.1, this notion does not depend on the choice of the representing sequence $(\xi_n)_n$.

DEFINITION 3.3. Under the above notation, we define the following:

(1) Let $K(A)$ be the set of all A-regular vectors of H_ω.

(2) Let $\mathrm{dom}(\widetilde{A}_\omega)$ be the set of $\xi \in K(A)$ for which $\lim_{t\to 0} \frac{1}{t}\big(u(t)_\omega - 1\big)\xi$ exists.

LEMMA 3.2. *$K(A)$ is a closed subspace of H_ω invariant under $u(t)_\omega$ for all $t \in \mathbb{R}$.*

Proof. It is clear that $K(A)$ is a subspace of H_ω, and that $K(A)$ is $u(t)_\omega$-invariant for all $t \in \mathbb{R}$. Let $\xi = (\xi_n)_\omega \in \overline{K(A)}$ and $\varepsilon > 0$. There exists $\eta = (\eta_n)_\omega \in K(A)$ such that $\|\xi - \eta\| < \varepsilon/3$. Let $t \in \mathbb{R}$. By the ω-equicontinuity of $\{f_n : t \mapsto e^{itA}\eta_n\}_{n=1}^{\infty}$, there exists $\delta > 0$ and $W_1 \in \omega$ such that for each $s \in (t - \delta, t + \delta)$ and $n \in W_1$, we have $\|e^{itA}\eta_n - e^{isA}\eta_n\| < \varepsilon/3$. Let $W_2 := \{n \in \mathbb{N};\, \|\xi_n - \eta_n\| < \varepsilon/3\} \in \omega$. Then, for $s \in (t - \delta, t + \delta)$ and $n \in W := W_1 \cap W_2 \in \omega$, we have

$$\begin{aligned}\|e^{itA}\xi_n - e^{isA}\xi_n\| &\leqslant \|e^{itA}(\xi_n - \eta_n)\| + \|e^{itA}\eta_n - e^{isA}\eta_n\| + \|e^{isA}(\eta_n - \xi_n)\| \\ &< \varepsilon.\end{aligned}$$

Therefore, $\xi = (\xi_n)_\omega$ is A-regular, and $\xi \in K(A)$. ∎

By Lemma 3.2, $v(t) := u(t)_\omega|_{K(A)}$ is a continuous one-parameter unitary group of $K(A)$. Therefore, by Stone's theorem, there exists a self-adjoint operator $\widetilde{A}_\omega$ with domain $\mathrm{dom}(\widetilde{A}_\omega)$ such that

$$i\widetilde{A}_\omega\xi = \lim_{t\to 0}\frac{1}{t}\big(v(t)-1\big)\xi, \quad \xi \in \mathrm{dom}(\widetilde{A}_\omega).$$

In the sequel, we will show that $\widetilde{A}_\omega\xi = (A\xi_n)_\omega$ for appropriate $(\xi_n)_n$ representing $\xi \in \mathrm{dom}(\widetilde{A}_\omega)$.

DEFINITION 3.4. Let A be a self-adjoint operator on H.

(1) A sequence $(\xi_n)_n \in \ell^\infty(\mathbb{N}, H)$ is called an A-*sequence* if $\xi_n \in \mathrm{dom}(A)$ for all $n \geqslant \mathbb{N}$. We denote the space of A-sequences by $\ell^\infty\big(\mathbb{N}, \mathrm{dom}(A)\big)$.

(2) An A-sequence $(\xi_n)_n$ is called *proper* if it satisfies the following condition:

$(*)$ For each $\varepsilon > 0$, there exists $a > 0$ and an A-sequence $(\eta_n)_n$ with the following properties:

(i) $\eta_n \in 1_{[-a,a]}(A)H$ for all $n \geqslant 1$.

(ii) $(A\xi_n)_n \in \ell^\infty(\mathbb{N}, H)$, and $\lim_{n\to\omega}\|\xi_n - \eta_n\|_A < \varepsilon$.

DEFINITION 3.5. As in [4], we let $\mathscr{D}_A$ be the set of all $\xi \in H_\omega$ which is represented by an A-sequence $(\xi_n)_n$ such that $(A\xi_n)_n$ is bounded, and let $H(A) = \overline{\mathscr{D}_A}$. We also define related subspaces: define $\widehat{\mathscr{D}}_A$ to be the space of all $\xi \in H_\omega$ which is represented by a proper A-sequence and also define $\mathscr{D}_0$ to be the set of all $\xi \in H_\omega$ which has a representative $(\xi_n)_n$ satisfying $\xi_n \in 1_{[-a,a]}(A)H$ for all $n \in \mathbb{N}$, where $a > 0$ is a constant independent of n.

It is clear that $\mathscr{D}_0 \subset \widehat{\mathscr{D}}_A \subset \mathscr{D}_A$.

The main result of the paper is that $\widehat{\mathscr{D}}_A = \mathscr{D}_A, K(A) = H(A)$, and $A_\omega = \widetilde{A}_\omega = \overline{A_\omega|_{\mathscr{D}_0}}$.

In this section we will show that

THEOREM 3.1. $\mathrm{dom}(\widetilde{A}_\omega) = \widehat{\mathscr{D}}_A \subset K(A)$, *and* $\mathscr{D}_0$ *is a core for* $\widetilde{A}_\omega$.

We need several lemmata. The following lemma justifies the choice of proper A-sequences to consider the ultrapower.

LEMMA 3.3. $\widehat{\mathscr{D}}_A \subset \mathrm{dom}(\widetilde{A}_\omega)$, *and for* $\xi \in \widehat{\mathscr{D}}_A$ *with a proper representative* $(\xi_n)_n$, *we have*

$$\widetilde{A}_\omega\xi = (A\xi_n)_\omega.$$

In particular, $(A\xi_n)_\omega = (A\xi_n')_\omega$ *if both* $(\xi_n)_n, (\xi_n')_n$ *are proper* A*-sequences representing the same vector* $\xi \in \widehat{\mathscr{D}}_A$.

P r o o f. We first show that $\widehat{\mathscr{D}}_A \subset K(A)$. Since $K(A)$ is closed and every element in $\widehat{\mathscr{D}}_A$ can be approximated by vectors of the form $(\eta_n)_\omega$, where $\eta_n \in$

$1_{[-a,a]}(A)H$ $(n \in \mathbb{N})$ for a fixed $a > 0$, it suffices to show that $\{t \mapsto e^{itA}\eta_n\}_{n=1}^{\infty}$ is ω-equicontinuous for such $(\eta_n)_\omega$. Let $\varepsilon > 0$ and $t \in \mathbb{R}$ be given. Let $A = \int_{\mathbb{R}} \lambda de(\lambda)$ be the spectral resolution of A. We have

$$\begin{aligned}\|e^{itA}\eta_n - e^{isA}\eta_n\|^2 &= \int_{\mathbb{R}} |e^{i(t-s)\lambda} - 1|^2 d\|e(\lambda)\eta_n\|^2 \\ &= 2\int_{\mathbb{R}} \Big(1 - \cos\big((t-s)\lambda\big)\Big) d\|e(\lambda)\eta_n\|^2 \\ &\leqslant \int_{[-a,a]} (t-s)^2\lambda^2 d\|e(\lambda)\eta_n\|^2 \\ &\leqslant (t-s)^2 a^2 \|\eta_n\|^2.\end{aligned}$$

Therefore, let $\delta > 0$ be such that $\delta^2 a^2 \sup_{n\geqslant 1}\|\eta_n\|^2 < \varepsilon^2$. Then, for each $n \in \mathbb{N}$ and $s \in (t-\delta, t+\delta)$, $\|e^{itA}\eta_n - e^{isA}\eta_n\| < \varepsilon$ holds. Therefore $(\eta_n)_\omega$ is A-regular and $\widehat{\mathscr{D}}_A \subset K(A)$ holds.

Next, let $\zeta := (iA\xi_n)_\omega$. We show that $\frac{1}{t}\big(v(t)-1\big)\xi$ converges to ζ as $t \to 0$. Let $\varepsilon > 0$. We may find $a > 0$ and $(\eta_n)_n$ satisfying the conditions in $(*)$ of Definition 3.4. Let $\eta = (\eta_n)_\omega$. Then we have

$$\begin{aligned}\left\|\frac{1}{t}\big(v(t)-1\big)\xi - \zeta\right\| \leqslant \left\|\frac{1}{t}\big(v(t)-1\big)(\xi - \eta)\right\| + \left\|\frac{1}{t}\big(v(t)-1\big)\eta - (iA\eta_n)_\omega\right\| \\ + \|(iA\eta_n)_\omega - (iA\xi_n)_\omega\|.\end{aligned}$$

By the condition $(*)$, the last term satisfies $\|(iA\eta_n)_\omega - (iA\xi_n)_\omega\| < \varepsilon$.

Now estimate the first term:

$$\begin{aligned}\left\|\frac{1}{t}\big(v(t)-1\big)(\xi - \eta)\right\|^2 &= \lim_{n\to\omega} \frac{1}{t^2}\int_{\mathbb{R}} |e^{it\lambda} - 1|^2 d\|e(\lambda)(\xi_n - \eta_n)\|^2 \\ &\leqslant \lim_{n\to\omega} \frac{1}{t^2}\int_{\mathbb{R}} t^2\lambda^2 d\|e(\lambda)(\xi_n - \eta_n)\|^2 \\ &= \|(A\xi_n)_\omega - (A\eta_n)_\omega\|^2 < \varepsilon^2.\end{aligned}$$

Using $\eta_n \in 1_{[-a,a]}(A)H$ $(n \geqslant 1)$, we then estimate the second term:

$$\begin{aligned}\left\|\frac{1}{t}\big(v(t)-1\big)\eta - (iA\eta_n)_\omega\right\|^2 &= \lim_{n\to\omega}\int_{-a}^{a}\left|\frac{e^{it\lambda}-1}{t} - i\lambda\right|^2 d\|e(\lambda)\eta_n\|^2 \\ &= \lim_{n\to\omega}\int_{-a}^{a}\left\{\left(\frac{\cos(t\lambda)-1}{t}\right)^2 + \left(\frac{\sin(t\lambda)}{t} - \lambda\right)^2\right\} d\|e(\lambda)\eta_n\|^2 \\ &= \lim_{n\to\omega}\int_{-a}^{a} F(t,\lambda) d\|e(\lambda)\eta_n\|^2,\end{aligned}$$

where

$$F(t,\lambda)=\lambda^2\left(2\frac{1-\cos(t\lambda)}{(t\lambda)^2}-2\frac{\sin(t\lambda)}{t\lambda}+1\right).$$

Therefore, for each t with $|t|a<\pi/2$, we have

$$\begin{aligned}\sup_{|\lambda|\leqslant a}F(t,\lambda)&\leqslant 2a^2\sup_{|\lambda|\leqslant a}\left(1-\frac{\sin(t\lambda)}{t\lambda}\right)\\&=2a^2\sup_{|x|\leqslant |t|a}\left(1-\frac{\sin x}{x}\right)\\&=2a^2\left(1-\frac{\sin(ta)}{ta}\right).\end{aligned}$$

Consequently, for $|t|<\pi/(2a)$,

$$\begin{aligned}\lim_{n\to\omega}\int_{-a}^{a}F(t,\lambda)d\|e(\lambda)\eta_n\|^2&\leqslant\lim_{n\to\omega}\int_{-a}^{a}2a^2\left(1-\frac{\sin(ta)}{ta}\right)d\|e(\lambda)\eta_n\|^2\\&=2a^2\left(1-\frac{\sin(ta)}{ta}\right)\|(\eta_n)_\omega\|^2\to 0\quad\text{as } t\to 0.\end{aligned}$$

Therefore we have

$$\overline{\lim_{t\to 0}}\left\|\frac{1}{t}\big(v(t)-1\big)\eta-(iA\eta_n)_\omega\right\|\leqslant 2\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, the claim is proved. ■

Now we show that the order of integration and ultralimit can be interchanged for the ω-equicontinuous family $\{F_n\colon\mathbb{R}\to H\}_{n=1}^{\infty}$ under some additional conditions.

LEMMA 3.4. *Let $F_n\in C(\mathbb{R},H)\cap L^1(\mathbb{R},H)$ $(n\in\mathbb{N})$ be a family of H-valued ω-equicontinuous maps satisfying the following two conditions*:

$$\int_{\mathbb{R}}\sup_{n\geqslant 1}\|F_n(t)\|dt<\infty,\qquad \sup_{n\geqslant 1}\|F_n(t)\|<\infty\quad(t\in\mathbb{R}).\tag{3.1}$$

$$\lim_{a\to\infty}\lim_{n\to\omega}\int_{\mathbb{R}\setminus[-a,a]}\|F_n(t)\|dt=0.\tag{3.2}$$

Then we have

$$\Big(\int_{\mathbb{R}}F_n(t)dt\Big)_\omega=\int_{\mathbb{R}}\big(F_n(t)\big)_\omega dt.$$

REMARK 3.1. *Note that, by the ω-equicontinuity of $\{F_n\}_{n=1}^{\infty}$, $t\mapsto\big(F_n(t)\big)_\omega$ is continuous. In particular, it is measurable.*

P r o o f. By (3.1), we have

$$\int_{\mathbb{R}} \big(F_n(t)\big)_\omega dt = \lim_{a\to\infty} \int_{-a}^{a} \big(F_n(t)\big)_\omega dt.$$

By (3.2), we also have

$$\Big(\int_{\mathbb{R}} F_n(t)dt\Big)_\omega = \lim_{a\to\infty} \Big(\int_{-a}^{a} F_n(t)dt\Big)_\omega.$$

Therefore, we have only to show that $\int_{-a}^{a} \big(F_n(t)\big)_\omega dt = \big(\int_{-a}^{a} F_n(t)dt\big)_\omega$ for all $a > 0$. By the ω-equicontinuity of $\{F_n\}_{n=1}^{\infty}$, there exists a partition of the interval $[-a, a]$ such that $t_0 = -a < t_1 < t_2 < \ldots < t_N = a$, and $W \in \omega$ so that for each $0 \leqslant i \leqslant N-1$, $n \in W$, and $\alpha, \beta \in [t_i, t_{i+1}]$, we have

$$\|F_n(\alpha) - F_n(\beta)\| < \varepsilon/4a.$$

This in particular implies that $\big\|\big(F_n(\alpha)\big)_\omega - \big(F_n(\beta)\big)_\omega\big\| < \varepsilon/4a$. Therefore, by the definition of the Riemann integral, we have

$$\Big\|\sum_{i=0}^{N-1}(t_{i+1}-t_i)F_n(t_i) - \int_{-a}^{a} F_n(t)dt\Big\| < \varepsilon/2 \quad (n \in W),$$

and

$$\Big\|\sum_{i=0}^{N-1}(t_{i+1}-t_i)\big(F_n(t_i)\big)_\omega - \int_{-a}^{a} \big(F_n(t)\big)_\omega dt\Big\| < \varepsilon/2.$$

Using $\big(\sum_{i=0}^{N-1}(t_{i+1}-t_i)F_n(t_i)\big)_\omega = \sum_{i=0}^{N-1}(t_{i+1}-t_i)\big(F_n(t_i)\big)_\omega$, we have

$$\Big\|\int_{-a}^{a} \big(F_n(t)\big)_\omega dt - \Big(\int_{-a}^{a} F_n(t)dt\Big)_\omega\Big\| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the claim is proved. ■

LEMMA 3.5. *Let $\xi = (\xi_n)_\omega \in K(A)$ and let $f \in L^1(\mathbb{R})$. Then we have*

$$\Big(\int_{\mathbb{R}} f(t)e^{itA}\xi_n dt\Big)_\omega = \int_{\mathbb{R}} \big(f(t)e^{itA}\xi_n\big)_\omega dt.$$

P r o o f. Note that $t \mapsto f(t)(e^{itA}\xi_n)_\omega$ is measurable thanks to Lemma 3.1. Let $C := \sup_n \|\xi_n\|$. First assume that $f \in L^1(\mathbb{R}) \cap C(\mathbb{R})$. It suffices to show that $\{F_n : t \mapsto f(t)e^{itA}\xi_n\}_{n=1}^{\infty}$ is ω-equicontinuous and satisfies the conditions (3.1) and (3.2) in Lemma 3.4. It follows that $\sup_n \int_{\mathbb{R}} \|F_n(t)\|dt = \int_{\mathbb{R}} |f(t)|dt \cdot \|\xi\| < \infty$, $\sup_n \|F_n(t)\| = |f(t)| < \infty$, and

$$\lim_{a\to\infty}\lim_{n\to\omega} \int_{\mathbb{R}\setminus[-a,a]} \|F_n(t)\|dt = \lim_{a\to\infty} \int_{\mathbb{R}\setminus[-a,a]} |f(t)|dt \cdot \|\xi\| = 0.$$

Therefore (3.1) and (3.2) in Lemma 3.4 are satisfied. We show the ω-equicontinuity of $\{F_n\}_{n=1}^{\infty}$. Suppose $\varepsilon > 0$ and $t \in \mathbb{R}$ are given. By the A-regularity of ξ and continuity of f, there exists $\delta > 0$ and $W \in \omega$ such that for each $s \in (t-\delta, t+\delta)$ and $n \in W$, we have

$$\|e^{itA}\xi_n - e^{isA}\xi_n\| < \frac{\varepsilon}{2\big(|f(t)|+1\big)}, \quad |f(t)-f(s)| < \frac{\varepsilon}{2(C+1)}.$$

Then it follows that

$$\begin{aligned}&\|f(t)e^{itA}\xi_n - f(s)e^{isA}\xi_n\|\\ &\leqslant |f(t)|\cdot\|e^{itA}\xi_n - e^{isA}\xi_n\| + |f(t)-f(s)|\cdot\|e^{isA}\xi_n\| < \varepsilon/2+\varepsilon/2 = \varepsilon.\end{aligned}$$

Therefore $\{F_n\}_{n=1}^{\infty}$ is ω-equicontinuous. By Lemma 3.4, the claim follows.

Next, suppose $f \in L^1(\mathbb{R})$. Let $\varepsilon > 0$. There exists $g \in L^1(\mathbb{R}) \cap C(\mathbb{R})$ such that $\|f-g\|_1 < \varepsilon/2(C+1)$. Then we have

$$\Big\|\Big(\int_{\mathbb{R}} f(t)e^{itA}\xi_n dt - \int_{\mathbb{R}} g(t)e^{itA}\xi_n dt\Big)_{\omega}\Big\| \leqslant \lim_{n\to\omega}\int_{\mathbb{R}} |f(t)-g(t)|\|\xi_n\|dt < \varepsilon/2,$$

$$\Big\|\int_{\mathbb{R}} \big(g(t)e^{itA}\xi_n\big)_{\omega} dt - \int_{\mathbb{R}} \big(f(t)e^{itA}\xi_n\big)_{\omega} dt\Big\| \leqslant \|g-f\|_1\cdot\|\xi\| < \varepsilon/2,$$

whence by applying the above argument for g we have

$$\Big\|\Big(\int_{\mathbb{R}} f(t)e^{itA}\xi_n dt\Big)_{\omega} - \int_{\mathbb{R}} \big(f(t)e^{itA}\xi_n\big)_{\omega} dt\Big\| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the claim is proved. ■

LEMMA 3.6. $\mathrm{dom}(\widetilde{A}_{\omega}) = \widehat{\mathscr{D}}_A$.

Proof. By Lemma 3.3, it suffices to show that $\mathrm{dom}(\widetilde{A}_{\omega}) \subset \widehat{\mathscr{D}}_A$. Let $e(\cdot)$ (resp. $\widetilde{e}(\cdot)$) be the spectral measure associated with A (resp. $\widetilde{A}_{\omega}$). We first show the following:

CLAIM. *For a given* $\xi \in \mathrm{dom}(\widetilde{A}_{\omega})$ *and* $\varepsilon > 0$, *there exists* $a > 0$ *and* $(\eta_n)_n \in \ell^{\infty}(\mathbb{N}, H)$ *with the properties*: $\eta_n \in 1_{[-a,a]}(A)H$ $(n \in \mathbb{N})$, $\|\xi - (\eta_n)_{\omega}\| < \varepsilon$, *and* $\|\widetilde{A}_{\omega}\xi - (A\eta_n)_{\omega}\| < \varepsilon$.

Note that in general $\widetilde{e}(B)$ is not the ultrapower of $e(B)$ for a Borel set B. Therefore we need some extra work (cf. [1], Section 4). As $\bigcup_{a>0} 1_{[-a,a]}(\widetilde{A}_{\omega})K(A)$ is a core for $\widetilde{A}_{\omega}$, there exists $a > 0$, $\eta = (\eta_n)_{\omega} \in 1_{[-a/2,a/2]}(\widetilde{A}_{\omega})K(A)$ such that $\|\xi - \eta\| < \varepsilon$ and $\|\widetilde{A}_{\omega}\xi - \widetilde{A}_{\omega}\eta\| < \varepsilon$. Let $f \in L^1(\mathbb{R})$ be a function with the following properties: $\mathrm{supp}(\hat{f}) \subset [-a,a]$, $\hat{f} = 1$ on $[-a/2,a/2]$, $0 \leqslant \hat{f}(\lambda) \leqslant 1$ $(\lambda \in \mathbb{R})$.

Here, $\hat{f}(\lambda) = \int_{\mathbb{R}} e^{i\lambda t} f(t)dt$ is the Fourier transform of f. For instance, one may choose the de la Vallée-Poussin kernel $D_{a/2}$ (see [1], Definition 4.12). Let

$$\eta' := \int_{\mathbb{R}} f(t)e^{it\widetilde{A}_\omega}\eta dt.$$

Then we have (by the spectral condition of η and $\hat{f} = 1$ on $[-a/2, a/2]$)

$$\begin{aligned}\eta' &= \int_{\mathbb{R}}\int_{\mathbb{R}} f(t)e^{it\lambda} d\big(\widetilde{e}(\lambda)\eta\big)dt = \int_{\mathbb{R}} \big(\int_{\mathbb{R}} f(t)e^{it\lambda}dt\big) d\big(\widetilde{e}(\lambda)\eta\big)\\ &= \int_{\mathbb{R}} \hat{f}(\lambda) d\big(\widetilde{e}(\lambda)\eta\big) = \hat{f}(\widetilde{A}_\omega)\eta = \eta.\end{aligned}$$

Furthermore, by Lemma 3.5, we have

$$\eta = \eta' = \big(\int_{\mathbb{R}} f(t)e^{itA}\eta_n dt\big)_\omega = \big(\hat{f}(A)\eta_n\big)_\omega,$$

and $\eta_n' := \hat{f}(A)\eta_n \in 1_{[-a,a]}(A)H$ for each $n \geqslant 1$. Therefore, $(\eta_n')_n$ is the required sequence, as $\widetilde{A}_\omega\eta = (A\eta_n')_\omega$ (cf. Lemma 3.3).

Assume now that $\xi \in \mathrm{dom}(\widetilde{A}_\omega)$ with $\|\xi\| = 1$. We show that $\xi \in \widehat{\mathscr{D}}_A$, i.e., it has a proper representative. Let $\varepsilon > 0$. We use the following argument similar to Lemma 3.9 (i) in [1]. By the above Claim, for each $k \in \mathbb{N}$, put $\varepsilon = 2^{-k-1}$ in the above argument to find $a_k (\leqslant a_{k+1} \leqslant a_{k+2} \leqslant \dots)$ and $(\eta_n^{(k)})_n \in \ell^\infty(\mathbb{N}, H)$ satisfying $\eta_n^{(k)} \in 1_{[-a_k, a_k]}(A)H$ $(n \in \mathbb{N})$, and

$$\|\xi - (\eta_n^{(k)})_\omega\| < \frac{1}{2^{k+1}}, \quad \|\widetilde{A}_\omega\xi - (A\eta_n^{(k)})_\omega\| < \frac{1}{2^{k+1}} \quad (k \in \mathbb{N}).$$

Furthermore, we may assume $\|\eta_n^{(k)}\| \leqslant 2$ for each $n, k \in \mathbb{N}$. Then for each $k \in \mathbb{N}$ we have

$$\|(\eta_n^{(k+1)})_\omega - (\eta_n^{(k)})_\omega\| < \frac{1}{2^k}, \quad \|(A\eta_n^{(k+1)})_\omega - (A\eta_n^{(k)})_\omega\| < \frac{1}{2^k}.$$

Let

$$G_k := \left\{ n \in \mathbb{N};\ \|\eta_n^{(k+1)} - \eta_n^{(k)}\| < \frac{1}{2^k},\ \|A\eta_n^{(k+1)} - A\eta_n^{(k)}\| < \frac{1}{2^k} \right\} \quad (k \in \mathbb{N}).$$

Then $G_k \in \omega$ $(k \in \mathbb{N})$ holds, and since ω is free, it follows that $F_k := \bigcap_{i=1}^k G_i \cap \{n \in \mathbb{N}; n \geqslant k\} \in \omega$ $(k \in \mathbb{N})$. Since $\{F_k\}_{k=1}^\infty$ is decreasing with empty intersection, $\mathbb{N} = (\mathbb{N} \setminus F_1) \sqcup \bigsqcup_{j=1}^\infty (F_j \setminus F_{j+1})$. Then define $(\xi_n)_n$ by

$$\xi_n := \begin{cases} \eta_n^{(1)} & (n \in \mathbb{N} \setminus F_1), \\ \eta_n^{(k)} & (n \in F_k \setminus F_{k+1}). \end{cases}$$

Then $\sup_{n\geqslant 1}\|\xi_n\|\leqslant 2<\infty$. Fix $k\geqslant 1$. If $n\in F_k=\bigsqcup_{j=k}^{\infty}(F_j\setminus F_{j+1})$, there is a unique $j\geqslant k$ for which $n\in F_j\setminus F_{j+1}$ holds, so that $\xi_n=\eta_n^{(j)}$. Then we have

$$\|\xi_n-\eta_n^{(k)}\|=\|\eta_n^{(j)}-\eta_n^{(k)}\|\leqslant\sum_{i=k}^{j-1}\|\eta_n^{(i+1)}-\eta_n^{(i)}\|\leqslant\sum_{i=k}^{j-1}\frac{1}{2^i}<\frac{1}{2^{k-1}},$$

so that $F_k\in\omega$ implies

$$\|(\xi_n)_\omega-(\eta_n^{(k)})_\omega\|<\frac{1}{2^{k-1}}\quad(k\in\mathbb{N}).$$

Similarly,

$$\|(A\xi_n)_\omega-(A\eta_n^{(k)})_\omega\|<\frac{1}{2^{k-1}}\quad(k\in\mathbb{N}).$$

In particular, for each $k\in\mathbb{N}$ we have

$$\|\xi-(\xi_n)_\omega\|\leqslant\|\xi-(\eta_n^{(k)})_\omega\|+\|(\eta_n^{(k)})_\omega-(\xi_n)_\omega\|<\frac{1}{2^{k-2}}.$$

Letting $k\to\infty$, we obtain $\xi=(\xi_n)_\omega$. We show that $(\xi_n)_n$ is a proper A-sequence. Suppose $\varepsilon>0$ is given. Take k such that $\varepsilon>2^{-k+1}$, and put $a=a_k>0,\eta_n:=\eta_n^{(k)}$. Then, by construction, $\eta_n\in 1_{[-a,a]}(A)H$ $(n\in\mathbb{N})$, $\|(\xi_n)_\omega-(\eta_n)_\omega\|<\varepsilon$, and $\|(A\xi_n)_\omega-(A\eta_n)_\omega\|<\varepsilon$ holds. Therefore, changing ξ_n to be zero if necessary for n belonging to a set I with $I\notin\omega$, we may assume that $(A\xi_n)_n$ is bounded, and $(\xi_n)_n$ is a proper A-sequence. This completes the proof. ∎

LEMMA 3.7. *Let $(\xi_n)_n\in\ell^\infty(\mathbb{N},H)$ be a sequence such that $\xi=(\xi_n)_\omega\in K(A)$. Then $(\widetilde{A}_\omega-i)^{-1}\xi=\big((A-i)^{-1}\xi_n\big)_\omega$ and $(\widetilde{A}_\omega+i)^{-1}\xi=\big((A+i)^{-1}\xi_n\big)_\omega$.*

Proof. Since $v(t)=e^{it\widetilde{A}}=(e^{itA})_\omega|_{K(A)}$ $(t\in\mathbb{R})$, by the resolvent formula and Lemma 3.5, we have

$$\begin{aligned}(\widetilde{A}_\omega-i)^{-1}\xi&=i\int_0^\infty e^{-t}e^{-it\widetilde{A}_\omega}\xi dt=i\int_0^\infty e^{-t}(e^{-itA}\xi_n)_\omega dt\\&=\Big(i\int_0^\infty e^{-t}e^{-itA}\xi_n dt\Big)_\omega=\big((A-i)^{-1}\xi_n\big)_\omega.\end{aligned}$$

The latter identity follows similarly. ∎

REMARK 3.2. *Note that $(\widetilde{A}_\omega-i)^{-1}\xi=\big((A-i)^{-1}\xi_n\big)_\omega$ holds even if $(\xi_n)_n$ is not proper. The only requirement is A-regularity: $(\xi_n)_\omega\in K(A)$.*

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. The assertion $\mathrm{dom}(\widetilde{A}_\omega)=\widehat{\mathscr{D}}_A$ is proved in Lemma 3.6. Then, for every $\xi\in\widehat{\mathscr{D}}_A$ and $\varepsilon>0$, there exists $\eta\in\mathscr{D}_0$ such that $\|\xi-\eta\|_{\widetilde{A}_\omega}<\varepsilon$ holds (cf. Lemma 3.3). Therefore, $\widetilde{A}_\omega$ is the closure of $\widetilde{A}_\omega|_{\mathscr{D}_0}$. ∎

4. ALTERNATIVE DESCRIPTION OF A_ω

Now we are ready to show

THEOREM 4.1. *Under the same notation as in Section* 3, *the following holds*:

(1) $K(A) = H(A)$, *and* $A_\omega = \widetilde{A}_\omega$. *Moreover*, $\mathscr{D}_0$ *is a core for* A_ω.

(2) *For a representative* $(\xi_n)_n$ *of* $\xi \in \mathrm{dom}(A_\omega)$, $A_\omega\xi = (A\xi_n)_\omega$ *holds if and only if it is a proper* A*-sequence* (see Definition 3.4).

P r o o f. (1) By construction, it is clear that $\widetilde{A}_\omega$ is a p.u. of A in $K(A) \subset H_\omega$. Therefore, by the maximality of A_ω, Theorem 1.1 (2), $K(A) \subset H(A)$ and $\widetilde{A}_\omega = A_\omega|_{K(A)}$. Consequently, if we show that $K(A) = H(A)$, then $\widetilde{A}_\omega = A_\omega$ holds. To show $H(A) \subset K(A)$, suppose $(\xi_n)_n$ is a representing sequence of $\xi \in \mathscr{D}_A$ with $(A\xi_n)_n \in \ell^\infty(\mathbb{N}, H)$. We show that $\{f_n : t \mapsto e^{itA}\xi_n\}_{n=1}^\infty$ is ω-equicontinuous. Let $C := \sup_n \|A\xi_n\|$. Then for $t, s \in \mathbb{R}$, as in the analysis in Section 3,

$$\|e^{itA}\xi_n - e^{isA}\xi_n\|^2 = \int_{\mathbb{R}} |e^{it\lambda} - e^{is\lambda}|^2 d\|e(\lambda)\xi_n\|^2 \leqslant (t-s)^2\|A\xi_n\|^2 \leqslant C^2(t-s)^2,$$

which tends to zero as $(t - s) \to 0$ uniformly in n. Thus, we infer that $\{f_n\}_{n=1}^\infty$ is ω-equicontinuous. Therefore $\mathscr{D}_A \subset K(A)$, and taking the closure, $H(A) \subset K(A)$ holds. Consequently, $H(A) = K(A)$. By Theorem 3.1, $\mathscr{D}_0$ is a core for $A_\omega = \widetilde{A}_\omega$.

(2) This follows from (1), Theorem 3.1, Lemma 3.3, and a simple observation that if $A_\omega\xi = (A\xi_n)_\omega$ and if $(\xi'_n)_n$ is another proper A-sequence representing ξ, then for every $\varepsilon > 0$ there is $a > 0$ and an A-sequence $(\eta_n)_n$ with $\eta_n \in 1_{[-a,a]}(A)H$ $(n \in \mathbb{N})$ such that $\lim_{n\to\omega} \|\xi_n - \eta_n\|_A = \lim_{n\to\omega} \|\xi'_n - \eta_n\|_A < \varepsilon$, so that $(\xi_n)_n$ is proper as well. ■

REMARK 4.1. *Finally*, *let us return to Example* 1.1. *We note that* $(\xi_n)_n$ *is proper*, *while* $(\xi'_n)_n$ *is not. The first claim is obvious. For the latter*, *if it were proper*, *then so would be* $\left(\frac{1}{n}\eta_n\right)_n$. *But if* $\left(\frac{1}{n}\eta_n\right)_n$ *were proper*, *there would exist an* A*-sequence* $(\zeta_n)_n$ *and* $a > 0$ *for which* $\zeta_n \in 1_{[-a,a]}(A)H$ $(n \in \mathbb{N})$, $\lim_{n\to\omega} \left\|\zeta_n - \frac{1}{n}\eta_n\right\| < 1/2$ *and* $\lim_{n\to\omega} \|A\zeta_n - \eta_n\| < 1/2$ *hold. Let* $n_0 \in \mathbb{N}$ *be such that* $n_0 > |a|$. *Then*, *for* $n \geqslant n_0$, $\eta_n \in 1_{\{n\}}(A)H$, *so* $\eta_n \perp \zeta_n$. *Thus*

$$\lim_{n\to\omega} \|A\zeta_n - \eta_n\|^2 = \lim_{n\to\omega} \|A\zeta_n\|^2 + 1 < \frac{1}{4},$$

which is a contradiction. Thus $\left(\frac{1}{n}\eta_n\right)_n$, *whence* $(\xi'_n)_n$, *is not proper. Note also that* $(\eta_n)_\omega$ *is perpendicular to* $H(A)$ *and*, *in particular*, $(A\xi'_n)_\omega \notin H(A)$. *To see this*, *let* $(\xi_n)_n$ *be an* A*-sequence such that* $(A\xi_n)_n$ *is bounded. Then*

$$\begin{aligned} |\lim_{n\to\omega}\langle \eta_n, \xi_n\rangle| &= |\lim_{n\to\omega}\langle \eta_n, (A-i)^{-1}(A+i)\xi_n\rangle| \\ &\leqslant \lim_{n\to\omega} \frac{1}{|n+i|}\|(A+i)\xi_n\| = 0. \end{aligned}$$

Thus $(\eta_n)_\omega \in \mathscr{D}_A^\perp = H(A)^\perp$.

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REFERENCES

[1] H. Ando and U. Haagerup, *Ultraproducts of von Neumann algebras*, arXiv: 12112.5457v2.
[2] A. Kishimoto, *Rohlin property for flows*, Contemp. Math. 335 (2003), pp. 195–207.
[3] A. Krupa and B. Zawisza, *Applications of ultrapowers in analysis of unbounded selfadjoint operators*, Bull. Polish Acad. Sci. Math. 32 (1984), pp. 581–588.
[4] A. Krupa and B. Zawisza, *Ultrapowers of unbounded selfadjoint operators*, Studia Math. 87 (2) (1987), pp. 101–120.
[5] T. Masuda and R. Tomatsu, *Rohlin flows on von Neumann algebras*, preprint, arXiv: 1206.0955v2.
[6] H. Saigo, *Categorical non-standard analysis*, preprint, arXiv: 1009.0234.
[7] K. Schmüdgen, *Unbounded self-adjoint operators on Hilbert space*, Grad. Texts in Math., Springer, 2012.

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