PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 35, Fasc. 2 (2015), pp. 301–312

CONFIDENCE INTERVALS FOR AVERAGE SUCCESS PROBABILITIES*

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Abstract. We provide Buehler-optimal one-sided and valid two-sided confidence intervals for the average success probability of a possibly inhomogeneous fixed length Bernoulli chain, based on the number of observed successes. Contrary to some claims in the literature, the one-sided Clopper–Pearson intervals for the homogeneous case are not completely robust here, not even if applied to hypergeometric estimation problems.

2010 AMS Mathematics Subject Classification: Primary: 62F25; Secondary: 62F35.

Key words and phrases: Bernoulli convolution, binomial distribution inequality, Clopper–Pearson, hypergeometric distribution, inhomogeneous Bernoulli chain, Poisson-binomial distribution, robustness.

1. INTRODUCTION AND RESULTS

The purpose of this paper is to provide optimal one-sided (Theorem 1.2) and valid two-sided (Theorems 1.1 and 1.3) confidence intervals for the average success probability of a possibly inhomogeneous fixed length Bernoulli chain, based on the number of observed successes. For this situation, intervals proposed in the literature known to us are, if at all clearly specified, in the one-sided case either not optimal or erroneously claimed to be valid (see Remarks 1.3 and 1.8 below), and in the two-sided case either improved here (see Remark 1.11) or not previously proven to be valid.

To be more precise, let B_p for $p \in [0,1]$, $\mathrm{B}_{n,p}$ for $n \in \mathbb{N}_0$ and $p \in [0,1]$, and $\mathrm{BC}_p := \ast_{j=1}^n \mathrm{B}_{p_j}$ for $n \in \mathbb{N}_0$ and $p \in [0,1]^n$ denote the Bernoulli, binomial, and Bernoulli convolution (or Poisson-binomial) laws with the indicated parameters. For $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$ let $]a,b] := \{x\colon a < x \leqslant b\}$ and let the other intervals be defined analogously. Then, for $n \in \mathbb{N}$ and $\beta \in \,]0,1[$, and writing $\overline{p} := \frac{1}{n}\sum_{j=1}^n p_j$ for $p \in [0,1]^n$, we are interested in β-confidence regions for the estimation problem

$$\big((\mathrm{BC}_p\colon p \in [0,1]^n)\,, [0,1]^n \ni p \mapsto \overline{p}\big), \tag{1.1}$$

* This research was partially supported by DFG grant MA 1386/3-1.

that is, in functions $\mathrm{K}\colon \{0,\ldots,n\} \to 2^{[0,1]}$ satisfying $\mathrm{BC}_p\,(\mathrm{K} \ni \overline{p}) \geqslant \beta$ for $p \in [0,1]^n$. Clearly, every such K is also a β-confidence region for the binomial estimation problem

$$\big((\mathrm{B}_{n,p}\colon p \in [0,1]), \mathrm{id}_{[0,1]}\big), \tag{1.2}$$

that is, satisfies $\mathrm{B}_{n,p}\,(\mathrm{K} \ni p) \geqslant \beta$ for $p \in [0,1]$, but the converse is false by Remark 1.2 below. However, a classical Chebyshev–Hoeffding result easily yields the following basic fact.

THEOREM 1.1. *Let $n \in \mathbb{N}$ and $\beta \in \,]0,1[$. For $m \in \{0,\ldots,n\}$, let K'_m be a β-confidence region for $\big((\mathrm{B}_{m,p}\colon p \in [0,1]), \mathrm{id}_{[0,1]}\big)$. Then a β-confidence region* K *for* (1.1) *is given by*

$$\mathrm{K}(x) := \bigcup_{\substack{l\in\{0,\ldots,x\},\\ m\in\{x-l,\ldots,n-l\}}} \left(\tfrac{m}{n}\mathrm{K}'_m(x-l) + \tfrac{l}{n}\right) \supseteq \mathrm{K}'_n(x) \quad \textit{for } x \in \{0,\ldots,n\}.$$

Proofs of the three theorems of this paper are presented in Section 2 below.

If the above K'_m are taken to be one-sided intervals of Clopper and Pearson [5], then the resulting K turns out to be Buehler-optimal and, if β is not unusually small, the formula for K simplifies drastically, as stated in Theorem 1.2 below for uprays:

A set $J \subseteq [0,1]$ is an *upray in* $[0,1]$ if $x \in J, y \in [0,1], x \leqslant y$ jointly imply $y \in J$. This is equivalent to J being of the form $[a,1]$ or $]a,1]$ for some $a \in [0,1]$. A function $\mathrm{K}\colon \{0,\ldots,n\} \to 2^{[0,1]}$ is an *upray* if each of its values $\mathrm{K}(x)$ is an upray in $[0,1]$.

For $\beta \in \,]0,1[$ and with

$$g_n(x) := g_{n,\beta}(x) := \text{ the } p \in [0,1] \text{ with } \mathrm{B}_{n,p}(\{x,\ldots,n\}) = 1-\beta$$

for $n \in \mathbb{N}$ and $x \in \{1,\ldots,n\}$, which is well defined due to the strict isotonicity of $p \mapsto \mathrm{B}_{n,p}(\{x,\ldots,n\})$ and which yields, in particular, the special values

$$g_n(1) = 1-\beta^{1/n} \quad \text{and} \quad g_n(n) = (1-\beta)^{1/n} \tag{1.3}$$

and the fact that

$$g_{n,\beta}(x) \ \text{ is strictly } \begin{Bmatrix}\text{increasing}\\ \text{decreasing}\end{Bmatrix} \text{ in } \begin{Bmatrix} x \\ \beta \end{Bmatrix}, \tag{1.4}$$

the Clopper–Pearson β-confidence uprays $\mathrm{K}_{\mathrm{CP},n}\colon \{0,\ldots,n\} \to 2^{[0,1]}$ are given by the formula

$$\mathrm{K}_{\mathrm{CP},n}(x) := \mathrm{K}_{\mathrm{CP},n,\beta}(x) := \begin{cases} [0,1] & \text{if } x = 0, \\ ]g_n(x),1] & \text{if } x \in \{1,\ldots,n\} \end{cases} \tag{1.5}$$

for $n \in \mathbb{N}_0$, and in particular

$$\mathrm{K}_{\mathrm{CP},n}(1) =]1-\beta^{1/n}, 1] \quad \text{and} \quad \mathrm{K}_{\mathrm{CP},n}(n) =](1-\beta)^{1/n}, 1]$$

for $n \in \mathbb{N}$.

An upray $\mathrm{K}\colon \{0,\ldots,n\} \to 2^{[0,1]}$ is *isotone* if it is isotone with respect to the usual order on $\{0,\ldots,n\}$ and the order reverse to set inclusion on $2^{[0,1]}$, that is, if we have the implication

$$x, y \in \{0,\ldots,n\},\ x < y \Rightarrow \mathrm{K}(x) \supseteq \mathrm{K}(y),$$

and *strictly isotone* if "$\supseteq$" above can be sharpened to "$\supsetneq$". For example, each of the above $\mathrm{K}_{\mathrm{CP},n}$ is strictly isotone by (1.4) and (1.5). An isotone β-confidence upray K for (1.1) is *(Buehler-)optimal* (see Buehler [2] and the recent discussion by Lloyd and Kabaila [11], prompted by rediscoveries by Wang [16]) if every other isotone β-confidence upray K^* for (1.1) satisfies $\mathrm{K}(x) \subseteq \mathrm{K}^*(x)$ for every $x \in \{0,\ldots,n\}$. Finally, a not necessarily isotone β-confidence upray K for (1.1) is *admissible* in the set of all confidence uprays for (1.1) if for every other β-confidence upray K^* for (1.1) with $\mathrm{K}^*(x) \subseteq \mathrm{K}(x)$ for each $x \in \{0,\ldots,n\}$ we have $\mathrm{K}^* = \mathrm{K}$.

Let us put

$$\beta_n := \mathrm{B}_{n,1/n}(\{0,1\}) \quad \text{for } n \in \mathbb{N},$$

so that $\beta_1 = 1$, $\beta_2 = \frac{3}{4}$, $\beta_3 = \frac{20}{27}$, and $\beta_n \downarrow \frac{2}{\mathrm{e}} = 0.735\ldots$, with the strict antitonicity of (β_n) following from Jogdeo and Samuels [9] (see [9], Theorem 2.1 with $m_n := n, p_n := \frac{1}{n}, r := 0$), so that we have in particular

$$\beta_n \leqslant \tfrac{3}{4} \quad \text{for } n \geqslant 2.$$

THEOREM 1.2. *Let $n \in \mathbb{N}$ and $\beta \in]0,1[$, and let* K *be as in Theorem* 1.1 *with the $\mathrm{K}'_m := \mathrm{K}_{\mathrm{CP},m}$ as defined in* (1.5). *Then* K *is the optimal isotone β-confidence upray for* (1.1), *is admissible in the set of all β-confidence uprays for* (1.1), *is strictly isotone*, *and has the effective level $\inf_{p\in[0,1]^n} \mathrm{BC}_p\,(\mathrm{K} \ni \overline{p}) = \beta$. We have*

$$\mathrm{K}(x) = \begin{cases} [0,1] & \text{if } x = 0, \\ \left]\frac{1-\beta}{n}, 1\right] & \text{if } x = 1, \\ ]g_n(x), 1] & \text{if } x \in \{2,\ldots,n\} \text{ and } \beta \geqslant \beta_n. \end{cases} \tag{1.6}$$

REMARK 1.1. *Nestedness is preserved by the construction in Theorem* 1.1: *Suppose that we apply Theorem* 1.1 *to several $\beta \in]0,1[$ and that we accordingly write $\mathrm{K}'_{m,\beta}$ and K_β in place of K'_m and* K. *If now $\beta, \tilde\beta \in]0,1[$ with $\beta < \tilde\beta$ are such that $\mathrm{K}'_{m,\beta}(x) \subseteq \mathrm{K}'_{m,\tilde\beta}(x)$ holds for $m \in \{0,\ldots,n\}$ and $x \in \{0,\ldots,m\}$*, *then*, *obviously*, *$\mathrm{K}_\beta(x) \subseteq \mathrm{K}_{\tilde\beta}(x)$ holds for $x \in \{0,\ldots,n\}$. By the second line in* (1.4) *and by* (1.5), *the Clopper–Pearson uprays are nested*, *and hence so are the uprays of Theorem* 1.2. *Analogous remarks apply to the confidence downrays of Remark* 1.6 *and to the two-sided confidence intervals of Theorem* 1.3.

REMARK 1.2. *Let* $n \geqslant 2$ *and* $\beta \in \,]0,1[$. *As noted by Agnew* [1] *but ignored by later authors* (*compare Remark* 1.8 *below*), $\mathrm{K}_{\mathrm{CP},n}$ *is not a* β*-confidence region for* (1.1). *This is obvious from Theorem* 1.2 *and* $\mathrm{K}_{\mathrm{CP},n}(1) \subsetneq \mathrm{K}(1)$, *by using either the optimality of* K *and the isotonicity of* $\mathrm{K}_{\mathrm{CP},n}$, *or the admissibility of* K *and* $\mathrm{K}_{\mathrm{CP},n}(x) \subseteq \mathrm{K}(x)$ *for every* x. *If* $\beta \geqslant \beta_n$, *then Theorem* 1.2 *further implies that the effective level of* $\mathrm{K}_{\mathrm{CP},n}$ *as a confidence region for* (1.1) *is*

$$\gamma_n := 1 - n(1-\beta^{1/n}) \in \,]1+\log(\beta), \beta[,$$

as for $p \in [0,1]^n$ *with* $\overline{p} \notin \left]\frac{1-\beta}{n}, g_n(1)\right]$, *formula* (1.6) *yields* $\mathrm{BC}_p(\mathrm{K}_{\mathrm{CP},n} \ni \overline{p}) = \mathrm{BC}_p(\mathrm{K} \ni \overline{p}) \geqslant \beta$, *and considering* $p_1 = ng_n(1) \leqslant 1$ *and* $p_2 = \ldots = p_n = 0$ *in the second step below yields*

$$\begin{aligned} \inf_{\overline{p} \in](1-\beta)/n, g_n(1)]} \mathrm{BC}_p(\mathrm{K}_{\mathrm{CP},n} \ni \overline{p}) &= \inf_{\overline{p} \in](1-\beta)/n, g_n(1)]} \prod_{j=1}^n (1-p_j) \\ &= 1 - ng_n(1) = \gamma_n. \end{aligned}$$

Since $\gamma_n \downarrow 1 + \log(\beta) < \beta$ *for* $n \to \infty$, *it follows for* $\beta > \frac{2}{\mathrm{e}}$ *that the* $\mathrm{K}_{\mathrm{CP},n}$ *are not even asymptotic* β*-confidence regions for* (1.1).

REMARK 1.3. *The only previous* β*-confidence upray for* (1.1) *known to us was provided by Agnew* [1] (see [1], Section 3) *as* $\mathrm{K}_{\mathrm{A}}(x) := [g_{\mathrm{A}}(x), 1]$ *with* $g_{\mathrm{A}}(0) := 0$ *and* $g_{\mathrm{A}}(x) := g_n(x) \wedge \frac{x-1}{n}$ *for* $x \in \{1,\ldots,n\}$. *But* K_{A} *is strictly worse than the optimal isotone* K *from Theorem* 1.2, *since* K_{A} *is isotone as well*, *with* $\mathrm{K}_{\mathrm{A}}(1) = [0,1] \supsetneq \mathrm{K}(1)$. *On the other hand*, *Lemma* 2.2 *below shows that actually* $g_{\mathrm{A}}(x) = g_n(x)$ *for* $\beta \geqslant \beta_n$ *and* $x \in \{2,\ldots,n\}$, *which is a precise version of an unproven claim in the cited reference.*

REMARK 1.4. *The condition* $\beta \geqslant \beta_n$ *in* (1.6) *cannot be omitted. Indeed*, *for* $n \in \mathbb{N}$, *let* $A_n := \big\{\beta \in \,]0,1[:$ *If* K *is as in Theorem* 1.2, *then* $\mathrm{K}(x) = \,]g_n(x), 1]$ *for* $x \in \{2,\ldots,n\}\big\}$. *Then* $[\beta_n, 1[\, \subseteq A_n$ *by Theorem* 1.2. *Numerically*, *we found*, *for example*, *also* $\beta_n - 0.001 \in A_n$ *for* $2 \leqslant n \leqslant 123$, *but* $\mathrm{K}(2) \supsetneq \,]g_n(2), 1]$ *for* $\beta = \beta_n - 0.001$ *and* $124 \leqslant n \leqslant 3000$.

REMARK 1.5. *The* β*-confidence upray* K *for* (1.1) *from Theorem* 1.2 *considered merely as a* β*-confidence interval shares with* $\mathrm{K}_{\mathrm{CP},n}$ *as a* β*-confidence interval for* (1.2) *the defect of not being admissible in the set of* all β*-confidence intervals*, *since with* $c := \big(\inf \mathrm{K}(n)\big) \vee \big(1-(1-\beta)^{1/n}\big)$ *and*

$$\mathrm{K}^*(x) := \begin{cases} [0,c] \subsetneq \mathrm{K}(0) & \text{if } x = 0, \\ \mathrm{K}(x) & \text{if } x \in \{1,\ldots,n\}, \end{cases}$$

we have $\mathrm{BC}_p(\mathrm{K}^* \ni \overline{p}) = \mathrm{BC}_p(\mathrm{K} \ni \overline{p}) \geqslant \beta$ *if* $\overline{p} \leqslant c$, *and*, *if* $\overline{p} > c$, $\mathrm{BC}_p(\mathrm{K}^* \ni \overline{p}) = \mathrm{BC}_p(\{1,\ldots,n\}) = 1 - \prod_{j=1}^n (1-p_j) \geqslant 1 - (1-\overline{p})^n > 1 - (1-c)^n \geqslant \beta$.

REMARK 1.6. *Since* K *is a* β*-confidence region for* (1.1) *iff* $\{0,\ldots,n\} \ni x \mapsto 1-\mathrm{K}(n-x)$ *is one*, *Theorem* 1.2 *and Remarks* 1.1–1.5 *yield obvious analogs for* downrays, *that is*, *confidence regions with each value being* $[0,b[$ *or* $[0,b]$ *for some* $b \in [0,1]$: *A downray* $\Lambda\colon \{0,\ldots,n\} \to 2^{[0,1]}$ *is* isotone *if* $\Lambda(x) \subseteq \Lambda(y)$ *holds for* $x < y$. *The Clopper–Pearson downrays* $\Lambda_{\mathrm{CP},n} := \Lambda_{\mathrm{CP},n,\beta}$ *defined by* $\Lambda_{\mathrm{CP},n,\beta}(x) := 1 - \mathrm{K}_{\mathrm{CP},n,\beta}(n-x)$ *are isotone*, *and Theorem* 1.2 *remains valid if we replace* $\mathrm{K}_{\mathrm{CP},m}$ *by* $\Lambda_{\mathrm{CP},m}$, *upray by downray*, *and* (1.6) *by*

$$\mathrm{K}(x) = \begin{cases} [0, 1-g_n(n-x)[& \text{if } x \in \{0,\ldots,n-2\} \text{ and } \beta \geqslant \beta_n, \\ \left[0, 1-\frac{1-\beta}{n}\right[& \text{if } x = n-1, \\ [0,1] & \text{if } x = n. \end{cases} \tag{1.7}$$

REMARK 1.7. *Let* $n \in \mathbb{N}$, $x \in \{0,\ldots,n\}$, *and* `beta` $\geqslant 3/4$ *be given. Then*, *by Theorem* 1.2, *an* `R` *code for computing the lower* `beta`*-confidence bound is*

```
(x==1)*((1-beta)/n)+
(x!=1)*binom.test(x,n,alt="g",conf.level=beta)$conf.int[1]
```

and, *by Remark* 1.6, *the corresponding code for the upper* `beta`*-confidence bound is the following*:

```
(x==n-1)*(1-(1-beta)/n)+
(x!=n-1)*binom.test(x,n,alt="l",conf.level=beta)$conf.int[2]
```

For example, *in* [3] (p. 249, lines 13–21) *we have* `n` $=7$ *and* `x` $=6$, *yielding here for* `beta` $=0.99, 0.98$, *and* 0.95 *the lower confidence bounds* $0.356\ldots$, $0.404\ldots$, *and* $0.479\ldots$, *respectively*, *so that the bounds claimed in* [3] *are indeed valid*, *but only now proven to be valid by Theorem* 1.2 (*compare Remark* 1.8).

REMARK 1.8. *Papers erroneously claiming the Clopper–Pearson uprays or downrays to be* β*-confidence regions for* (1.1) *include*: *Kappauf and Bohrer* [10] (p. 652, lines 3–5), *Byers et al.* [3] (p. 249, the first column, lines 15–18), *and Cheng et al.* [4] (p. 7, lines 10–8 from the bottom). *The analogous claim of Ollero and Ramos* [12] (p. 247, lines 9–12) *for a certain subfamily of* $(\mathrm{BC}_p\colon p \in [0,1]^n)$, *which includes the hypergeometric laws with sample size parameter* n, *is refuted in Remark* 1.10 *below. The common source of error in these papers seems to be an unclear remark of Hoeffding* [8] (p. 720, the first paragraph of Section 5) *related to the fact that*, *by Theorem* 4 *in* [8] *or by David* [6], *certain tests for* $\Theta_0 \subseteq [0,1]$ *in the binomial model* $(\mathrm{B}_{n,p}\colon p \in [0,1])$ *keep their level as tests for* $\widetilde{\Theta_0} := \{p \in [0,1]^n \colon \overline{p} \in \Theta_0\}$ *in* $(\mathrm{BC}_p\colon p \in [0,1]^n)$. *Let us further note that Ollero and Ramos* [12] *could have cited Vatutin and Mikhailov* [15] *concerning the representability of hypergeometric laws as Bernoulli convolutions.*

REMARK 1.9. *The core of the unclear remark in* [8] *mentioned in Remark* 1.8 *is* “that the usual (one-sided and two-sided) tests for the constant probability of ‘success’ in n independent (Bernoulli) trials can be used as tests for the average probability of success when the probability of success varies from trial to trial.”

We specify and generalise this in the following way. Let $n \in \mathbb{N}$, $p_1 \leqslant p_2 \in [0,1]$, $\gamma_-, \gamma_+ \in [0,1]$, $c_- \leqslant \lfloor np_1 \rfloor - 1$, *and* $c_+ \geqslant \lceil np_2 \rceil + 1$. *Then the randomised test*

$$\psi := \mathbf{1}_{\{0,\ldots,c_- -1\}} + \gamma_- \mathbf{1}_{\{c_-\}} + \gamma_+ \mathbf{1}_{\{c_+\}} + \mathbf{1}_{\{c_+ +1,\ldots,n\}}$$

for the hypothesis $[p_1, p_2]$ *in the binomial model* $(\mathrm{B}_{n,p} \colon p \in [0,1])$ *keeps its level as a randomised test for* $\{p \in [0,1]^n \colon \overline{p} \in [p_1,p_2]\}$ *in the model* $(\mathrm{BC}_p \colon p \in [0,1]^n)$ *because for every* p *with* $\overline{p} \in [p_1,p_2]$ *it follows from Theorem* 4 *in* [8] *that we have*

$$\begin{aligned}\mathrm{BC}_p\psi &= \gamma_- \mathrm{BC}_p(\{0,\ldots,c_-\}) + (1-\gamma_-)\mathrm{BC}_p(\{0,\ldots,c_- - 1\}) \\ &\quad + \gamma_+ \mathrm{BC}_p(\{c_+,\ldots,n\}) + (1-\gamma_+)\mathrm{BC}_p(\{c_+ + 1,\ldots,n\}) \\ &\leqslant \mathrm{B}_{n,\overline{p}}\psi.\end{aligned}$$

But this statement does not always apply to the one-sided tests based on the Clopper–Pearson uprays. Indeed, let $n=2$ *and* $\beta \in]0,1[$. *Let* $r \in [0,1]$, $H := [0,r]$, *and* $\psi := \mathbf{1}_{\{\mathrm{K}_{\mathrm{CP},n} \cap H = \emptyset\}}$, *so that we have* $\sup_{p\in H} \mathrm{B}_{n,p}\psi \leqslant 1-\beta$. *But if, for example,* $r = 1-\sqrt{\beta}$, *the test simplifies to* $\psi = \mathbf{1}_{\{1,2\}}$, *and for* $p := (r-\varepsilon, r+\varepsilon)$ *for an* $\varepsilon > 0$ *small enough, we have* $\overline{p} \in H$ *and* $\mathrm{BC}_p\psi = 1 - \mathrm{BC}_p(\{0\}) = 1 - \beta + \varepsilon^2 > 1-\beta$.

REMARK 1.10. *Clopper–Pearson uprays can be invalid for hypergeometric estimation problems: For* $N \in \mathbb{N}_0$, $n \in \{0,\ldots,N\}$, *and* $p \in \{\frac{j}{N} \colon j \in \{0,\ldots,N\}\}$, *let* $\mathrm{H}_{n,p,N}$ *denote the hypergeometric law of the number of red balls drawn in a simple random sample of size* n *from an urn containing* Np *red and* $N(1-p)$ *blue balls, so that we have* $\mathrm{H}_{n,p,N}(\{k\}) = \binom{Np}{k}\binom{N(1-p)}{n-k}/\binom{N}{n}$ *for* $k \in \mathbb{N}_0$. *For* $\beta \in]0,1[$ *and fixed* n *and* N, *in general,* $\mathrm{K}_{\mathrm{CP},n}$ *is not a* β*-confidence region for the estimation problem* $\big((\mathrm{H}_{n,p,N} \colon p \in \{\frac{j}{N} \colon j \in \{0,\ldots,N\}\}), p \mapsto p\big)$ *because if, for example,* $n \geqslant 2$ *and* $\beta = \big(1-\frac{1}{N}\big)^n$, *then for* $p = g_n(1)$ *we have* $p = 1-\beta^{1/n} = \frac{1}{N}$, *and so* $\mathrm{H}_{n,p,N}(\mathrm{K}_{\mathrm{CP},n} \ni p) = \mathrm{H}_{n,p,N}(\{0\}) = \binom{N(1-p)}{n}/\binom{N}{n} = \prod_{j=0}^{n-1} \frac{N(1-p)-j}{N-j} < (1-p)^n = \beta$.

In contrast to Remark 1.2, we have the following positive result for the two-sided Clopper–Pearson β-confidence intervals $\mathrm{M}_{\mathrm{CP},n}$ for (1.2), as defined in (1.8) below.

THEOREM 1.3. *Let* $n \in \mathbb{N}$, $\beta \in]0,1[$, *and*

$$\mathrm{M}_{\mathrm{CP},n}(x) := \mathrm{K}_{\mathrm{CP},n,(1+\beta)/2}(x) \cap \Lambda_{\mathrm{CP},n,(1+\beta)/2}(x) \quad \text{for } x \in \{0,\ldots,n\} \tag{1.8}$$

with $\mathrm{K}_{\mathrm{CP},n,(1+\beta)/2}$ *as in* (1.5) *and* $\Lambda_{\mathrm{CP},n,(1+\beta)/2}$ *as in Remark* 1.6. *If* $\beta \geqslant 2\beta_n - 1$ *or* $n = 1$, *hence, in particular, if* $\beta \geqslant \frac{1}{2}$, *then* $\mathrm{M}_{\mathrm{CP},n}$ *is a* β*-confidence interval for* (1.1).

REMARK 1.11. *The interval* $\mathrm{M}_{\mathrm{CP},n}$ *of Theorem* 1.3 *improves on the two-sided interval for* (1.1) *obtained by Agnew* [1] *in the obvious way from his one-sided ones.*

REMARK 1.12. *In contrast to Remark* 1.4, *we do not know whether the condition "*$\beta \geqslant 2\beta_n - 1$ *or* $n = 1$*" in Theorem* 1.3 *might be omitted.*

REMARK 1.13. *The robustness property of the two-sided Clopper–Pearson intervals given by Theorem* 1.3 *does not extend to every other two-sided interval for* (1.2), *for example, if* $n = 2$, *not to the Sterne* [13] *type* β*-confidence interval* $\mathrm{K}_{\mathrm{S},n}$ *for* (1.2) *of Dümbgen* [7] (p. 5, C_α^{St}).

Indeed, for $\beta \in \,]0,1[$ *and* $n \in \mathbb{N}$, $\mathrm{K}_{\mathrm{S},n}$ *is given by*

$$\begin{aligned}\mathrm{K}_{\mathrm{S},n}(x) &\coloneqq \mathrm{K}_{\mathrm{S},n,\beta}(x)\\ &\coloneqq \Big\{p \in [0,1]\colon \mathrm{B}_{n,p}\big(\{k\colon \mathrm{B}_{n,p}(\{k\}) \leqslant \mathrm{B}_{n,p}(\{x\})\}\big) > 1-\beta\Big\}.\end{aligned}$$

If, for example, $n = 2$ *and* $\beta > \beta_2$, *we have in particular* $\mathrm{K}_{\mathrm{S},2}(0) = [0, 1 - g_2(2)[$, $\mathrm{K}_{\mathrm{S},2}(1) = \,]g_2(1), 1 - g_2(1)[$, *and* $\mathrm{K}_{\mathrm{S},2}(2) = \,]g_2(2), 1]$, *and indeed* $\mathrm{K}_{\mathrm{S},2}$ *is not valid for* (1.1) *because for* $p \in [0,1]^2$ *with* $\overline{p} = g_2(1)$ *and* $p_1 \neq p_2$ *we have*

$$\mathrm{BC}_p(\mathrm{K}_{\mathrm{S},2} \ni \overline{p}) = \mathrm{BC}_p(\{0\}) = \prod_{j=1}^{2}(1-p_j) < (1-\overline{p})^2 = \big(1 - g_2(1)\big)^2 = \beta.$$

For $n = 2$ *and* $\beta > \beta_2$ *we get a* β*-confidence interval for* (1.2), *say* $\tilde{\mathrm{K}}$, *from Theorem* 1.1 *by setting* $\mathrm{K}'_m \coloneqq \mathrm{K}_{\mathrm{S},m}$ *for* $m \in \{0,1,2\}$, *namely,*

$$\tilde{\mathrm{K}}(0) = [0, 1-(1-\beta)^{1/2}[, \quad \tilde{\mathrm{K}}(1) = \,\big]\tfrac{1-\beta}{2}, \tfrac{1+\beta}{2}\big[, \quad \tilde{\mathrm{K}}(2) = \,](1-\beta)^{1/2}, 1].$$

It can be seen that $\tilde{\mathrm{K}}(x) \subsetneq \mathrm{M}_{\mathrm{CP},2}(x)$ *for* $x \in \{0,1,2\}$, *with* $\mathrm{M}_{\mathrm{CP},2}$ *as defined in Theorem* 1.3. *We do not know whether these inclusions are true for every* n *and usual* β, *but in fact we do not even know whether* $\mathrm{K}_{\mathrm{S},n}(x) \subseteq \mathrm{M}_{\mathrm{CP},n}(x)$ *holds universally.*

2. PROOFS OF THE THEOREMS

Proof of Theorem 1.1. We obviously have $\mathrm{K}(x) \subseteq [0,1]$ and, by considering $l = 0$ and $m = n$, $\mathrm{K}(x) \supseteq \mathrm{K}'_n(x)$ for every x. If $\varphi\colon \{0,\ldots,n\} \to \mathbb{R}$ is any function and $\pi \in [0,1]$, then, by Hoeffding's ([8], Corollary 2.1) generalization of Tchebichef's second theorem in [14], the minimum of the expectation $\mathrm{BC}_p\varphi$ as a function of $p \in [0,1]^n$ subject to $\overline{p} = \pi$ is attained at some point p whose coordinates take on at most three values and with at most one of them distinct from zero and one. Given $p \in [0,1]^n$, the preceding sentence applied to $\pi \coloneqq \overline{p}$ and to φ being the indicator of $\{\mathrm{K} \ni \pi\}$ yields the existence of $r, s \in \{0, \ldots, n\}$ with $r + s \leqslant n$ and of an $a \in [0, 1]$ with $r + sa = n\pi$, and

$$\begin{aligned}\mathrm{BC}_p\left(\mathrm{K} \ni \overline{p}\right) &\geqslant (\delta_r * \mathrm{B}_{s,a})\left(\left\{x \in \{r, \ldots, r+s\} \colon \mathrm{K}(x) \ni \pi\right\}\right)\\ &\geqslant (\delta_r * \mathrm{B}_{s,a})\left(\left\{x \in \{r, \ldots, r+s\} \colon \tfrac{s}{n}\mathrm{K}'_s(x-r) + \tfrac{r}{n} \ni \pi\right\}\right)\\ &= \mathrm{B}_{s,a}(\mathrm{K}'_s \ni a)\\ &\geqslant \beta\end{aligned}$$

by bounding in the second step the union defining $\mathrm{K}(x)$ by the set with the index $(l, m) = (r, s)$. ∎

For proving Theorem 1.2, we use Lemma 2.2 prepared by Lemma 2.1. Let $F_{n,p}$ and $f_{n,p}$ denote the distribution and density functions of the binomial law $\mathrm{B}_{n,p}$.

LEMMA 2.1. *Let* $n \in \mathbb{N}$. *Then*

(2.1) $$F_{n,x/n}(x) < F_{n,1/n}(1) \quad \textit{for } x \in \{2, \ldots, n-1\}.$$

Proof. If $x \in \mathbb{N}$ with $x \leqslant \frac{n-1}{2}$, then for $p \in \left]\frac{x}{n}, \frac{x+1}{n}\right[$ we have $y := x + 1 - np > 0$, hence

$$\begin{aligned}\frac{f_{n-1,p}\,(x)}{f_{n,(x+1)/n}\,(x+1)} &= \frac{f_{n-1,p}\,(x)}{f_{n-1,(x+1)/n}\,(x)}\\ &= \frac{\bigl(1 + y/(n-x-1)\bigr)^{n-x-1}}{\bigl(1 + y/(np)\bigr)^{x}}\\ &> \frac{\bigl(1 + y/(n-x-1)\bigr)^{n-x-1}}{(1 + y/x)^{x}} \geqslant 1,\end{aligned}$$

using the isotonicity of $]0, \infty[\; \ni t \mapsto \left(1 + \frac{y}{t}\right)^t$ in the last step, and hence we get

$$F_{n,x/n}(x) - F_{n,(x+1)/n}(x+1) = n \int_{x/n}^{(x+1)/n} f_{n-1,p}\,(x)\,\mathrm{d}p - f_{n,(x+1)/n}\,(x+1) > 0;$$

consequently, (2.1) holds under the restriction $x \leqslant \frac{n+1}{2}$. If now $x \in \mathbb{N}$ with $\frac{n+1}{2} \leqslant x \leqslant n - 1$, then $1 \leqslant k := n - x < \frac{n}{2}$, and hence an inequality attributed to Simmons by Jogdeo and Samuels ([9], Corollary 4.2) yields $F_{n,k/n}(k-1) > 1 - F_{n,k/n}(k)$, so that

$$F_{n,x/n}(x) = 1 - F_{n,k/n}(k-1) < F_{n,k/n}(k) \leqslant F_{n,1/n}(1),$$

using (2.1) in the last step in a case already proved in the previous sentence. ∎

LEMMA 2.2. *Let* $n \in \mathbb{N}$, $\beta \in [\beta_n, 1[$, *and* $x \in \{2, \ldots, n\}$. *Then* $g_n(x) \leqslant \frac{x-1}{n}$.

Proof. Using Lemma 2.1, we get $F_{n,(x-1)/n}(x-1) \leqslant F_{n,1/n}(1) = \beta_n \leqslant \beta = F_{n,g_n(x)}(x-1)$, and hence the claim. ■

Proof of Theorem 1.2. To simplify the defining representation of K in the present case, let us put

$$g(x) := \min_{\substack{l \in \{0,\ldots,x-1\},\\ m \in \{x-l,\ldots,n-l\}}} \left(\tfrac{m}{n} g_m(x-l) + \tfrac{l}{n}\right) \qquad \text{for } x \in \{1, \ldots, n\}. \tag{2.2}$$

For $x \in \{0, \ldots, n\}$, we have, using (1.5),

$$\mathrm{K}(x) \supseteq \tfrac{n-x}{n} \mathrm{K}_{\mathrm{CP},n-x}(x-x) + \tfrac{x}{n} = \left[\tfrac{x}{n}, 1\right],$$

and hence, in particular, $\mathrm{K}(0) = [0, 1]$. For $x \in \{1, \ldots, n\}$, we have, with (l, m) denoting some pair where the minimum in (2.2) is attained,

$$\mathrm{K}(x) \supseteq \tfrac{m}{n} \mathrm{K}_{\mathrm{CP},m}(x-l) + \tfrac{l}{n} = \left]g(x), \tfrac{l+m}{n}\right] \supseteq \left]g(x), \tfrac{x}{n}\right]$$

and, using $g_x(x) < 1$ in the third step below,

$$\begin{aligned} \mathrm{K}(x) \setminus]g(x), 1] &\subseteq \bigcup_{m \in \{0,\ldots,n-x\}} \left(\tfrac{m}{n} \mathrm{K}_{\mathrm{CP},m}(x-x) + \tfrac{x}{n}\right) \subseteq \left[\tfrac{x}{n}, 1\right] \\ &\subseteq \left]\tfrac{x}{n} g_x(x-0) + \tfrac{0}{n}, 1\right] \subseteq]g(x), 1]. \end{aligned}$$

Combining the above yields

$$\mathrm{K}(x) = \begin{cases} [0, 1] & \text{if } x = 0, \\ ]g(x), 1] & \text{if } x \in \{1, \ldots, n\}, \end{cases} \tag{2.3}$$

so, in particular, K is indeed an upray, and (1.6) holds in its trivial first case. Using (1.3) and the isotonicity of $t \mapsto (\beta^t - 1)/t$ due to the convexity of $t \mapsto \beta^t$, we have

$$g(1) = \min_{m=1}^{n} \tfrac{m}{n} g_m(1) = \tfrac{1}{n} \min_{m=1}^{n} m(1 - \beta^{1/m}) = \tfrac{1-\beta}{n},$$

and hence (1.6) holds also in the second case. The last case is treated at the end of this proof.

K is strictly isotone since, for $x \in \{2,\ldots,n\}$, we get, using $g_m(x-1) < g_m(x)$ for $2 \leqslant x \leqslant m \leqslant n$ due to (1.4),

$$\begin{aligned} g(x) &= \min_{m\in\{x,\ldots,n\}} \tfrac{m}{n} g_m(x) \\ &\quad \wedge \min_{\substack{l\in\{1,\ldots,x-1\},\\ m\in\{x-(l-1)-1,\ldots,n-(l-1)-1\}}} \Big(\tfrac{m}{n} g_m\big(x-1-(l-1)\big) + \tfrac{l-1}{n} + \tfrac{1}{n}\Big) \\ &> \min_{m\in\{x-1,\ldots,n\}} \tfrac{m}{n} g_m(x-1) \wedge \min_{\substack{l\in\{0,\ldots,x-1\},\\ m\in\{x-1-l,\ldots,n-1-l\}}} \big(\tfrac{m}{n} g_m(x-1-l) + \tfrac{l}{n}\big) \\ &\geqslant g(x-1). \end{aligned}$$

By considering $p = (1-\beta, 0, \ldots, 0) \in [0,1]^n$ in the first step below, and using $\mathrm{K}(1) = \left]\frac{1-\beta}{n}, 1\right] \not\ni \frac{1-\beta}{n}$ and the isotonicity of K in the second, we get

$$\inf_{p\in[0,1]^n} \mathrm{BC}_p(\mathrm{K} \ni \overline{p}) \leqslant \mathrm{B}_{1-\beta}\Big(\mathrm{K} \ni \tfrac{1-\beta}{n}\Big) = \mathrm{B}_{1-\beta}(\{0\}) = \beta,$$

and hence, by Theorem 1.1, $\inf_{p\in[0,1]^n} \mathrm{BC}_p(\mathrm{K} \ni \overline{p}) = \beta$.

To prove the optimality of K, let us assume that $\tilde{\mathrm{K}}\colon \{0,\ldots,n\} \to 2^{[0,1]}$ is another isotone upray and that we have an $x' \in \{0,\ldots,n\}$ with

$$\tilde{\mathrm{K}}(x') \subsetneq \mathrm{K}(x'). \tag{2.4}$$

We have to show that $\inf_{p\in[0,1]^n} \mathrm{BC}_p(\tilde{\mathrm{K}} \ni \overline{p}) < \beta$. If $x' = 0$, then $\mathrm{K}(x') = [0,1]$ and, since $\tilde{\mathrm{K}}(0)$ is an upray in $[0,1]$, (2.4) yields $0 \notin \tilde{\mathrm{K}}(0)$, and hence

$$\inf_{p\in[0,1]^n} \mathrm{BC}_p(\tilde{\mathrm{K}} \ni \overline{p}) \leqslant \delta_0(\tilde{\mathrm{K}} \ni 0) = 0 < \beta.$$

If $x' \in \{1,\ldots,n\}$, by (2.3) and (2.2) we get $\mathrm{K}(x') = \left]\frac{m}{n} g_m(x'-l) + \frac{l}{n}, 1\right]$ for some $l \in \{0,\ldots,x'-1\}$ and $m \in \{x'-l,\ldots,n-l\}$, and since $g_m(x'-l) < 1$, we find an $a \in \left]g_m(x'-l), 1\right]$ with $\frac{m}{n} a + \frac{l}{n} \notin \tilde{\mathrm{K}}(x')$. Hence $\frac{m}{n} a + \frac{l}{n} \notin \tilde{\mathrm{K}}(y)$ for $y \in \{x',\ldots,n\}$ by the isotonicity of $\tilde{\mathrm{K}}$, and we obtain

$$\begin{aligned} \inf_{p\in[0,1]^n} \mathrm{BC}_p(\tilde{\mathrm{K}} \ni \overline{p}) &\leqslant \mathrm{B}_{m,a}\big(\{x \in \{0,\ldots,n\} : \tilde{\mathrm{K}}(x+l) \ni \tfrac{l+ma}{n}\}\big) \\ &\leqslant \mathrm{B}_{m,a}(\{0,\ldots,x'-l-1\}) \\ &< \mathrm{B}_{m,g_m(x'-l)}(\{0,\ldots,x'-l-1\}) \\ &= \beta. \end{aligned}$$

To prove the admissibility of K, assume that there was a β-confidence upray K^* for (1.1) with $\mathrm{K}^*(x) \subseteq \mathrm{K}(x)$ for each $x \in \{0,\ldots,n\}$ and $\mathrm{K}^*(x') \subsetneq \mathrm{K}(x')$ for some x'. Then, since K is strictly isotone,

$$\mathrm{K}^{**}(x) := \left.\begin{cases} \mathrm{K}(x) & \text{if } x \neq x', \\ \mathrm{K}^*(x') \cup \mathrm{K}(x'+1) & \text{if } x = x' < n, \\ \mathrm{K}^*(x') & \text{if } x = x' = n, \end{cases}\right\} \supseteq \mathrm{K}^*(x)$$

would define an isotone β-confidence upray for (1.1) with $\mathrm{K}^{**}(x') \subsetneq \mathrm{K}(x')$, contradicting the optimality of K.

To prove finally the last case of (1.6), let $n \geqslant 2$ and $\beta \geqslant \beta_n$, and let now $\tilde{\mathrm{K}}\colon \{0,\ldots,n\} \to 2^{[0,1]}$ be defined by the right-hand side of (1.6). If $p \in [0,1]^n$ with $\overline{p} \in \left[0, \frac{1-\beta}{n}\right]$, then

$$\mathrm{BC}_p(\tilde{\mathrm{K}} \ni \overline{p}) \geqslant \mathrm{BC}_p(\{0\}) = \prod_{j=1}^{n}(1-p_j) \geqslant 1 - \sum_{j=1}^{n} p_j = 1 - n\overline{p} \geqslant \beta.$$

If $p \in [0,1]^n$ with $\overline{p} \in \left]\frac{1-\beta}{n}, 1\right]$, then with $g_n(n+1) := 1$ either there is a $c \in \{2,\ldots,n\}$ with $\overline{p} \in]g_n(c), g_n(c+1)]$, or $\overline{p} \in \left]\frac{1-\beta}{n}, g_n(2)\right]$ and we put $c := 1$; in either case then $n\overline{p} \leqslant n g_n(c+1) \leqslant c \leqslant n$ by Lemma 2.2, and hence an application of Theorem 4, (26) from Hoeffding [8] in the second step below yields

$$\mathrm{BC}_p(\tilde{\mathrm{K}} \ni \overline{p}) = \mathrm{BC}_p(\{0,\ldots,c\}) \geqslant F_{n,\overline{p}}(c) \geqslant F_{n,g_n(c+1)}(c) \geqslant \beta.$$

Hence $\tilde{\mathrm{K}}$ is a β-confidence upray for (1.1) and satisfies $\tilde{\mathrm{K}}(x) \subseteq \mathrm{K}(x)$ for each x, and so the admissibility of K yields $\tilde{\mathrm{K}} = \mathrm{K}$, and hence (1.6) holds true. ∎

P r o o f o f T h e o r e m 1.3. Let $\gamma := \frac{1+\beta}{2}$, let K_γ be the γ-confidence upray from Theorem 1.2, and let Λ_γ be the analogous γ-confidence downray from Remark 1.6. Then, by subadditivity, $\mathrm{M}_\beta(x) := \mathrm{K}_\gamma(x) \cap \Lambda_\gamma(x)$ for $x \in \{0,\ldots,n\}$ defines a β-confidence interval for (1.1). If $n = 1$, then $\mathrm{M}_{\mathrm{CP},n} = \mathrm{M}_\beta$, hence the claim. So let $\beta \geqslant 2\beta_n - 1$, that is, $\gamma \geqslant \beta_n$. Then (1.6) and (1.7), with γ in place of β, yield $\mathrm{M}_{\mathrm{CP},n}(x) = \mathrm{M}_\beta(x)$ for $x \notin \{1, n-1\}$. So, if $\overline{p} \notin \big(\mathrm{M}_{\mathrm{CP},n}(1) \setminus \mathrm{M}_\beta(1)\big) \cup \big(\mathrm{M}_{\mathrm{CP},n}(n-1) \setminus \mathrm{M}_\beta(n-1)\big)$, we have $\mathrm{BC}_p(\mathrm{M}_{\mathrm{CP},n} \ni \overline{p}) = \mathrm{BC}_p(\mathrm{M}_\beta \ni \overline{p}) \geqslant \beta$. Otherwise, $\overline{p} \in \left]\frac{1-\gamma}{n}, g_{n,\gamma}(1)\right]$ or $\overline{p} \in \left[g_{n,\gamma}(n-1), 1 - \frac{1-\gamma}{n}\right[$. In the first case, we have $\overline{p} \in \left]\frac{1-\gamma}{n}, g_{n,\gamma}(1)\right] = \left]\frac{1-\gamma}{n}, 1-\gamma^{1/n}\right] \subseteq [0, 1-(1-\gamma)^{1/n}[= \mathrm{M}_{\mathrm{CP},n}(0)$ and from $\overline{p} \in \mathrm{M}_{\mathrm{CP},n}(0)$ and $\overline{p} \leqslant 1 - \gamma^{1/n}$ we get

$$\begin{aligned}\mathrm{BC}_p(\mathrm{M}_{\mathrm{CP},n} \ni \overline{p}) \geqslant \mathrm{BC}_p(\{0\}) &= \prod_{j=1}^{n}(1-p_j) \\ &\geqslant 1 - n\overline{p} \geqslant 1 - n(1-\gamma^{1/n}) \geqslant \gamma > \beta.\end{aligned}$$

In the second case, analogously, $\overline{p} \in \left[g_{n,\gamma}(n-1), 1 - \frac{1-\gamma}{n}\right[= \left[\gamma^{1/n}, 1 - \frac{1-\gamma}{n}\right[\subseteq \mathrm{M}_{\mathrm{CP},n}(n)$ and from $\overline{p} \in \mathrm{M}_{\mathrm{CP},n}(n)$ and $\overline{p} \geqslant \gamma^{1/n}$ we get $\mathrm{BC}_p(\mathrm{M}_{\mathrm{CP},n} \ni \overline{p}) \geqslant \mathrm{BC}_p(\{n\}) = \prod_{j=1}^{n} p_j \geqslant \overline{p}^n \geqslant \gamma > \beta$. ∎

Acknowledgments. We thank Jona Schulz for help with the proof of Lemma 2.1, and the referee for suggesting practical computations and to address nestedness.

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Received on 9.4.2014;
revised version on 18.12.2014