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EFFICIENT CLASSES OF RATIO-CUM-PRODUCT ESTIMATORS OF POPULATION MEAN IN STRATIFIED RANDOM SAMPLING

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Abstract. In this paper, classes of separate and combined ratio-product estimators are proposed for estimating the finite population mean in stratified random sampling. The expressions for biases and mean squared errors (MSEs) of the proposed classes are derived to the first order of approximation. It is also verified that the proposed classes of estimators, under their optimum conditions, are equivalent to the separate regression estimator. The proposed classes of estimators are compared with the other existing estimators by using the MSE criterion, and the conditions under which the proposed classes perform better are obtained. Theoretical results are validated with the help of an empirical study.

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1. INTRODUCTION

Stratification means division into layers (or groups). Auxiliary information (e.g., past data or some other information) related to the study variable may be utilized to classify the entire heterogeneous population into different groups such that (i) units within each group are as homogeneous as possible, and (ii) the group means are as widely different as possible. Thus the population under investigation is divided into different strata so as to obtain the homogeneity within each stratum, and the sample observations are drawn from each stratum generally by a well-known procedure of simple random sampling (SRS). It is also a well-established fact that the proper use of supplementary information on auxiliary variable(s) may lead to more efficient estimators of population parameter(s) (i.e., population mean in the present case) under consideration. The literature on survey sampling describes a great variety of techniques for using auxiliary information by means of ratio, product and regression methods of estimation.

Let y and x denote, respectively, the study variable and the auxiliary variable. Also, let $\bar{Y}$ and $\bar{X}$ denote the respective population means of y and x. Then the ratio (product) estimator of $\bar{Y}$ is more precise as compared to the usual unbiased estimator, and is equally efficient as the regression estimator provided the relationship between y and x is linear through the origin, and the variance of y is proportional to that of x. However, in many practical situations, the line does not pass through the neighborhood of the origin, and hence the performance of the ratio (product) estimator is not significant. This led various authors, including Srivastava [17], Walsh [22], Reddy [8], Gupta [2], Vos [21], Sukhatme et al. [18], Naik and Gupta [7], Upadhyaya and Singh [20], Singh [11], and Singh and Ruiz Espejo [12], to modify the ratio (product) estimator in SRS to get better alternatives. Also, several authors such as Kadilar and Cingi ([4], [5]), Singh and Vishwakarma ([13], [14]), Koyuncu and Kadilar [6], Shabbir and Gupta [9], and Tailor et al. [19] made attempts to develop estimators in stratified random sampling.

Keeping this fact in view, we have made an effort to propose classes of separate and combined ratio-product estimators for population mean $\bar{Y}$ using auxiliary information in stratified random sampling and analyzed their properties.

For this, we consider a finite population $U = \{U_1, U_2, \ldots, U_N\}$ consisting of N units, and the units are partitioned into L distinct strata with hth stratum containing N_h units $(h = 1, 2, \ldots, L)$ such that $\sum_{h=1}^{L} N_h = N$. Let n_h be the size of the sample drawn from the hth stratum by using the simple random sampling without replacement (SRSWOR) scheme such that $\sum_{h=1}^{L} n_h = n$. Let (y_{hi}, x_{hi}) be the observed values of (y, x) on the ith unit of the hth stratum $(i = 1, 2, \ldots, N_h)$.

Moreover, the population means of the variables y and x in the hth stratum are $\bar{Y}_h = \sum_{i=1}^{N_h} y_{hi}/N_h$, $\bar{X}_h = \sum_{i=1}^{N_h} x_{hi}/N_h$, and the corresponding sample means in the hth stratum are $\bar{y}_h = \sum_{i=1}^{n_h} y_{hi}/n_h$, $\bar{x}_h = \sum_{i=1}^{n_h} x_{hi}/n_h$.

The sample means of the variables y and x, in stratified random sampling, are given by

$$\bar{y}_{st} = \sum_{h=1}^{L} W_h \bar{y}_h \quad \text{and} \quad \bar{x}_{st} = \sum_{h=1}^{L} W_h \bar{x}_h,$$

where $W_h = N_h/N$ denotes the stratum weight. Also, $\bar{y}_{st}$ and $\bar{x}_{st}$ are the unbiased estimators of the population means $\bar{Y} = \sum_{h=1}^{L} W_h \bar{Y}_h$ and $\bar{X} = \sum_{h=1}^{L} W_h \bar{X}_h$, respectively.

The separate ratio estimator for the population mean $\bar{Y}$ is defined as

$$\bar{y}_{RS} = \sum_{h=1}^{L} W_h \bar{y}_h \left(\frac{\bar{X}_h}{\bar{x}_h}\right). \tag{1.1}$$

This estimator is preferred over the usual unbiased estimator $\bar{y}_{st}$ provided the variables y and x are strongly positively correlated. However, if the variables y and x are negatively correlated, the population mean $\bar{Y}$ is estimated by using a separate product estimator, which is given by

$$\bar{y}_{PS} = \sum_{h=1}^{L} W_h \bar{y}_h \left(\frac{\bar{x}_h}{\bar{X}_h} \right). \tag{1.2}$$

Also, the separate regression estimator for $\bar{Y}$ is defined as

$$\bar{y}_{lrs} = \sum_{h=1}^{L} W_h [\bar{y}_h + b_h(\bar{X}_h - \bar{x}_h)]. \tag{1.3}$$

Here, $b_h = s_{yxh}/s_{xh}^2$ denotes the sample regression coefficient of y on x in the hth stratum, where

$$s_{xh}^2 = \frac{1}{n_h - 1} \sum_{i=1}^{n_h} (x_{hi} - \bar{x}_h)^2 \quad \text{and} \quad s_{yxh} = \frac{1}{n_h - 1} \sum_{i=1}^{n_h} (y_{hi} - \bar{y}_h)(x_{hi} - \bar{x}_h).$$

Sometimes the population mean $\bar{X}_h$ of the auxiliary variable x in the hth stratum is not known in advance, rather the combined population mean $\bar{X}$ is known. In such a situation, it is not possible to use the separate ratio, product and regression estimators, but we can use the traditional combined ratio, product and regression estimators (see Singh [16]), which are given, respectively, by

$$\bar{y}_{RC} = \bar{y}_{st} \left(\frac{\bar{X}}{\bar{x}_{st}} \right), \tag{1.4}$$

$$\bar{y}_{PC} = \bar{y}_{st} \left(\frac{\bar{x}_{st}}{\bar{X}} \right), \tag{1.5}$$

$$\bar{y}_{lrc} = \bar{y}_{st} + b(\bar{X} - \bar{x}_{st}), \tag{1.6}$$

where $b = \sum_{h=1}^{L} W_h^2 \lambda_h s_{yxh} \big/ \sum_{h=1}^{L} W_h^2 \lambda_h s_{xh}^2$, $\lambda_h = 1/n_h - 1/N_h$.

It is well known that the variance of the stratified sample mean $\bar{y}_{st}$ under SRSWOR is given by

$$\operatorname{Var}(\bar{y}_{st}) = \sum_{h=1}^{L} W_h^2 \lambda_h S_{yh}^2 = \sum_{h=1}^{L} W_h^2 \lambda_h \bar{Y}_h^2 C_{yh}^2, \tag{1.7}$$

where $C_{yh} = S_{yh}/\bar{Y}_h$.

To the first order of approximation, the mean squared errors (MSEs) of $\bar{y}_{RS}$, $\bar{y}_{PS}$, $\bar{y}_{lrs}$, $\bar{y}_{RC}$, $\bar{y}_{PC}$ and $\bar{y}_{lrc}$ are given, respectively, by

$$\operatorname{MSE}(\bar{y}_{RS}) = \sum_{h=1}^{L} W_h^2 \lambda_h (S_{yh}^2 - 2R_h S_{yxh} + R_h^2 S_{xh}^2), \tag{1.8}$$

$$\operatorname{MSE}(\bar{y}_{PS}) = \sum_{h=1}^{L} W_h^2 \lambda_h (S_{yh}^2 + 2R_h S_{yxh} + R_h^2 S_{xh}^2), \tag{1.9}$$

$$\text{MSE}(\bar{y}_{lrs}) = \sum_{h=1}^{L} W_h^2 \lambda_h S_{yh}^2 (1 - \rho_{yxh}^2), \tag{1.10}$$

$$\text{MSE}(\bar{y}_{RC}) = \sum_{h=1}^{L} W_h^2 \lambda_h (S_{yh}^2 - 2RS_{yxh} + R^2 S_{xh}^2), \tag{1.11}$$

$$\text{MSE}(\bar{y}_{PC}) = \sum_{h=1}^{L} W_h^2 \lambda_h (S_{yh}^2 + 2RS_{yxh} + R^2 S_{xh}^2), \tag{1.12}$$

$$\text{MSE}(\bar{y}_{lrc}) = \sum_{h=1}^{L} W_h^2 \lambda_h S_{yh}^2 (1 - \rho_{yx}^2), \tag{1.13}$$

where $R_h = \bar{Y}_h / \bar{X}_h$, $R = \bar{Y}/\bar{X}$,

$$\rho_{yx} = \frac{\sum_{h=1}^{L} W_h^2 \lambda_h S_{yxh}}{\sqrt{\left(\sum_{h=1}^{L} W_h^2 \lambda_h S_{yh}^2\right)\left(\sum_{h=1}^{L} W_h^2 \lambda_h S_{xh}^2\right)}}, \quad \rho_{yxh} = \frac{S_{yxh}}{S_{yh} S_{xh}},$$

$$S_{yh}^2 = \frac{1}{N_h - 1} \sum_{i=1}^{N_h} (y_{hi} - \bar{Y}_h)^2, \quad S_{xh}^2 = \frac{1}{N_h - 1} \sum_{i=1}^{N_h} (x_{hi} - \bar{X}_h)^2,$$

$$S_{yxh} = \frac{1}{N_h - 1} \sum_{i=1}^{N_h} (y_{hi} - \bar{Y}_h)(x_{hi} - \bar{X}_h).$$

Theoretically, it has been established that, in general, the regression estimator is more efficient than the ratio and product estimators except when the regression line of the variable under study on the auxiliary variable passes through the neighborhood of the origin (see Cochran [1]). Further, we note that in many practical situations the regression line does not pass through the neighborhood of the origin. In these situations, the ratio (product) estimator does not perform so well as the linear regression estimator. Considering this fact, an attempt is made to improve the performance of suggested ratio-product estimators with their properties.

2. THE PROPOSED CLASS OF SEPARATE RATIO-PRODUCT ESTIMATORS

We propose the following class of separate ratio-product estimators for estimating population mean $\bar{Y}$:

$$\hat{\bar{Y}}_{RP}^{(s)} = \sum_{h=1}^{L} W_h \bar{y}_h \left[\alpha_h \left\{ \frac{\beta_h \bar{X}_h + \gamma_h}{\beta_h \bar{x}_h + \gamma_h} \right\} + (1 - \alpha_h) \left\{ \frac{\beta_h \bar{x}_h + \gamma_h}{\beta_h \bar{X}_h + \gamma_h} \right\} \right], \tag{2.1}$$

where α_h, β_h and γ_h are either real constants or functions of some known parameters of an auxiliary variable x, which are determined so that the MSE of $\hat{\bar{Y}}_{RP}^{(s)}$ is the minimum.

To obtain the bias and MSE of $\hat{\bar{Y}}^{(s)}_{RP}$, we write

$$\bar{y}_h = \bar{Y}_h(1 + e_{0h}) \quad \text{and} \quad \bar{x}_h = \bar{X}_h(1 + e_{1h})$$

such that $E(e_{0h}) = E(e_{1h}) = 0$, and under the SRSWOR, we have

$$E(e^2_{0h}) = \lambda_h C^2_{yh}, \quad E(e^2_{1h}) = \lambda_h C^2_{xh}, \quad E(e_{0h}e_{1h}) = \lambda_h \rho_{yxh} C_{yh} C_{xh}. \tag{2.2}$$

Expressing (2.1) in terms of e's, we have

$$\hat{\bar{Y}}^{(s)}_{RP} = \sum_{h=1}^{L} W_h \bar{Y}_h (1 + e_{0h})\{\alpha_h (1 + \delta_h e_{1h})^{-1} + (1 - \alpha_h)(1 + \delta_h e_{1h})\}, \tag{2.3}$$

where $\delta_h = \{\beta_h \bar{X}_h \big/ (\beta_h \bar{X}_h + \gamma_h)\}$. Now, expanding (2.3), multiplying out and retaining terms of e's up to the second degree, we obtain

$$\begin{aligned}
&\hat{\bar{Y}}^{(s)}_{RP} = \\
&= \sum_{h=1}^{L} W_h \bar{Y}_h (1 + e_{0h})\{\alpha_h (1 - \delta_h e_{1h} + \delta^2_h e^2_{1h}) + (1 + \delta_h e_{1h} - \alpha_h - \alpha_h \delta_h e_{1h})\} \\
&= \sum_{h=1}^{L} W_h \bar{Y}_h \{1 + e_{0h} + \delta_h e_{1h} + \delta_h e_{0h} e_{1h} + \alpha_h (\delta^2_h e^2_{1h} - 2\delta_h e_{0h} e_{1h} - 2\delta_h e_{1h})\} \\
&= \sum_{h=1}^{L} W_h \bar{Y}_h + \sum_{h=1}^{L} W_h \bar{Y}_h \{e_{0h} + \delta_h e_{1h} + \delta_h e_{0h} e_{1h} \\
&\qquad\qquad + \alpha_h (\delta^2_h e^2_{1h} - 2\delta_h e_{0h} e_{1h} - 2\delta_h e_{1h})\} \\
&= \bar{Y} + \sum_{h=1}^{L} W_h \bar{Y}_h \{e_{0h} + \delta_h e_{1h} + \delta_h e_{0h} e_{1h} + \alpha_h (\delta^2_h e^2_{1h} - 2\delta_h e_{0h} e_{1h} - 2\delta_h e_{1h})\}
\end{aligned}$$

or, equivalently,

$$\begin{aligned}
&\hat{\bar{Y}}^{(s)}_{RP} - \bar{Y} = \\
&= \sum_{h=1}^{L} W_h \bar{Y}_h \{e_{0h} + \delta_h e_{1h} + \delta_h e_{0h} e_{1h} + \alpha_h (\delta^2_h e^2_{1h} - 2\delta_h e_{0h} e_{1h} - 2\delta_h e_{1h})\}.
\end{aligned} \tag{2.4}$$

Taking the expectation on both sides of (2.4) and using results in (2.2), we obtain the bias of $\hat{\bar{Y}}^{(s)}_{RP}$ to the first order of approximation (i.e., to the terms of order $O(n^{-1}_h)$) as

$$\mathrm{Bias}(\hat{\bar{Y}}^{(s)}_{RP}) = \sum_{h=1}^{L} \frac{W_h \lambda_h \delta_h}{\bar{X}_h} \{\alpha_h R_h \delta_h S^2_{xh} + (1 - 2\alpha_h) S_{yxh}\}. \tag{2.5}$$

Squaring both sides of (2.4) and retaining terms of e's to the second degree, we have

$$(\hat{\hat{Y}}_{RP}^{(s)} - \bar{Y})^2 = \Big[\sum_{h=1}^{L} W_h \bar{Y}_h \{e_{0h} + (1-2\alpha_h)\delta_h e_{1h}\}\Big]^2$$

or, equivalently,

$$(2.6)\quad (\hat{\hat{Y}}_{RP}^{(s)} - \bar{Y})^2 = \\ = \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \{e_{0h}^2 + \delta_h^2 e_{1h}^2 (1-2\alpha_h)^2 + 2(1-2\alpha_h)\delta_h e_{0h} e_{1h}\} \\ + \sum_{h \neq l=1}^{L} W_h W_l \bar{Y}_h \bar{Y}_l \{e_{0h} + (1-2\alpha_h)\delta_h e_{1h}\}\{e_{0l} + (1-2\alpha_l)\delta_l e_{1l}\}.$$

Taking the expectation on both sides of (2.6) and using results in (2.2), we obtain the MSE of $\hat{\hat{Y}}_{RP}^{(s)}$ to the first order of approximation as

(2.7)

$$\mathrm{MSE}(\hat{\hat{Y}}_{RP}^{(s)}) = \sum_{h=1}^{L} W_h^2 \lambda_h \{S_{yh}^2 + 2(1-2\alpha_h) R_h \delta_h S_{yxh} + (1-2\alpha_h)^2 R_h^2 \delta_h^2 S_{xh}^2\}.$$

We note that sampling is carried out independently in each stratum, therefore the covariance terms are vanished (i.e., $E(e_{0h}e_{0l}) = E(e_{0h}e_{1l}) = E(e_{1h}e_{0l}) = E(e_{1h}e_{1l}) = 0$). The MSE of $\hat{\hat{Y}}_{RP}^{(s)}$ in (2.7) is minimized for

$$(2.8)\qquad \delta_h = \frac{\rho_{yxh} S_{yh}}{(2\alpha_h - 1) R_h S_{xh}} = \delta_{h(\mathrm{opt})} \text{ (say)}.$$

Substitution of $\delta_{h(\mathrm{opt})}$ in place of δ_h in (2.7) yields the minimum MSE of $\hat{\hat{Y}}_{RP}^{(s)}$ as

$$(2.9)\qquad \mathrm{MSE}(\hat{\hat{Y}}_{RP}^{(s)})_{\min} = \sum_{h=1}^{L} W_h^2 \lambda_h S_{yh}^2 (1-\rho_{yxh}^2) = \mathrm{MSE}(\bar{y}_{lrs}).$$

Thus we establish the following theorem.

THEOREM 2.1. *To the first order of approximation*,

$$(2.10)\qquad \mathrm{MSE}(\hat{\hat{Y}}_{RP}^{(s)}) \geqslant \sum_{h=1}^{L} W_h^2 \lambda_h S_{yh}^2 (1-\rho_{yxh}^2)$$

with equality holding if $\delta_h = \rho_{yxh} S_{yh} / \{(2\alpha_h - 1) R_h S_{xh}\}$.

REMARK 2.1. *The lower bound of the MSE of* $\hat{\hat{Y}}_{RP}^{(s)}$ *in* (2.10) *is the same as that of the MSE of the separate regression estimator* $\bar{y}_{lrs}$. *Hence, the asymptotic optimum estimator in the proposed class* $\hat{\hat{Y}}_{RP}^{(s)}$ *corresponds to the separate regression estimator*, *i.e.*,

$$(2.11)\qquad \hat{\hat{Y}}_{RP(\mathrm{opt})}^{(s)} = \bar{y}_{lrs}.$$

3. EFFICIENCY COMPARISONS FOR THE CLASS PROPOSED IN SECTION 2

From (1.7)–(1.10) and (2.7) we have the following:

(i) $\mathrm{MSE}(\hat{\hat{Y}}^{(s)}_{RP}) < \mathrm{Var}(\bar{y}_{st})$ if

$$\min\left\{\frac{1}{2}, \frac{2K_h+\delta_h}{2\delta_h}\right\} < \alpha_h < \max\left\{\frac{1}{2}, \frac{2K_h+\delta_h}{2\delta_h}\right\}. \tag{3.1}$$

(ii) $\mathrm{MSE}(\hat{\hat{Y}}^{(s)}_{RP}) < \mathrm{MSE}(\bar{y}_{RS})$ if

$$\min\left\{\frac{1+\delta_h}{2\delta_h}, \frac{2K_h+\delta_h-1}{2\delta_h}\right\} < \alpha_h < \max\left\{\frac{1+\delta_h}{2\delta_h}, \frac{2K_h+\delta_h-1}{2\delta_h}\right\}. \tag{3.2}$$

(iii) $\mathrm{MSE}(\hat{\hat{Y}}^{(s)}_{RP}) < \mathrm{MSE}(\bar{y}_{PS})$ if

$$\min\left\{\frac{\delta_h-1}{2\delta_h}, \frac{2K_h+\delta_h+1}{2\delta_h}\right\} < \alpha_h < \max\left\{\frac{\delta_h-1}{2\delta_h}, \frac{2K_h+\delta_h+1}{2\delta_h}\right\}. \tag{3.3}$$

(iv) $\mathrm{MSE}(\hat{\hat{Y}}^{(s)}_{RP}) = \mathrm{MSE}(\bar{y}_{lrs})$ if

$$\alpha_h = \frac{K_h+\delta_h}{2\delta_h}, \tag{3.4}$$

where

$$K_h = \frac{\rho_{yxh}C_{yh}}{C_{xh}} \quad \text{with } C_{yh} = \frac{S_{yh}}{\bar{Y}_h},\ \ C_{xh} = \frac{S_{xh}}{\bar{X}_h}.$$

4. THE PROPOSED CLASS OF COMBINED RATIO-PRODUCT ESTIMATORS

We propose the following class of combined ratio-product estimators for $\bar{Y}$:

(4.1)
$$\hat{\hat{Y}}^{(c)}_{RP} = \bar{y}_{st}\left[\alpha\left\{\frac{\sum\limits_{h=1}^{L} W_h(\beta_h\bar{X}_h+\gamma_h)}{\sum\limits_{h=1}^{L} W_h(\beta_h\bar{x}_h+\gamma_h)}\right\} + (1-\alpha)\left\{\frac{\sum\limits_{h=1}^{L} W_h(\beta_h\bar{x}_h+\gamma_h)}{\sum\limits_{h=1}^{L} W_h(\beta_h\bar{X}_h+\gamma_h)}\right\}\right],$$

where α, β_h and γ_h are either real constants or functions of some known parameters of an auxiliary variable x, which are determined so that the MSE of $\hat{\hat{Y}}^{(c)}_{RP}$ is the minimum.

To obtain the bias and MSE of $\hat{\hat{Y}}^{(c)}_{RP}$, we write

$$\bar{y}_h = \bar{Y}_h(1+e_{0h}) \quad \text{and} \quad \bar{x}_h = \bar{X}_h(1+e_{1h})$$

such that $E(e_{0h}) = E(e_{1h}) = 0$, and under the SRSWOR, we have

$$E(e_{0h}^2) = \lambda_h C_{yh}^2, \quad E(e_{1h}^2) = \lambda_h C_{xh}^2, \quad E(e_{0h}e_{1h}) = \lambda_h\rho_{yxh}C_{yh}C_{xh}. \tag{4.2}$$

Expressing (4.1) in terms of e's, we get

$$\hat{\bar{Y}}_{RP}^{(c)} = \bar{Y}(1+e_0)\{\alpha(1+e_1)^{-1} + (1-\alpha)(1+e_1)\}, \tag{4.3}$$

where

$$e_0 = \frac{1}{\bar{Y}}\sum_{h=1}^{L} W_h \bar{Y}_h e_{0h} \quad \text{and} \quad e_1 = \frac{1}{\bar{X}_M}\sum_{h=1}^{L} W_h \beta_h \bar{X}_h e_{1h}$$

with $\bar{X}_M = \sum_{h=1}^{L} W_h(\beta_h \bar{X}_h + \gamma_h)$. We will now assume that $|e_1| < 1$ so that we may expand $(1+e_1)^{-1}$ as a series in powers of e_1. Expanding (4.3), multiplying out and retaining terms of e's to the second degree, we obtain

$$\hat{\bar{Y}}_{RP}^{(c)} = \bar{Y}\{1 + e_0 + e_1 + e_0 e_1 + \alpha(e_1^2 - 2e_0e_1 - 2e_1)\},$$

$$\hat{\bar{Y}}_{RP}^{(c)} - \bar{Y} = \bar{Y}\{e_0 + e_1 + e_0 e_1 + \alpha(e_1^2 - 2e_0e_1 - 2e_1)\}. \tag{4.4}$$

Taking the expectation on both sides of (4.4) and using results of (4.2), we obtain the bias of $\hat{\bar{Y}}_{RP}^{(c)}$ to the first order of approximation as

$$\text{Bias}(\hat{\bar{Y}}_{RP}^{(c)}) = \frac{1}{\bar{X}_M}\sum_{h=1}^{L} W_h^2 \lambda_h \beta_h \{\alpha R_M \beta_h S_{xh}^2 + (1-2\alpha)S_{yxh}\}, \tag{4.5}$$

where $R_M = \bar{Y}/\bar{X}_M$. Squaring both sides of (4.4) and again retaining terms of e's to the second degree, we have

$$(\hat{\bar{Y}}_{RP}^{(c)} - \bar{Y})^2 = \bar{Y}^2\{e_0^2 + (1-2\alpha)^2 e_1^2 + 2(1-2\alpha)e_0e_1\}. \tag{4.6}$$

Taking the expectation on both sides of (4.6) and using results of (4.2), we obtain the MSE of $\hat{\bar{Y}}_{RP}^{(c)}$ to the first order of approximation as

(4.7)

$$\text{MSE}(\hat{\bar{Y}}_{RP}^{(c)}) = \sum_{h=1}^{L} W_h^2 \lambda_h \{S_{yh}^2 + 2(1-2\alpha)R_M \beta_h S_{yxh} + (1-2\alpha)^2 R_M^2 \beta_h^2 S_{xh}^2\}.$$

Minimization of (4.7) with respect to β_h gives the optimum value of β_h in the form

$$\beta_h = \frac{\rho_{yxh} S_{yh}}{(2\alpha - 1) R_M S_{xh}} = \beta_{h(\text{opt})} \text{ (say)}. \tag{4.8}$$

Substituting $\beta_h = \beta_{h(\text{opt})}$ in (4.7), we obtain the minimum MSE of $\hat{\bar{Y}}_{RP}^{(c)}$ as

$$\text{MSE}(\hat{\bar{Y}}_{RP}^{(c)})_{\min} = \sum_{h=1}^{L} W_h^2 \lambda_h S_{yh}^2 (1 - \rho_{yxh}^2) = \text{MSE}(\bar{y}_{lrs}). \tag{4.9}$$

Hence we establish the following theorem.

THEOREM 4.1. *To the first order of approximation*,

$$\text{MSE}(\hat{\bar{Y}}_{RP}^{(c)}) \geqslant \sum_{h=1}^{L} W_h^2 \lambda_h S_{yh}^2 (1-\rho_{yxh}^2) \tag{4.10}$$

with equality holding if $\beta_h = \rho_{yxh} S_{yh} \big/ \{(2\alpha-1) R_M S_{xh}\}$.

REMARK 4.1. *The lower bound of the MSE of* $\hat{\bar{Y}}_{RP}^{(c)}$ *in* (4.10) *is the same as that of the MSE of the separate regression estimator* $\bar{y}_{lrs}$. *Hence, the asymptotic optimum estimator in the proposed class* $\hat{\bar{Y}}_{RP}^{(c)}$ *corresponds to the separate regression estimator, i.e.*,

$$\hat{\bar{Y}}_{RP(\text{opt})}^{(c)} = \bar{y}_{lrs}. \tag{4.11}$$

5. EFFICIENCY COMPARISONS FOR THE CLASS PROPOSED IN SECTION 4

From (1.7), (1.11)–(1.13) and (4.7) we have the following:

(i) $\text{MSE}(\hat{\bar{Y}}_{RP}^{(c)}) < \text{Var}(\bar{y}_{st})$ if

$$\min\left\{\frac{1}{2}, \frac{2\rho_{yxh}S_{yh} + R_M\beta_h S_{xh}}{2R_M\beta_h S_{xh}}\right\} < \alpha < \max\left\{\frac{1}{2}, \frac{2\rho_{yxh}S_{yh} + R_M\beta_h S_{xh}}{2R_M\beta_h S_{xh}}\right\}. \tag{5.1}$$

(ii) $\text{MSE}(\hat{\bar{Y}}_{RP}^{(c)}) < \text{MSE}(\bar{y}_{RC})$ if

$$\min\left\{\frac{R_M\beta_h + R}{2R_M\beta_h}, \frac{2\rho_{yxh}S_{yh} + (R_M\beta_h - R)S_{xh}}{2R_M\beta_h S_{xh}}\right\} < \alpha < \max\left\{\frac{R_M\beta_h + R}{2R_M\beta_h}, \frac{2\rho_{yxh}S_{yh} + (R_M\beta_h - R)S_{xh}}{2R_M\beta_h S_{xh}}\right\}. \tag{5.2}$$

(iii) $\text{MSE}(\hat{\bar{Y}}_{RP}^{(c)}) < \text{MSE}(\bar{y}_{PC})$ if

$$\min\left\{\frac{R_M\beta_h - R}{2R_M\beta_h}, \frac{2\rho_{yxh}S_{yh} + (R_M\beta_h + R)S_{xh}}{2R_M\beta_h S_{xh}}\right\} < \alpha < \max\left\{\frac{R_M\beta_h - R}{2R_M\beta_h}, \frac{2\rho_{yxh}S_{yh} + (R_M\beta_h + R)S_{xh}}{2R_M\beta_h S_{xh}}\right\}. \tag{5.3}$$

(iv) $\text{MSE}(\hat{\bar{Y}}_{RP}^{(c)}) < \text{MSE}(\bar{y}_{lrc})$ if

$$\left\{\frac{S_{yh}\big(\rho_{yxh} - \sqrt{\rho_{yxh}^2 - \rho_{yx}^2}\big) + R_M\beta_h S_{xh}}{2R_M\beta_h S_{xh}}\right\} < \alpha < \left\{\frac{S_{yh}\big(\rho_{yxh} + \sqrt{\rho_{yxh}^2 - \rho_{yx}^2}\big) + R_M\beta_h S_{xh}}{2R_M\beta_h S_{xh}}\right\}. \tag{5.4}$$

6. EMPIRICAL STUDY

To demonstrate the performances of the proposed classes of estimators $\hat{\bar{Y}}_{RP}^{(s)}$ and $\hat{\bar{Y}}_{RP}^{(c)}$ over the other existing estimators, three population data sets have been considered. The description of the populations along with the values of various parameters is given in Table 1.

The MSEs along with the percent relative efficiencies (PREs) of various estimators of $\bar{Y}$ have been computed, and findings are presented in Table 2. The PREs are obtained for various suggested estimators of $\bar{Y}$ with respect to the stratified sample mean $\bar{y}_{st}$ using the formula

$$\mathrm{PRE}(\phi, \bar{y}_{st}) = \frac{\mathrm{Var}(\bar{y}_{st})}{\mathrm{MSE}(\phi)} \times 100,$$

where $\phi = \bar{y}_{st},\ \bar{y}_{RS},\ \bar{y}_{PS},\ \bar{y}_{lrs},\ \bar{y}_{RC},\ \bar{y}_{PC},\ \bar{y}_{lrc},\ \hat{\bar{Y}}_{RP(\mathrm{opt})}^{(s)},\ \hat{\bar{Y}}_{RP(\mathrm{opt})}^{(c)}$.

TABLE 1. Description of populations I–III

Population	Data sets	Stratum no. (h)	N_h	n_h	$\bar{Y}_h$	$\bar{X}_h$	S_{yh}^2	S_{xh}^2	S_{yxh}
I (1)	$N = 25, n = 10$	1	6	3	417.33	6.81	74775.47	15.97	1007.05
	$\bar{Y} = 410.841$	2	8	3	503.38	10.12	259113.70	132.66	5709.16
	$\bar{X} = 8.3796$	3	11	4	340.00	7.97	65885.60	38.44	1404.71
II (2)	$N = 25, n = 10$	1	6	3	135.00	366.67	80.00	2706.67	440.00
	$\bar{Y} = 102.6$	2	12	4	99.17	310.83	226.52	1881.06	618.94
	$\bar{X} = 325.998$	3	7	3	80.71	317.14	120.24	2890.48	444.05
III (3)	$N = 20, n = 8$	1	10	4	149.70	1622.99	181.17	10438.71	-1072.80
	$\bar{Y} = 126.15$	2	10	4	102.60	2035.95	158.76	10662.63	-655.25
	$\bar{X} = 1829.47$								

(Header: "Values of parameters for the hth stratum" spans columns N_h to S_{yxh}.)

Sources:
(1) Singh and Chaudhary [10]. y: total number of trees; x: area under orchards in hectares.
(2) Singh and Singh Mangat [15]. y: juice quantity; x: weight of cane.
(3) Japan Meteorological Society [3]. y: study variable; x: auxiliary variable.

From Table 2 it is observed that:

(i) In both the populations I and II, the proposed classes of estimators $\hat{\bar{Y}}_{RP}^{(s)}$ and $\hat{\bar{Y}}_{RP}^{(c)}$ perform well as compared to the stratified sample mean ($\bar{y}_{st}$), separate ratio estimator ($\bar{y}_{RS}$), combined ratio estimator ($\bar{y}_{RC}$), and combined regression estimator ($\bar{y}_{lrc}$).

(ii) In population III, the proposed classes of estimators $\hat{\bar{Y}}_{RP}^{(s)}$ and $\hat{\bar{Y}}_{RP}^{(c)}$ perform well as compared to the stratified sample mean ($\bar{y}_{st}$), separate product estimator ($\bar{y}_{PS}$), combined product estimator ($\bar{y}_{PC}$), and combined regression estimator ($\bar{y}_{lrc}$).

TABLE 2. MSEs and PREs of various estimators of $\bar{Y}$

	Population I		Population II		Population III	
Estimator	MSE	PRE	MSE	PRE	MSE	PRE
$\bar{y}_{st}$	8274.88	100.00	11.26	100.00	12.75	100.00
$\bar{y}_{RS}$	1014.64	815.55	3.28	343.23	*	*
$\bar{y}_{PS}$	*	*	*	*	7.19	177.17
$\bar{y}_{lrs}$	842.62	**982.04**	1.74	**648.64**	7.10	**179.47**
$\bar{y}_{RC}$	1159.01	713.96	3.47	324.28	*	*
$\bar{y}_{PC}$	*	*	*	*	7.57	168.33
$\bar{y}_{lrc}$	948.89	872.06	2.78	404.62	7.44	171.32
$\hat{\bar{Y}}^{(s)}_{RP(\mathrm{opt})}$	842.62	**982.04**	1.74	**648.64**	7.10	**179.47**
$\hat{\bar{Y}}^{(c)}_{RP(\mathrm{opt})}$	842.62	**982.04**	1.74	**648.64**	7.10	**179.47**

* Data is not applicable.

Bold values signify the maximum PRE.

(iii) The performances of both the classes $\hat{\bar{Y}}^{(s)}_{RP}$ and $\hat{\bar{Y}}^{(c)}_{RP}$ are the same and are equal to that of the separate regression estimator ($\bar{y}_{lrs}$), as was expected from the results of Sections 2 and 4.

7. CONCLUSION

In this paper, classes of separate and combined ratio-product estimators were developed for estimating the mean of a study variable. It is worth mentioning that, for specific choices of β_h and γ_h in (2.1) and (4.1), several estimators could be developed. Some of them, in the optimum cases, are equivalent to the separate regression estimator ($\bar{y}_{lrs}$), while others are not.

The theoretical results discussed in Sections 2 and 4 were numerically justified by the data sets considered in Table 1. Also, it follows from Table 2 that there is considerable gain in efficiency by the proposed classes of estimators $\hat{\bar{Y}}^{(s)}_{RP}$ and $\hat{\bar{Y}}^{(c)}_{RP}$ over the other existing estimators. Hence, the proposed classes $\hat{\bar{Y}}^{(s)}_{RP}$ and $\hat{\bar{Y}}^{(c)}_{RP}$ are more appropriate, as compared to the other existing estimators, for estimating the unknown mean $\bar{Y}$ of the study variable y.

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