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SOME PROPERTIES OF THE EMPTINESS TIME OF A DAM

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Abstract. We investigate the emptiness time $T(c)$ of a dam initiated by content $c \geqslant 0$, assuming that the total input process is a compound Poisson one and that the release rate function is given. We prove that if the release function is increasing (decreasing), then the expected value $\mathrm{E}T(c)$, $c \geqslant 0$, is concave (convex). We also give an estimation of the expected value of the emptiness time under some deformations of inputs.

1. Introduction. Let us consider the content process of a dam generated by a compound Poisson process and a general release function. Let $\boldsymbol{\tau} = (\tau_1, \tau_2, \ldots)$ denote the sequence of input instances forming the Poisson process with some parameter λ and let $\boldsymbol{A} = (A_1, A_2, \ldots)$ denote the sequence of inputs being a sequence of independent random variables with some distribution function F. The release rate function is denoted by $r = r(z)$, $z \geqslant 0$, and it is assumed that $r(0) = 0$, $r(z) > 0$, $z > 0$.

In this note we consider the emptiness time of the dam initiated by content $c \geqslant 0$. The unconditional emptiness time of the dam is equal to $T(A) = T$, where A is a random variable independent of the input process and distributed according to F.

It is obvious that for $r(z) = 1$, $z > 0$, the content process of the dam reduces to the virtual waiting process in the queueing system $M/G/1$ generated by $\boldsymbol{\tau}$ and $\boldsymbol{A}$. In that case we have

$$T(c) = c + T_1 + T_2 + \ldots + T_{N(c)}, \quad c \geqslant 0, \tag{1}$$

where $N(c)$, $c \geqslant 0$, is the Poisson process with parameter λ, and $T_1, T_2, \ldots$ are independent random variables distributed as T.

Equation (1) implies that in the queueing process the value $\mathrm{E}T(c) = (1 + \lambda \mathrm{E}T)c$, $c \geqslant 0$, is a linear function in c. Thus $\mathrm{E}T = (1 + \lambda \mathrm{E}T)\mathrm{E}A$, and $\mathrm{E}T = \mathrm{E}A/(1 - \lambda \mathrm{E}A)$ depends only upon the expectations $\mathrm{E}\tau_1 = 1/\lambda$ and $\mathrm{E}A$.

This note is an attempt to generalize these results. We prove that if the function r is increasing (decreasing), then the function $ET(c)$, $c \geqslant 0$, is concave (convex). This property enables us to characterize the consequence of the deformations of the inputs on the emptiness time of the dam. This result extends the paper [3] where the consequences of deformations of the content process were analyzed.

2. The emptiness time. Let us introduce after Harrison and Resnick [2] the functions necessary to a description of the emptiness time process:

$$R(c) = \int_0^c \frac{1}{r(u)} du, \quad z(c, t) = R^{-1}\big(R(c)-t\big), \quad c \geqslant 0,\ 0 \leqslant t \leqslant R(c),$$

gives the content $z(c, t)$ of the dam initiated by content c and with no inputs in the time interval $[0, R(c)]$.

The stochastic process $T(c)$, $c \geqslant 0$, satisfies the equality

$$T(c) = \begin{cases} R(c) & \text{if } \tau_1 > R(c), \\ \tau_1 + T_1\big(z(c, \tau_1)+A_1\big) & \text{if } \tau_1 \leqslant R(c), \end{cases} \tag{2}$$

where $T_1(c)$, $c \geqslant 0$, is the emptiness time process generated by the jump moments $\boldsymbol{\tau}_1 = (\tau_2, \tau_3, \ldots)$, the inputs $\boldsymbol{A}_1 = (A_2, A_3, \ldots)$, and the release function r.

It is easy to prove that (1) and (2) are equivalent in the case $r(z) = 1$, $z > 0$.

In this note we deal with the expected value $\theta(c) = ET(c)$, $c \geqslant 0$. Taking in (2) the expected value with respect to τ_1 we get

$$\begin{aligned} E_{\tau_1} T(c) &= R(c)e^{-\lambda R(c)} + \int_0^{R(c)} \big(u + T_1\big(z(c, u)+A_1\big)\big)\lambda e^{-\lambda u} du \\ &= \frac{1}{\lambda}(1-e^{-\lambda R(c)}) + \int_0^{R(c)} T_1\big(z(c, u)+A_1\big)\lambda e^{-\lambda u} du, \quad c \geqslant 0. \end{aligned}$$

Hence, for the function $\theta(c)$, $c \geqslant 0$, we get the integral equation

$$\theta(c) = \frac{1}{\lambda}(1-e^{-\lambda R(c)}) + \int_0^{R(c)} E_A\,\theta\big(z(c, u)+A\big)\lambda e^{-\lambda u} du, \quad c \geqslant 0. \tag{3}$$

Equation (3) may be solved in a standard manner (see [1]). To that purpose define the nonnegative operator φ to operate on nondecreasing function $f(c)$, $c \geqslant 0$, having the form

$$\varphi(f)(c) = \int_0^{R(c)} E_A f\big(z(c, u)+A\big)\lambda e^{-\lambda u} du, \quad c \geqslant 0.$$

If we define $\varphi^{*0}(f) = f$, $\varphi^{*n}(f) = \varphi(\varphi^{*(n-1)}(f))$, $n = 1, 2, \ldots$, then the solution of (3), provided it exists, takes form

$$\theta(c) = \sum_{n=0}^{\infty} \varphi^{*n}(1-e^{-\lambda R(\cdot)})(c), \quad c \geqslant 0. \tag{4}$$

The equation solution (4) is rather no suitable for subsequent considerations including the simplest case of the release function $r(z) = 1$, $z > 0$. In the sequel we reformulate equation (3) to the differential form.

LEMMA 1. *The function* $\theta(c)$, $c \geqslant 0$, *satisfies the equation*

$$\theta'(c) = \frac{1}{r(c)}(1-\lambda\theta(c)+\lambda E_A\theta(c+A)), \quad c \geqslant 0.$$

Proof. For the function $z = z(c, u)$, $c \geqslant 0$, $0 \leqslant u \leqslant R(c)$, we have

$$\frac{dz}{dc} = \frac{r(z)}{r(c)}, \quad \frac{dz}{du} = -r(z).$$

Differentiation of (3) gives

$$\begin{aligned}\theta'(c) &= e^{-\lambda R(c)}\frac{1}{r(c)} + \mathrm{E}_A\theta\big(z(c, R(c))+A\big)\lambda e^{-\lambda R(c)}\frac{1}{r(c)} + \\ &\quad + \int_0^{R(c)} \mathrm{E}_A\theta'\big(z(c, u)+A\big)\frac{r(z)}{r(c)}\lambda e^{-\lambda u}\,du \\ &= \frac{1}{r(c)}\Bigg[e^{-\lambda R(c)} + \lambda\mathrm{E}_A\theta(A)e^{-\lambda R(c)} - \\ &\quad - \int_0^{R(c)}\left(\frac{d}{du}\mathrm{E}_A\theta\big(z(c, u)+A\big)\right)\lambda e^{-\lambda u}\,du\Bigg].\end{aligned} \tag{5}$$

Integrating by parts we obtain

$$\begin{aligned}&\int_0^{R(c)}\left(\frac{d}{du}\mathrm{E}_A\theta\big(z(c, u)+A\big)\right)e^{-\lambda u}\,du \\ &\quad = \mathrm{E}_A\theta(A)e^{-\lambda R(c)} - \mathrm{E}_A\theta(c+A) + \int_0^{R(c)} \mathrm{E}_A\theta\big(z(c, u)+A\big)\lambda e^{-\lambda u} \\ &\quad = \mathrm{E}_A\theta(A)e^{-\lambda R(c)} - \mathrm{E}_A\theta(c+A) + \theta(c) - \frac{1}{\lambda}(1-e^{-\lambda R(c)}).\end{aligned}$$

Substituting the above into (5) we get Lemma 1.

Let $T(a, b)$, $a \geqslant b \geqslant 0$, denote the first passage time in the content process of the dam from state a to state b. But $T(a, 0) = T(a)$ and $T(c+a) = T(c+a, c)+T(c)$, $c \geqslant 0$, so that

$$\theta(c+a) = \theta(c+a, c)+\theta(c) \geqslant \theta(c), \quad \text{where } \theta(c+a, c) = \mathrm{E}T(c+a, c) \geqslant 0.$$

THEOREM 1. *If the function r is increasing (decreasing), then the function $\theta(c)$, $c \geqslant 0$, is concave (convex).*

Proof. In the proof we restrict our considerations to the case of the increasing function r. Let us consider the difference $D = \mathrm{E}_A\theta'(c+A)-\theta'(c)$. From Lemma 1 we have

$$D = \mathrm{E}_A \frac{1}{r(c+A)}(1-\lambda\theta(c+A)+\lambda \mathrm{E}_{A'}\theta(c+A+A'))- \tag{6}$$

$$-\frac{1}{r(c)}(1-\lambda\theta(c)+\lambda \mathrm{E}_{A'}\theta(c+A'))$$

$$\leqslant \frac{\lambda}{r(c)}\mathrm{E}_A \mathrm{E}_{A'}(\theta(c+A+A')-\theta(c+A')-\theta(c+A)+\theta(c))$$

$$= \frac{\lambda}{r(c)}\mathrm{E}_A \mathrm{E}_{A'}(\theta(c+A+A', c)-\theta(c+A, c)-\theta(c+A', c))$$

$$= \frac{\lambda}{r(c)}\mathrm{E}_A \mathrm{E}_{A'}(\theta(c+A+A', c+A)-\theta(c+A', c)).$$

For any release function r and $a \geqslant 0$ define $r_a(0) = 0$, $r_a(z) = r(z+a)$, $z > 0$. Let us consider the process $T_a(c)$, $c \geqslant 0$, generated by the release function r_a and the random sequences A and τ. The characteristics of this process will be indexed by the parameter a.

The equality $R_a(b) = R(a+b)-R(a)$ implies $z_a(b, t)+a = z(a+b, t)$, $t \geqslant 0$, and, in consequence, $T_a(c+b, c) = T(c+a+b, c+a)$, $a, b, c \geqslant 0$. For the increasing function r and every $a \geqslant 0$ we have $r_a(z) \geqslant r(z)$, $z \geqslant 0$. Thus $z_a(c, t) \leqslant z(c, t)$, $t \geqslant 0$, whence $T_a(c+b, c) \leqslant T(c+b, c)$, $a, b, c \geqslant 0$. Finally,

$$\theta(c+a+b, c+a) = \theta_a(c+b, c) \leqslant \theta(c+b, c).$$

Substituting the above into (6) we have $D \leqslant 0$, which, by Lemma 1, completes the proof of Theorem 1.

3. The deformation of inputs. Let us consider the sequence of inputs $A_n^{**} = A_n+\Delta_n$, $n = 1, 2, \ldots$, being a deformation of the sequence A. Assume that (A_n, Δ_n), $n = 1, 2, \ldots$, are independent, $A_n+\Delta_n \geqslant 0$, $\mathrm{E}\Delta_n|A_n = 0$, $n = 1, 2, \ldots$ The emptiness time process of the dam under the deformation assumption is indexed by two asterisks. Theorem 1 and Jensen's inequality applied to (3) lead to the following result:

THEOREM 2. *If in the model of the dam the release function r is increasing (decreasing), then the deformation of inputs decreases (increases) the emptiness time of the dam in expectation:*

$$\mathrm{E}T^{**}(c) \leqslant \mathrm{E}T(c) \quad (\mathrm{E}T^{**}(c) \geqslant \mathrm{E}T(c)), \quad c \geqslant 0.$$

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