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HOW RANDOM IS RANDOM?

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Abstract. A sequence $\{V_n\}$ of r.v.'s is *asymptotically quasi-deterministic* (AQD) if there exist deterministic functions $\beta_1(n) < \beta_2(n)$ and a constant $C > 0$ such that $\beta_1(n) < V_n < \beta_2(n)$ except for finitely many n with probability 1 and

$$\limsup_{n\to\infty} (\beta_2(n)-\beta_1(n)) \leqslant C.$$

A few surprising examples of AQD sequences are given.

1. Introduction. In 1975 P. Erdős and the author of the present paper* investigated [1] the length of the longest head-run of a coin tossing sequence of size N. In order to formulate the results, we introduce the following notation.

The logarithms appearing in Sections 1 and 2 are meant to be the base 2.

Let $X_1, X_2, \ldots$ be a sequence of i.i.d. r.v.'s with $P(X_1 = 0) = P(X_1 = 1) = 1/2$ and let $S_0 = 0$, $S_n = X_1 + X_2 + \ldots + X_n$ $(n = 1, 2, \ldots)$,

$$I(N, K) = \max_{0 \leqslant n \leqslant N-K} (S_{n+K} - S_n) \quad (1 \leqslant K \leqslant N).$$

Finally, let Z_N be the largest integer for which $I(N, Z_N) = Z_N$. This r.v. Z_N is the length of the longest head-run up to N. The properties of Z_N can be described by the following results:

THEOREM A. *Let ε be an arbitrary positive number. Then*

$$Z_N \geqslant [\log N - \log\log\log N + \log\log e - 2 - \varepsilon] = \alpha_1(N, \varepsilon) = \alpha_1$$

except for finitely many N with probability 1.

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THEOREM B. *Let ε be an arbitrary positive number. Then*

$$Z_N < [\log N - \log\log\log N + \log\log e - 1 + \varepsilon] = \alpha_2(N, \varepsilon) = \alpha_2$$

infinitely often with probability 1.

THEOREM C. *Let $\alpha_3(N)$ be a sequence of positive numbers for which*

$$\sum_{N=1}^{\infty} 2^{-\alpha_3(N)} = \infty.$$

Then $Z_N > \alpha_3(N)$ infinitely often with probability 1.

THEOREM D. *Let $\alpha_4(N)$ be a sequence of positive numbers for which*

$$\sum_{N=1}^{\infty} 2^{-\alpha_4(N)} < \infty.$$

Then $Z_N \leqslant \alpha_4(N)$ except for finitely many N with probability 1.

Theorems A and D clearly imply

CONSEQUENCE 1. *For any $\varepsilon > 0$ we have $\alpha_1(N, \varepsilon) \leqslant Z_N \leqslant \alpha_4(N)$ except for finitely many N with probability* 1.

Evaluating the value of $\alpha_1(N, 0.1)$ and that of $\alpha_4(N) = \log N + 1.1 \log\log N$ for $N = 2^{2^{20}} \sim 10^{315,653}$, we get

$$\alpha_1 = 1,048,569, \quad \alpha_4 = 1,048,598, \quad \text{and} \quad \alpha_4 - \alpha_1 = 29.$$

This means that by flipping a coin $2^{2^{20}}$ times the length of the longest head-block "must be" between 1,048,569 and 1,048,598. The fact that the interval (α_1, α_4) is very short means that the sequence $\{Z_N\}$ is "almost deterministic". In this paper we collect some further nearly deterministic sequences.

2. The area of the largest head-square. Let $\{X_{ij}\}$ $(i = 0, 1, 2, \dots;\ j = 0, 1, 2, \dots)$ be a double array of i.i.d. r.v.'s with $P(X_{ij} = 0) = P(X_{ij} = 1) = 1/2$, let

$$S(n, m, K) = \sum_{j=m}^{m+K-1} \sum_{i=n}^{n+K-1} X_{ij},$$

$$I(N, K) = \max_{\substack{0 \leqslant n \leqslant N+1-K \\ 0 \leqslant m \leqslant N+1-K}} S(n, m, K) \quad (1 \leqslant K \leqslant N),$$

and define Y_N as the largest integer for which $I(N, Y_N) = Y_N^2$. Here Y_N^2 is the area of the "largest head-square" in the square $[0, N] \times [0, N]$. In order to describe the properties of the sequence $\{Y_N\}$ we introduce the following notation:

$$f(N) = \sqrt{2\log N - 2}, \quad a(N) = \sqrt{2\log N} - [\sqrt{2\log N}],$$

$$\beta_1(N, \varepsilon) = \beta_1(N) = \begin{cases} [f(N)] & \text{if } a(N) \leqslant \varepsilon, \\ [f(N)]+1 & \text{if } a(N) > \varepsilon, \end{cases}$$

$$\beta_2(N) = \beta_2(N, \varepsilon) = \begin{cases} [f(N)]+3 & \text{if } a(N) \leqslant 1-\varepsilon, \\ [f(N)]+4 & \text{if } a(N) > 1-\varepsilon, \end{cases}$$

where $0 < \varepsilon < 1$.

Then we have

THEOREM 1. *Let* ε *be an arbitrary positive number smaller than* 1. *Then*

$$\beta_1(N, \varepsilon) < Y_N < \beta_2(N, \varepsilon) \tag{1}$$

except for finitely many N *with probability* 1.

Observe that

$$\beta_2(N) - \beta_1(N) = \begin{cases} 3 & \text{if } a(N) \leqslant \varepsilon, \\ 2 & \text{if } \varepsilon < a(N) < 1-\varepsilon. \\ 3 & \text{if } a(N) \geqslant 1-\varepsilon. \end{cases}$$

Hence Theorem 1 can be reformulated as

THEOREM 1*. *For any* ε $(0 < \varepsilon < 1)$

$$Y_N = \begin{cases} [f(N)]+1 \text{ or } [f(N)]+2 & \text{if } a(N) \leqslant \varepsilon, \\ [f(N)]+2 & \text{if } \varepsilon < a(N) < 1-\varepsilon, \\ [f(N)]+2 \text{ or } [f(N)]+3 & \text{if } a(N) \geqslant 1-\varepsilon \end{cases}$$

except for finitely many N *with probability* 1.

It is worth-while to introduce the following definitions:

Definition 1. A sequence $\{V_N\}$ of r.v.'s is *asymptotically quasi-deterministic* (AQD) if there exist deterministic functions $\beta_1(N) < \beta_2(N)$ and a constant $C > 0$ such that $\beta_1(N) < V_N < \beta_2(N)$ except for finitely many N with probability 1 and

$$\limsup_{N\to\infty} (\beta_2(N) - \beta_1(N)) < C.$$

Definition 2. A sequence of $\{V_N\}$ of r.v.'s is *asymptotically deterministic* (AD) if there exist deterministic functions $\beta_1(N) < \beta_2(N)$ such that $\beta_1(N) < V_N < \beta_2(N)$ except for finitely many N with probability 1 and

$$\lim_{N\to\infty} (\beta_2(N) - \beta_1(N)) = 0.$$

It is trivial that our sequence $\{Y_N\}$ is *not* AD. Theorem 1 states that it is AQD.

A well-known (but surprising) example of an AD sequence is given by our

THEOREM E. *Let* $N_1, N_2, \ldots$ *be a sequence of independent normal* (0, 1) *r.v.'s and let* $T_n = \max\{N_1, N_2, \ldots, N_n\}$. *Then*

$$\lim_{n\to\infty} (T_n - (2\log n)^{1/2}) = 0$$

with probability 1, *i.e.* $\{T_n\}$ *is an AD sequence.*

Proof of Theorem 1. In order to prove the inequality $Y_N > \beta_1(N)$ we show that among the squares

$$T_{ij}(N) = [i, i+\beta_1(N)] \times [j, j+\beta_1(N)] \quad (i, j = 0, 1, 2, \ldots, N-\beta_1(N))$$

there exists at least one having heads at each of its lattice points. This implies the existence of integers i, j such that

$$0 \leqslant i, j \leqslant N+1-\beta_1(N) \quad \text{and} \quad S(i, j, \beta_1(N)+1) = (\beta_2(N)+1)^2.$$

In fact, we prove a bit stronger statement. Namely, the existence of integers μ, ν such that $0 \leqslant \mu, \nu \leqslant l$ and

$$S\big(\mu(\beta_1(N)+1), \nu(\beta_1(N)+1), \beta_1(N)+1\big) = (\beta_1(N)+1)^2,$$

where l is the largest integer for which $l(\beta_1(N)+1) \leqslant N-\beta_1(N)$. Since

$$P\{S(i, j, \beta_1(N)+1) = (\beta_1(N)+1)^2\} = 2^{-(\beta_1(N)+1)^2}$$

and

$$\beta_1(N) \leqslant (2\log N)^{1/2} - 1 - \varepsilon,$$

we have

$$P\{S\big(\mu(\beta_1(N)+1), \nu(\beta_1(N)+1), \beta_1(N)+1\big) < (\beta_1(N)+1)^2, 0 \leqslant \mu, \nu < l\}$$
$$= (1-2^{-(\beta_1(N)+1)^2})^{(l+1)^2} \leqslant \exp(-2^{-(\beta_1(N)+1)^2}(l+1)^2)$$
$$\leqslant \exp(-2^{-((2\log n)^{1/2}-\varepsilon)^2}(l+1)^2) \leqslant \exp(-C(\log N)^{-1} \cdot 2^{2\varepsilon(2\log N)^{1/2}}) \leqslant N^{-2}$$

if N is large enough and C is a suitable positive constant. Now, the first part of Theorem 1 follows from the Borel-Cantelli lemma.

In order to prove the inequality $Y_N < \beta_2(N)$, we have to show that

$$S(i, j, \beta_2(N)) < (\beta_2(N))^2 \quad (i, j = 0, 1, 2, \ldots, N+1-\beta_2(N)).$$

Since $\beta_2(N) \geqslant (2 \log N)^{1/2}+\varepsilon$, we get

$$\begin{aligned} P_N &= P\{S(i, j, \beta_2(N)) \\ &= (\beta_2(N))^2 \text{ for at least one } i, j\ (0 \leqslant i, j \leqslant N+1-\beta_2(N))\} \\ &\leqslant (N+2-\beta_2(N))^2 \cdot 2^{-(\beta_2(N))^2} \\ &\leqslant N^2 \cdot 2^{-((2\log N)^{1/2}+\varepsilon)^2} \leqslant 2^{-2\varepsilon(2\log N)^{1/2}} \end{aligned}$$

if N is large enough.

Let $N_k = 2^{k^{1/2}}$. Then, obviously,

$$\sum_{k=1}^{\infty} P_{N_k} < \infty$$

and the Borel-Cantelli lemma implies that $Y_{N_k} < \beta_2(N_k)$ except for finitely many k with probability 1. Let $N_k \leqslant N < N_{k+1}$. Then our statement follows from the trivial inequalities

$$Y_N \leqslant Y_{N_{k+1}} < \beta_2(N_{k+1}, \varepsilon) \leqslant \beta_2(N_k, 2\varepsilon) \leqslant \beta_2(N, 2\varepsilon).$$

We note that Theorem 1 does not give the best possible result. We can get a sharper result replacing the ε in (1) by a sequence $\{\varepsilon_N\}$ tending to 0. In fact, without any new idea one can prove

THEOREM 2. *We have*

$$\beta_1(N, \varepsilon_N) < Y_N < \beta_2(N, \varepsilon_N)$$

except for finitely many N with probability 1, *where*

$$\varepsilon_N = C\frac{\log\log N}{(2\log N)^{1/2}} \quad \text{and} \quad C > \frac{1}{2}.$$

We do not claim either that Theorem 2 is the best possible result.

3. On the length of the longest increasing run. Let $U_1, U_2, \ldots$ be a sequence of independent r.v.'s uniformly distributed on the interval (0, 1). We are interested in the length of the longest increasing run of the sequence $U_1, U_2, \ldots, U_N$. More formally, let Q_N be the largest integer for which there exists an integer R_N such that

$$U_{R_N+1} < U_{R_N+2} < \ldots < U_{R_N+Q_N} \quad \text{and} \quad R_N+Q_N \leqslant N,$$

where Q_N is the length of the longest increasing run. From here on log means the natural logarithm. In his recent paper [3] the author proved

THEOREM F. *Let*

$$f(n) = \frac{\log n}{b_n} - \frac{1}{2} \quad \text{and} \quad a(n) = f(n) - [f(n)],$$

where b_n is the unique solution of the equation $b_n e^{b_n} = e^{-1}\log n$. For any ε ($0 < \varepsilon < 1$) put also

$$l_n = l_n(\varepsilon) = \begin{cases} [f(n)]-3 & \text{if } a(n) \leqslant \varepsilon, \\ [f(n)]-2 & \text{if } a(n) > \varepsilon, \end{cases}$$

and

$$U_n = U_n(\varepsilon) = \begin{cases} [f(n)]+2 & \text{if } a(n) < 1-\varepsilon, \\ [f(n)]+3 & \text{if } a(n) > 1-\varepsilon. \end{cases}$$

Then for any ε ($0 < \varepsilon < 1$) we have $l_N < Q_N < U_N$ except for finitely many N with probability 1.

We note that

$$b_n = \log\log n - \log\log\log n - 1 + o(1).$$

Since $U_n(\varepsilon) - l_n(\varepsilon) \leqslant 6$ ($0 < \varepsilon < 1$; $n = 1, 2, \ldots$), Theorem F means that $\{Q_N\}$ is an AQD sequence. Clearly enough, it is not an AD sequence.

In this case it is also not clear how far is this result from the best possible one. However, another theorem of [3] shows that one cannot get a much sharper bound.

4. On the densest interval of a Poisson process. Let $\{\pi_\lambda(t) = \pi(t), t \geqslant 0\}$ be a Poisson process of parameter $\lambda > 0$. Consider the process

$$Q_T(\lambda) = Q_T = \max_{0 \leqslant t \leqslant T-1} (\pi(t+1) - \pi(t)).$$

Following the method of proof of Theorem F one can prove

THEOREM 3. *Let*

$$f(t) = \frac{\log t}{b_t} - \frac{1}{2} \quad \text{and} \quad a(t) = f(t) - [f(t)],$$

where $b_t = b_t(\lambda)$ is the unique solution of the equation

$$b_t e^{b_t} = \frac{\log t}{e^{\lambda}}.$$

For any ε ($0 < \varepsilon < 1$) put also

$$l_t = l_t(\varepsilon, \lambda) = \begin{cases} [f(t)]-3 & \text{if } a(t) \leqslant \varepsilon, \\ [f(t)]-2 & \text{if } a(t) > \varepsilon, \end{cases}$$

and

$$U_t = U_t(\varepsilon, \lambda) = \begin{cases} [f(t)]+2 & \text{if } a(t) < 1-\varepsilon, \\ [f(t)]+3 & \text{if } a(t) > 1-\varepsilon. \end{cases}$$

Then for any ε ($0 < \varepsilon < 1$) and $\lambda > 0$ we have an r.v. $T_0 = T_0(\varepsilon, \lambda, w)$ such that $P(T_0 < \infty) = 1$ and $l_t < Q_t < U_t$ provided that $t \geqslant T_0$.

We note that

$$b_t = \log\log t - \log\log\log t - \log\lambda - 1 + o(1)$$

and

$$(2)\quad f(t) = \frac{\log t}{\log\log t} + \frac{\log t \log\log\log t}{(\log\log t)^2} + \frac{\log t}{(\log\log t)^2}\big(1 + \log\lambda + o(1)\big).$$

Theorem 3 states that $\{Q_t, t \geqslant 0\}$ is an AQD process (clearly, not an AD). It is worth-while to mention that its lower and upper bounds $\big(l_t(\varepsilon, \lambda)$ and $U_t(\varepsilon, \lambda)\big)$ do not depend very strongly on the value of λ (cf. (2)).

5. On the continuity modulus of the Wiener process. Let $\{W(t), t \geqslant 0\}$ be a Wiener process and consider the continuity modulus

$$c(h) = \sup_{0 \leqslant t \leqslant 1-h} \frac{W(t+h) - W(t)}{h^{1/2}}.$$

The well-known result of P. Lévy is the following

THEOREM G. *We have*

$$\lim_{h \to 0} \frac{c(h)}{(2\log 1/h)^{1/2}} = 1$$

with probability 1.

This theorem means that the process $(2\log 1/h)^{-1/2} c(h)$ is an AD process. A stronger result (see [2]) states that the process $c(h)$ itself is an AD process:

THEOREM H. *We have*

$$\lim_{n \to 0} \big(c(h) - (2\log 1/h)^{1/2}\big) = 0$$

with probability 1.

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