PROBABILITY
AND
MATHEMATICAL STATISTICS

Vol. 4, Fasc. 2 (1984), p. 207-219

ASYMPTOTIC EXPANSIONS FOR CONDITIONAL DISTRIBUTIONS: THE LATTICE CASE

BY

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Abstract. It is shown that the conditional distribution of $X_1+\ldots+X_n$, given $Y_1+\ldots+Y_n = y$, admits an asymptotic expansion whenever (X_1, Y_1), $(X_2, Y_2), \ldots$ is a sequence of independent identically distributed lattice random vectors and y lies in a set $A(n)$ for which $P\{Y_1+\ldots+Y_n \notin A(n)\}$ can be neglected. Explicit formulas are given for the terms of order $n^{-1/2}$ and n^{-1}.

1. Introduction. Let $\mathfrak{P}$ be a family of probability measures on the Borel field $\mathscr{B}^k$ of some Euclidean space $\boldsymbol{R}^k$, and for fixed $P \in \mathfrak{P}$ let $Z_1, Z_2, \ldots$ be a sequence of independent k-variate random vectors with distribution P. Partition the vectors Z_i according to $Z_i = (X_i, Y_i)$, where X_i is p-variate, Y_i is q-variate, and $p+q = k$. We consider the conditional distribution $Q(P, n, y)$ of $X_1+\ldots+X_n$, given $Y_1+\ldots+Y_n = y$, in the following two cases:

(i) The set of all integral k-vectors Z^k is the minimal lattice for Z_1 (i.e. $Z_1 \in Z^k$ almost surely and Z^k is the minimal additive subgroup of $\boldsymbol{R}^k$ with this property).

(ii) Z^q is the minimal lattice for Y_1, and Z_1 satisfies a uniform Cramér condition in its first argument X_1:

For every $\varepsilon > 0$ there exists $\delta > 0$ such that for $t_1 \in \boldsymbol{R}^p$, $t_2 \in \boldsymbol{R}^q$, $\|t_1\| \geqslant \varepsilon$, we have

$$|\mathrm{E}\exp(it_1^{\mathrm{T}} X_1 + it_2^{\mathrm{T}} Y_1)| \leqslant 1-\delta. \tag{1.1}$$

We shall obtain asymptotic expensions for the distribution functions and the point probabilities in case (i), and for probabilities of convex sets in case (ii). This will be done with an error term uniform in $P \in \mathfrak{P}$ and y in a subset $A(P, n)$ of Z^q such that

$$\sup\{P\{Y_1+\ldots+Y_n \notin A(P, n)\}\colon P \in \mathfrak{P}\}$$

can be neglected.

Asymptotic results on $Q(P, n, y)$ were first obtained by Steck [10]. He proves weak convergence of suitably standardized conditional distributions to the normal law. Higher order approximations for conditional distributions are derived by Michel [6] for the case where for m sufficiently large the distribution of $Z_1+\ldots+Z_m$ is dominated by the k-variate Lebesgue measure. Our proofs are based on Michel's method. For $p=1$, explicit formulas are given for the terms of order $n^{-1/2}$ and n^{-1} of the expansions.

Asymptotic expansions for conditional distributions are a basic tool to investigate the asymptotic behavior of asymptotically similar tests in exponential models (see [7] and [3]).

As a side result, we obtain asymptotic expansions for certain distributions by writing these distributions as $Q(P, n, y)$ with suitably chosen P and y:

Example 1.1. (a) Let P be the distribution of $(U, U+V)$, where U and V are independent Poisson variables. Then $Q(P, n, y)$ is a binomial distribution with parameters n and $\mathrm{E}U/\mathrm{E}(U+V)$.

(b) If P is the distribution of $(U, U+V)$, where U and V are independent Bernoulli variables with $\mathrm{E}U=\mathrm{E}V$, then $Q(P, n, y)$ is a hypergeometric distribution with parameters $2n$, n, y. Approximations for hypergeometric distributions can be found in [8]. If $\mathrm{E}U=p_1\neq p_2=\mathrm{E}V$, then $Q(P, n, y)$ is no longer hypergeometric. For

$$\theta = p_1(1-p_2)/(p_2(1-p_1))$$

we obtain

$$Q(P, n, y)\{k\} = \binom{n}{k}\binom{n}{y-k}\theta^k\left[\sum_{j=0}^{y}\binom{n}{j}\binom{n}{y-j}\theta^j\right]^{-1}, \quad k=0,\ldots,y.$$

Asymptotic normality of these distributions was shown by Hannan and Harkness [2].

Example 1.2. Let P_1, P_2 be probability measures on $\mathscr{B}$ satisfying the usual Cramér condition

$$\limsup_{|t|\to\infty} \left|\int e^{itx} P_j(dx)\right| < 1, \quad j=1, 2. \tag{1.2}$$

Consider a sequence $U_1, U_2, \ldots$ of independent random variables, some of which have the distribution P_1, the others have the distribution P_2. Asymptotic expansions for the distribution of $U_1+\ldots+U_n$ can be obtained from our Theorem 2.3:

Let $(X_1, Y_1), (X_2, Y_2), \ldots$ be a sequence of independent identically distributed bivariate random vectors such that

(a) $P\{Y_1=1\}=1-P\{Y_1=2\}=p\in(0, 1)$;

(b) the conditional distribution of X_1, given $Y_1=j$, is P_j $(j=1, 2)$.

If k terms in the sequence $U_1, \ldots, U_n$ have the distribution P_1 and $n-k$

terms have the distribution P_2, then the distribution of $U_1 + \ldots + U_n$ equals $Q(P, n, k)$.

Here the uniform Cramér condition (1.1) is satisfied, i.e. by (1.2) for any $\varepsilon > 0$ we have

$$\sup\{|\mathrm{E}\exp(itX_1 + isY_1)|:\ |t| \geqslant \varepsilon,\ s \in \boldsymbol{R}\}$$
$$= \sup\{|p\int \exp(itx + is)P_1(dx) + (1-p)\int \exp(itx + 2is)P_2(dx)|:\ |t| \geqslant \varepsilon,\ s \in \boldsymbol{R}\}$$
$$\leqslant \sup\{p|\int e^{itx}P_1(dx)| + (1-p)|\int e^{itx}P_2(dx)|:\ |t| \geqslant \varepsilon\} < 1.$$

The above result extends easily to more than two possible distributions of $U_1, U_2, \ldots$

Another application of our results is the approximation of the surprise index (see [9]):

Example 1.3. Let $U_1, U_2, \ldots$ be a sequence of independent identically distributed Z^1-valued random variables. Write

$$p_n(k) = P\{U_1 + \ldots + U_n = k\}.$$

The surprise index of the event $\{X_1 + \ldots + X_n = k\}$ is the number

$$S_{n,k} = \sum_{j \in Z^1} p_n^2(j)/p_n(k).$$

Let V and W be independent random variables having the same distribution as U_1, and let P be the distribution of $(V, V-W)$. Then

$$Q(P, n, 0)\{k\} = P\{V = W = k\}/P\{V = W\} = p_n^2(k)[\sum_{j \in z^1} p_j^2(k)]^{-1}$$
$$= p_n(k)/S_{n,k}.$$

Using asymptotic expansions for $Q(P, n, 0)$ and $p_n(k)$ we can easily compute asymptotic expansions for

$$S_{n,k} = p_n(k)/Q(P, n, 0)\{k\}.$$

2. The results. Fix an integer $s \geqslant 3$. For $P, Q \in \mathfrak{P}$ define

$$d(P, Q) = \sup\{|P(A) - Q(A)|:\ A \in \mathscr{B}^k\}.$$

The following assumptions are made throughout this section:

Assumption 1. The family $\mathfrak{P}$ is compact in the topology induced by d.

Assumption 2. For all $P \in \mathfrak{P}$ there exists M such that $\int \|z\|^r P(dz) \leqslant M$, where $r = \max(2s-1, p+1)$.

Remark 2.1. For $P \in \mathfrak{P}$ we denote by $\Sigma(P)$ the covariance matrix of P. For all $P \in \mathfrak{P}$ the matrix $\Sigma(P)$ is nonsingular, for otherwise Z^k would not be the minimal lattice supporting P or (1.1) would fail. By Assumption 2 the map $P \to \Sigma(P)$ is continuous. Hence Assumption 1 implies that there exist c

and C $(0 < c < C < \infty)$ such that for all $P \in \mathfrak{P}$ and all eigenvalues λ of $\Sigma(P)$ we have $c < \lambda < C$.

Remark 2.2. For $P \in \mathfrak{P}$ we denote the characteristic function of P by f_P. Let $\varepsilon > 0$ and

$$A(\varepsilon) = \{z \in R^k \colon \varepsilon \leqslant |z_j| \leqslant \pi,\ j = 1, \ldots, k\}.$$

Then

$$\sup\{|f_P(z)| \colon z \in A(\varepsilon),\ P \in \mathfrak{P}\} < 1. \tag{2.1}$$

We need the following notation. For $P \in \mathfrak{P}$ and partition $\Sigma = \Sigma(P)$ let

$$\Sigma = \begin{pmatrix} \Sigma_{00} & \Sigma_{01} \\ \Sigma_{10} & \Sigma_{11} \end{pmatrix}$$

and

$$\Lambda = \Sigma^{-1} = \begin{pmatrix} \Lambda_{00} & \Lambda_{01} \\ \Lambda_{10} & \Lambda_{11} \end{pmatrix},$$

where Σ_{00}, Λ_{00} are (p, p)-matrices and Σ_{11}, Λ_{11} are (q, q)-matrices. Let

$$\tilde{\Sigma} = \Sigma_{00} - \Sigma_{01} \Sigma_{11}^{-1} \Sigma_{10},$$

which is the (symmetric and positive definite) inverse of Λ_{00}.

For a positive integer m and a symmetric positive definite (m, m)-matrix A let φ_A denote the Lebesgue density of an m-variate normal random vector with zero mean and covariance matrix A.

We put

$$\mu(P) = \int z P(dz),$$

$$\tilde{z}(P, n) = (\tilde{x}(P, n), \tilde{y}(P, n)) = n^{-1/2}(z - n\mu(P)), \qquad \tilde{x}(P, n) \in R^p,\ \tilde{y}(P, n) \in R^q,$$

$$A(P, n) = \{y \in Z^q \colon \tilde{y}(P, n)^{\mathrm{T}} \Sigma_{11}^{-1}(P) \tilde{y}(P, n) \leqslant (s - 3/2) \log n\}.$$

For nonnegative k-dimensional integral vectors v denote the v-th cumulant of P by $\chi_v(P)$, and for $j = 0, \ldots, s-2$ let $P_j(-\Phi_{0:\Sigma(P)} \colon \{\chi_v(P)\})$ be the finite signed measure defined in [1] (p. 53, Lemma 7.2) which has Lebesgue density $P_j(-\varphi_{0:\Sigma(P)} \colon \{\chi_v(P)\})$. Let P' be the distribution of Y_1, and for nonnegative q-dimensional integral vectors v let $\bar{\chi}_v(P)$ denote the v-th cumulant of P'. For $j = 0, \ldots, s-2$ let us define $P_j(-\varphi_{0:\Sigma_{11}(P)} \colon \{\bar{\chi}_v(P)\})$ and $P_j(-\varphi_{0:\Sigma_{11}(P)} \colon \{\bar{\chi}_v(P)\})$ as above, and for $j = 0, \ldots, s-2$ and fixed $z = (x, y) \in R^k$, $x \in R^p$, $y \in R^q$, determine $H_j\,(P, y, x)$ by the formal identity

$$\sum_{j=0}^{\infty} n^{-j/2} P_j(-\varphi_{0:\Sigma(P)} \colon \{\chi_v(P)\})(z) \tag{2.2}$$

$$= \sum_{j=0}^{\infty} n^{-j/2} P_j(-\varphi_{0:\Sigma_{11}(P)} \colon \{\bar{\chi}_v(P)\})(y) \sum_{j=0}^{\infty} n^{-j/2} H_j(P, y, x).$$

THEOREM 2.1. *If Assumptions* 1 *and* 2 *are satisfied in case* (i), *then uniformly for* $P \in \mathfrak{P}$ *and* $y \in A(P, n)$

$$(2.3) \qquad \sum_{x \in z^p} |Q(P, n, y)\{x\} - q(P, n, \tilde{y}(P, n), \tilde{x}(P, n))| = o(n^{-(s-2)/2}),$$

where

$$q(P, n, y, x) = n^{-p/2} \sum_{j=0}^{s-2} n^{-j/2} H_j(P, y, x).$$

Proof. For short we write $\tilde{z}$, $\tilde{x}$, $\tilde{y}$ instead of $\tilde{z}(P, n)$, $\tilde{x}(P, n)$, and $\tilde{y}(P, n)$, respectively. Note first that uniformly for $z \in Z^k$ and $P \in \mathfrak{P}$

$$(2.4) \qquad (1+\|z\|^{k+r-1})|P\{Z_1 + \ldots + Z_n = z\} - \\ - n^{-k/2} \sum_{j=0}^{r-3} n^{-j/2} P_j(-\varphi_{0:\Sigma(P)}: \{\chi_v(P)\})(z)| = O(n^{-(k+r-2)/2}).$$

The proof of (2.4) follows the pattern of the proof of Theorem 22.1 in [1]. The crucial point is (22.10) on p. 232 in [1]. An upper bound for I_1 can be found by Theorem 9.10 in [1], and an upper bound for I_2 can be derived from (2.1). Similarly we infer that uniformly for $P \in \mathfrak{P}$ and $y \in Z^q$

$$(2.5) \qquad |P\{Y_1 + \ldots + Y_n = y\} - n^{-q/2} \sum_{j=0}^{r-3} n^{-j/2} P_j(-\varphi_{0:\Sigma_{11}(P)}: \{\bar{\chi}_v(P)\})(\tilde{y})| \\ = O(n^{-(q+r-2)/2}).$$

Furthermore, there exist $D > 0$ and N such that for all $P \in \mathfrak{P}$, $n > N$, and $y \in A(P, n)$

$$(2.6) \qquad \sum_{j=0}^{r-3} n^{-j/2} P_j(-\varphi_{0:\Sigma_{11}(P)}: \{\bar{\chi}_v(P)\})(\tilde{y}) \geqslant Dn^{-(s-3/2)/2}.$$

The relation

$$|a/b - c/d| \leqslant d^{-1}(|b-d|\, a/b + |a-c|)$$

for $a \geqslant 0$ and $b, d > 0$ implies that uniformly for $P \in \mathfrak{P}$ and $y \in A(P, n)$

$$\sum_{x \in Z^p} |P\{Z_1 + \ldots + Z_n = (x, y)\}/P\{Y_1 + \ldots + Y_n = y\} - \\ - n^{-p/2} \sum_{j=0}^{r-3} n^{-j/2} P_j(-\varphi_{0:\Sigma(P)}: \{\chi_v(P)\})(\tilde{x}, \tilde{y}) \times \\ \times [\sum_{j=0}^{r-3} n^{-j/2} P_j(-\varphi_{0:\Sigma_{11}(P)}: \{\bar{\chi}_v(P)\})(\tilde{y})]^{-1}| \\ = O(n^{-(r-2-s+3/2)/2})(1 + n^{-p/2} \sum_{x \in Z^p} (1+\|x\|^{k+r-1})^{-1}) = o(n^{-(s-2)/2}).$$

Finally, there exists a polynomial R in $z=(x, y)\in \mathbb{R}^k$, $x\in \mathbb{R}^p$, $y\in \mathbb{R}^q$, such that for all $P\in \mathfrak{P}$ and sufficiently large n

$$\Big|\sum_{j=0}^{r-3} n^{-j/2} P_j\big(-\varphi_{0:\Sigma(P)}: \{\chi_v(P)\}\big)(z)\times$$

$$\times\Big[\sum_{j=0}^{r-3} n^{-j/2} P_j\big(-\varphi_{0:\Sigma_{11}(P)}: \{\bar{\chi}_v(P)\}\big)(y)\Big]^{-1}-\sum_{j=0}^{s-2} n^{-j/2} H_j(P, y, x)\Big|$$
$$\leqslant R(z)\, H_0(P, y, x)\, n^{-(s-1)/2}.$$

Putting

$$H_0(P, y, x)=\varphi_{\tilde{\Sigma}(P)}\big(x-\Sigma_{01}(P)\,\Sigma_{11}^{-1}(P)\, y\big)$$

we infer that uniformly for $P\in \mathfrak{P}$ and $y\in A(P, n)$

$$n^{-(s-1)/2} \sum_{x\in Z^p} n^{-p/2} R\big((\tilde{x}, \tilde{y})\big) H_0(P, \tilde{y}, \tilde{x})=o(n^{-(s-2)/2}),$$

which completes the proof.

A nonuniform version of Theorem 2.1 can be found in [5]. Relation (2.3) yields asymptotic expansions for the distribution function of $Q(P, n, y)$. For a nonnegative p-dimensional integral vector $\alpha=(\alpha_1, \ldots, \alpha_p)$ let S_α be the p-variate Bernoulli polynomial of order α defined by

$$S_\alpha(x_1, \ldots, x_p)=\prod_{i=1}^{p} B_{\alpha_i}(x_i),$$

where B_m for $m=0, 1, \ldots$ is the m-th Bernoulli polynomial. These polynomials are defined by the relations

$$B_0(x)\equiv 1,$$
$$B'_{m+1}(x)=B_m(x), \qquad x\in(0, 1),\ m=0, 1, \ldots,$$
$$\int_0^1 B_m(x)\, dx=0, \qquad m=1, 2, \ldots,$$
$$B_m(x+1)=B_m(x), \qquad x\in \mathbb{R},\ m=1, 2, \ldots$$

For $f\colon \mathbb{R}^p\to \mathbb{R}$ we write $D^\alpha f(x)$ instead of

$$(\partial^{|\alpha|}/\partial^{\alpha_1} u_1 \ldots \partial^{\alpha_p} u_p)\, f(u)|_{u=x}.$$

THEOREM 2.2. *If Assumptions* 1 *and* 2 *are satisfied in case* (i) *then uniformly for* $P\in \mathfrak{P}$, $y\in A(P, n)$, *and* $x\in \mathbb{R}^p$

$$Q(P, n, y)(-\infty, x]-\Lambda\big(P, n, \tilde{y}(P, n)\big)\big(\tilde{x}(P, n)\big)=o(n^{-(s-2)/2}),$$

where

$$\Lambda(P, n, y)(x)=\sum(-1)^{|\alpha|} n^{-|\alpha|/2} S_\alpha\big(n^{1/2} x+n\mu_1(P)\big) D^\alpha \tilde{q}(P, n, y, x).$$

The summation extends over all nonnegative p-dimensional integral vectors α *such that* $|\alpha| \leqslant s-2$, $\mu_1(P) = \mathrm{E}X_1$, *and*

$$\tilde{q}(P, n, y, u) = \int_{\tilde{x}(P,n)<u} q(P, n, y, \tilde{x}(P, n))dx.$$

Proof. We apply Theorem A.4.3 of [1], p. 258, for $r = s-1$. To this end we have to find upper bounds for $|D^\alpha q(P, n, y, u)|(1+\|u\|^m)$ which hold uniformly for $P \in \mathfrak{P}$ and $y \in A(P, n)$. Note first that for all m and nonnegative integral vectors α there exists j such that, for all $P \in \mathfrak{P}$, $y \in \boldsymbol{R}^q$, $u \in \boldsymbol{R}^p$, and for $n = 1, 2, \ldots,$

$$|D^\alpha q(P, n, y, u)|(1+\|u\|^m) \leqslant j(1+\|y\|+\|u\|)^j H_0(P, y, u).$$

If $y \in A(P, n)$, then $\|\tilde{y}\|^2 \leqslant C(s-3/2)\log n$, where $\tilde{y} = \tilde{y}(P, n)$. Note that for all $y \in \boldsymbol{R}^q$,

$$\sup\{\|u\|^j H_0(P, y, u)\colon u \in \boldsymbol{R}^p\} = \sup\{\|u+\Sigma_{01}(P)\Sigma_{11}^{-1}(P)y\|^j \varphi_{\tilde{\Sigma}(P)}(u)\colon u \in \boldsymbol{R}^p\}.$$

Consequently, there exists j such that, for all $P \in \mathfrak{P}$, $u \in \boldsymbol{R}^p$, $n = 1, 2, \ldots,$ and $y \in A(P, n)$,

$$|D^\alpha q(P, n, y, u)|(1+\|u\|^m) \leqslant j\log^j n.$$

In the definition of Λ_r in [1], p. 259, (A.4.20), we can omit all terms of order $o(n^{-(s-2)/2})$. This proves the theorem.

For $p = 1$ and $s = 4$ we infer that uniformly for $P \in \mathfrak{P}$, $x \in Z$, and $y \in A(P, n)$

$$Q(P, n, y)(-\infty, x) = \Phi\left(\sigma^{-1}(\tilde{x}-\Sigma_{01}(P)\Sigma_{11}^{-1}(P)y)\right)+$$
$$+\sigma^{-1}\varphi\left(\sigma^{-1}(\tilde{x}-\Sigma_{01}(P)\Sigma_{11}^{-1}(P)\tilde{y})\right)\left[n^{-1/2}(R_1(P, \tilde{x}, \tilde{y})-\tfrac{1}{2})+\right.$$
$$\left.+n^{-1}\left(R_2(P, \tilde{y}, \tilde{x})-\tfrac{1}{2}W_1(P, \tilde{y}, \tilde{x})-\tfrac{1}{12}\sigma^{-1}(\tilde{x}-\Sigma_{01}(P)\Sigma_{11}^{-1}(P)\tilde{y})\right)\right]+o(n^{-1})$$

with R_1, R_2, W_1 given by (3.3), (3.4), and (3.1), respectively.

COROLLARY 2.1. *For fixed* $M > 0$ *there exist polynomials* $Q_1(P, y, x)$ *and* $Q_2(P, y, x)$ *in* $(x, y) \in \boldsymbol{R}^{q+1}$ *such that uniformly for* $P \in \mathfrak{P}$, $y \in A(P, n)$, *and* $x \in Z$ *with*

$$|\tilde{x}(P, n)-\Sigma_{01}(P)\Sigma_{11}^{-1}(P)\tilde{y}(P, n)| \leqslant M$$

we have

$$Q(P, n, y)(-\infty, x) = \Phi\left(\sigma^{-1}(\tilde{x}(P, n)-\Sigma_{01}(P)\Sigma_{11}^{-1}(P)\tilde{y}(P, n))+\right.$$
$$\left.+n^{-1/2}Q_1(P, \tilde{y}(P, n), \tilde{x}(P, n))+n^{-1}Q_2(P, \tilde{y}(P, n), \tilde{x}(P, n))\right)+o(n^{-1}),$$

where

$$Q_1(P, y, x) = R_1(P, y, x) - \tfrac{1}{2},$$
$$Q_2(P, y, x) = R_2(P, y, x) - \tfrac{1}{2} W_1(P, y, x) - \\ - \tfrac{1}{12}\sigma^{-1}\big(x - \Sigma_{01}(P)\Sigma_{11}^{-1}(P)y\big) + \\ + \tfrac{1}{2}\sigma^{-1}\big(x - \Sigma_{01}(P)\Sigma_{11}^{-1}(P)y\big)Q_1^2(P, y, x)$$

and R_1, R_2, *and* W_1 *are given by* (3.3), (3.4), *and* (3.1), *respectively.*

If the distribution $Q(P, n, y)$ is smooth, then an approximation of the distribution function of $Q(P, n, y)$ by $\tilde{q}(P, n, y, \cdot)$ should be possible. In the following we consider case (ii), i.e. Y_1 has the minimal lattice Z^q and X_1 satisfies the uniform Cramér condition (1.1).

THEOREM 2.3 *If Assumptions* 1 *and* 2 *are satisfied in case* (ii), *then uniformly for* $P \in \mathfrak{P}$, $y \in A(P, n)$ *and convex measurable* $C \subset \mathbb{R}^p$

$$Q(P, n, y)(C) = \int_C q\big(P, n, \tilde{y}(P, n), \tilde{x}(P, n)\big)dx + o(n^{-(s-2)/2}).$$

Proof. Denote by P_n the distribution of $n^{-1/2}(x - n\mu_1)$ under $Q(P, n, y)$ and by Q_n the signed measure with Lebesgue density $n^{p/2} q\big(P, n, \tilde{y}(P, n), \cdot\big)$. For $A \subset \mathbb{R}^p$ and $\varepsilon > 0$ let ∂A be the boundary of A and

$$A^\varepsilon = \{x \in \mathbb{R}^p \colon (\exists\, x' \in A)\ \|x - x'\| < \varepsilon\}.$$

All error terms in this proof hold uniformly for $P \in \mathfrak{P}$, $y \in A(P, n)$, and convex measurable $C \subset \mathbb{R}^p$. To prove our assertion

$$P_n(C) = Q_n(C) + o(n^{-(s-2)/2})$$

it suffices to show that the following relations hold:

(A) $\sup\{|Q_n((\partial C)^\varepsilon)|/\varepsilon \colon \varepsilon > 0\} = o(n^{1/2})$;

(B) for all nonnegative integral p-vectors α with $|\alpha| \leqslant p+1$,

$$\int 1_{\{\|t\| \leqslant n^{(s-1)/2}\}}(t)\,\Big|D^\alpha \int \exp(it^{\mathrm{T}}x)(P_n - Q_n)(dx)\Big|\,dt = o(n^{-(s-2)/2})$$

(see [1], p. 97, Corollary 11.5, and p. 98, Lemma 11.6).

Relation (A) follows from the equality

$$\sup\{|q(P, n, \tilde{y}(P, n), u)|/H_0(P, \tilde{y}(P, n), u) \colon u \in \mathbb{R}^p\} = o(n^{1/2})$$

and from Sazonov's lemma (see [1], p. 24, Corollary 3.2).

For the proof of (B) we use the equations

$$D^\alpha \int \exp(it^{\mathrm{T}}x) P_n(dx) = D^\alpha \mathrm{E}\exp\big(it^{\mathrm{T}} n^{-1/2}(X_1 + \ldots + X_n - n\mu_1(P)\big) \times \\ \times 1_{\{Y_1 + \ldots + Y_n = y\}} / P\{Y_1 + \ldots + Y_n = y\}$$

and

(2.7) $$D^{\alpha}\mathrm{E}\exp\bigl(it^{\mathrm{T}}n^{-1/2}(X_1+\ldots+X_n-n\mu_1(P))\bigr)\mathbb{1}_{\{Y_1+\ldots+Y_n=y\}} =(2\pi n^{1/2})^{-q}\int \mathbb{1}_{\{|v_i|\leqslant \pi n^{1/2}, i=1,\ldots,q\}}(v)\,D^{\alpha}f_n(t,v)\exp\bigl(iv^{\mathrm{T}}\tilde{y}(P,n)\bigr)\,dv,$$

where f_n is the characteristic function of $n^{-1/2}(Z_1+\ldots+Z_n-\mu(P))$. If we replace $D^{\alpha}f_n(t,v)$ by an asymptotic expansion, then we obtain an asymptotic expansion for the left-hand side of (2.7). This is done in the following lemma. Since this lemma is used in [4] in a slightly more general situation, we state all assumptions in detail.

LEMMA 2.1. *Let $\mathfrak{P}$ be a family of probability measures satisfying Assumption 1 and let $r_0 \geqslant 0$ be an integer for which*

$$\sup\{\int \|z\|^{r_0+s}P(dz)\colon P\in\mathfrak{P}\} < \infty.$$

Assume that for all $P\in\mathfrak{P}$ the covariance matrix $\Sigma(P)$ is non-singular. Let ψ_n be the characteristic function of

$$\sum_{j=0}^{s-3} n^{-j/2}P_j\bigl(-\Phi_{0:\Sigma(P)}\colon \{\chi_v(P)\}\bigr).$$

Then for nonnegative integral p-vectors α with $|\alpha|\leqslant s$ there exists a positive ε such that, for $t\in\mathbf{R}^p$, $\|t\|\leqslant \varepsilon n^{1/2}$, $P\in\mathfrak{P}$, $y\in\mathbf{Z}^q$, and $n=1,2,\ldots,$

$$\Bigl|D^{\alpha}\Bigl(\mathrm{E}\exp\bigl(it^{\mathrm{T}}n^{-1/2}(X_1+\ldots+X_n-n\mu_1(P))\bigr)\mathbb{1}_{\{Y_1+\ldots+Y_n=y\}}- -(2\pi n^{1/2})^{-q}\int\psi_n(t,v)\exp\bigl(-iv^{\mathrm{T}}\tilde{y}(P,n)\bigr)dv\Bigr)\Bigr| \leqslant \exp(-\varepsilon\|t\|^2)\bigl(1+\|\tilde{y}(P,n)\|^{r_0}\bigr)^{-1}\varepsilon^{-1}n^{-(s+q-2)/2}.$$

Proof. Theorem 9.10 of [1], p. 81, implies that there exists a positive ε_1 such that, for $t\in\mathbf{R}^p$, $\|t\|\leqslant\varepsilon_1 n^{1/2}$, $v\in\mathbf{R}^q$, $\|v\|\leqslant\varepsilon_1 n^{1/2}$, a nonnegative integral k-vector β with $|\beta|\leqslant s+r_0$, and for $n=1,2,\ldots,$

$$\bigl|D^{\beta}\bigl(f_n(t,v)-\psi_n(t,v)\bigr)\bigr|\leqslant\varepsilon_1^{-1}\exp(-\varepsilon_1\|t\|^2-\varepsilon_1\|v\|^2)\,n^{-(s-2)/2}.$$

Using (2.7) we infer that there exists $\varepsilon_2>0$ such that, for $t\in\mathbf{R}^p$, $\|t\|\leqslant\varepsilon_2 n^{1/2}$, $P\in\mathfrak{P}$, and a nonnegative integral p-vector α with $|\alpha|\leqslant s$,

$$\Bigl|D^{\alpha}\Bigl(\mathrm{E}\exp\bigl(it^{\mathrm{T}}n^{-1/2}(X_1+\ldots+X_n-n\mu_1(P))\bigr)\mathbb{1}_{\{Y_1+\ldots+Y_n=y\}}- -(2\pi n^{1/2})^{-q}\int\psi_n(t,v)\exp\bigl(-iv^{\mathrm{T}}\tilde{y}(P,n)\bigr)dv\Bigr)\Bigr| \leqslant\bigl(1+\|\tilde{y}(P,n)\|^{r_0}\bigr)^{-1}\max\{\varepsilon_2^{-1}\exp(-\varepsilon_2\|t\|^2)\,n^{-(s+q-2)/2}+I_1(\beta)+I_2(\beta)\},$$

where

$$I_1(\beta)=(2\pi n^{1/2})^{-q}\int\mathbb{1}_{\{|v_i|\leqslant\pi n^{1/2}, i=1,\ldots,q,\ \|v\|\geqslant\varepsilon_1 n^{1/2}\}}(v)\,|D^{\beta}f_n(t,v)|\,dv,$$
$$I_2(\beta)=(2\pi n^{1/2})^{-q}\int\mathbb{1}_{\{\|v\|\geqslant\varepsilon_1 n^{1/2}\}}(v)\,|D^{\beta}\psi_n(t,v)|\,dv,$$

and the maximum is taken over all nonnegative integral k-vectors β with $|\beta|\leqslant s+r_0$.

There exists a positive ε_3 such that for all $t \in R^p$, $P \in \mathfrak{P}$, $n = 1, 2, \ldots$, and all these β's we have

$$I_2(\beta) \leqslant \exp\{-\varepsilon_3 n\}.$$

In order to prove the same relation for $I_1(\beta)$ we note that

$$f_n(t, v) = f_P(n^{-1/2} t, n^{-1/2} v)^n \exp\left(-it^{\mathrm{T}} n^{1/2} \mu_1(P) - iv^{\mathrm{T}} n^{1/2} \mu_2(P)\right). \tag{2.8}$$

Since for all positive δ we have

$$\sup\{|f_p(0, v)|:\ \delta \leqslant |v_j| \leqslant \pi,\ j = 1, \ldots, q,\ P \in \mathfrak{P}\} < 1$$

and since $\{f_P:\ P \in \mathfrak{P}\}$ is equicontinuous, there exists $\varepsilon_4 > 0$ such that

$$\sup\{|f_P(t, v)|:\ \|t\| \leqslant \varepsilon_4,\ \|v\| \geqslant \varepsilon_1,\ |v_i| \leqslant \pi,\ i = 1, \ldots, q,\ P \in \mathfrak{P}\} < 1.$$

Consequently, there exists $\varepsilon_5 > 0$ such that for $t \in R^p$, $\|t\| \leqslant \varepsilon_5 n^{1/2}$, $P \in \mathfrak{P}$, $n = 1, 2, \ldots$, and for all relevant β's we get

$$I_1(\beta) \leqslant \exp\{-\varepsilon_5 n\}.$$

This proves the lemma.

We apply Lemma 2.1 with $r_0 = 0$ and r instead of s and obtain upper bounds for

$$\int 1_{\{\|t\| \leqslant \varepsilon n^{1/2}\}} |D^\alpha \int \exp(it^{\mathrm{T}} x)(P_n - Q_n)(dx)|\, dt.$$

We apply the inequality

$$|a/b - c/d| \leqslant d^{-1}(|b-d|\, |a|/b - |a-c|),$$

where

$$a = \int 1_{\{\|t\| \leqslant \varepsilon n^{1/2}\}} D^\alpha \mathrm{E} \exp\left(it^{\mathrm{T}} n^{-1/2}(X_1 + \ldots + X_n - n\mu(P))\right) \times$$
$$\times 1_{\{Y_1 + \ldots + Y_n = y\}}\, dt,$$

$$c = \int 1_{\{\|t\| \leqslant \varepsilon n^{1/2}\}} D^\alpha \left((2\pi n^{1/2})^{-q} \int \psi_n(t, v) \exp\left(-iv^{\mathrm{T}} \tilde{y}(P, n)\right) dv\right) dt,$$

$$b = P\{Y_1 + \ldots + Y_n = y\},$$

$$d = n^{-q/2} \sum_{j=0}^{r-3} n^{-j/2} P_j\left(-\varphi_{0:\Sigma_{11}(P)}:\ \{\chi_v(P)\}\right)(\tilde{y}(P, n)).$$

Together with the inequality $|a|/|b| \leqslant (|c| + |a-c|)/(|d| - |b-d|)$ we obtain

$$|a/b - c/d| = o(n^{-(s-2)/2}).$$

Hence relation (B) holds if

$$\int 1_{\{n^{(s-1)/2} \geqslant \|t\| \geqslant \varepsilon n^{1/2}\}} |D^\alpha \int \exp(it^{\mathrm{T}} x)(P_n - Q_n)(dx)|\, dt = o(n^{-(s-2)/2}).$$

Obviously,

$$\int 1_{\{n^{(s-1)/2} \geq \|t\| \geq \varepsilon n^{1/2}\}} |D^\alpha \int e^{itx} Q_n(dx)|\, dt = o(n^{-(s-2)/2}).$$

For the proof of the same relation for P_n we use the uniform Cramér condition (1.1). From (2.7) we obtain

$$|D^\alpha \int \exp(it^{\mathrm{T}} x) P_n(dx)| \leqslant \sup\{|D^\alpha f_n(t, v)|: v \in R^q\}/P\{Y_1 + \ldots + Y_n = y\}.$$

Equation (2.8) and condition (1.1) yield that sup $\{|D^\alpha f_n(t, v)|: v \in R^q\}$ converges to zero exponentially. Using (2.5) and (2.6) we see that $P\{Y_1 + \ldots + Y_n = y\}$ does not converge to zero exponentially. This proves the equality

$$\int 1_{\{n^{(s-1)/2} \geq \|t\| \geq \varepsilon n^{1/2}\}} |D^\alpha \int \exp(it^{\mathrm{T}} x) P_n(dx)|\, dt = o(n^{-(s-2)/2}).$$

Now the proof of Theorem 2.3 is complete.

3. Formulas. To write the formulas in an economic way we need the following notation:

For positive integers m and $i_1, \ldots, i_m \in \{0, \ldots, q\}$ let

$$\sigma_{i_1,\ldots,i_m} = \int (z^{(i_1)} - \mu^{(i_1)}(P)) \ldots (z^{(i_n)} - \mu^{(i_n)}(P)) P(dz),$$

where $z^{(0)}, \ldots, z^{(q)}$ and $\mu^{(0)}(P), \ldots, \mu^{(q)}(P)$ are the components of the vectors z and $\mu(P)$, respectively. Write

$$\Sigma^{-1} = A = (a_{ij})_{i,j=0,\ldots,q},$$

$$(i, j, l) = a_{00}^{3/2} \sigma_{i,j,l}, \quad i, j, l = 0, \ldots, q,$$

$$(i, j, l, m) = a_{00}^2 \sigma_{i,j,l,m}, \quad i, j, l, m = 0, \ldots, q.$$

We note that $\sigma = a_{00}^{-1/2}$.

If in the brackets an index, say i, is replaced by a dot, this means multiplication by $a_{11}^{-1} a_{1i}$ and summation over $i = 0, \ldots, q$. If a pair of indices i, j is replaced by a pair of plus signs or asterisks, this means multiplication by $a_{00}^{-1} a_{ij}$ and summation over $i, j = 0, \ldots, q$. For example,

$$(\cdot, \cdot, \cdot) = a_{00}^{-3/2} \sum_{i,j,l=0}^{q} a_{0i} a_{0j} a_{0l} \sigma_{i,j,l},$$

$$(\cdot, +, +) = a_{00}^{-1/2} \sum_{i,j,l=0}^{q} a_{0i} a_{jl} \sigma_{i,j,l}.$$

We shall use the following convention: if in a product an index occurs at least twice, this means summation over this index starting from 0 in case of a Roman type index, and from 1 in case of a Greek type index.

For $H_i(P, y, x)$ introduced in (2.2) we define W_i by

$$H_i(P, y, x) = W_i(P, y, x) H_0(P, y, x), \quad i = 1, 2.$$

Then

(3.1) $$W_1(P, y, x) = u^3(\cdot\cdot, \cdot)/6 - u(\cdot, +, +)/2 + (u^2-1)\sigma r_\beta(\beta, \cdot, \cdot)/2 + \\ + u\sigma^2 r_\beta r_\gamma(\beta, \gamma, \cdot)/2,$$

(3.2)
$$\begin{aligned} W_2(P, y, x) = {} & (u^4-3)\big((\cdot, \cdot, \cdot, \cdot)-3\big)/24 - (u^2-1)\big((\cdot, \cdot, +, +)-3-q\big)/4 + \\ & + u^3\sigma r_\beta(\beta, \cdot, \cdot, \cdot)/6 - u\sigma r_\beta(\beta, \cdot, +, +)/2 + \\ & + (u^2-1)\sigma^2 r_\beta r_\gamma(\beta, \gamma, \cdot, \cdot)/4 + u\sigma^3 r_\beta r_\gamma r_\delta(\beta, \gamma, \delta, \cdot)/6 - \\ & - (u^2-1) y_\beta r_\beta/4 + (u^2-1)\sigma^4 r_\beta r_\gamma r_\delta r_\varepsilon(\beta, \gamma, \cdot)(\delta, \varepsilon, \cdot)/72 - \\ & - u\sigma^3 r_\beta r_\gamma r_\delta(\beta, \gamma, +)(\delta, \cdot, +)/2 + \\ & + (u^3-u)\sigma^3 r_\beta r_\gamma r_\delta(\beta, \cdot, \cdot)(\gamma, \delta, \cdot)/4 - \\ & - (u^2-1)\sigma^2 r_\beta r_\gamma(\beta, \gamma, +)(+, \cdot, \cdot)/4 + \\ & + (u^2-1)\sigma^2 r_\beta r_\gamma(\beta, \cdot, +)(\gamma, \cdot, +)/2 + (u^4-3)\sigma^2 r_\beta r_\gamma(\beta, \gamma, \cdot)(\cdot, \cdot, \cdot)/12 + \\ & + (u^4-2u^2-1)\sigma^2 r_\beta r_\gamma(\beta, \cdot, \cdot)(\gamma, \cdot, \cdot)/8 - \\ & - (u^2-1)\sigma^2 r_\beta r_\gamma(\beta, \gamma, \cdot)(\cdot, +, +)/4 + u\sigma r_\beta(\beta, \cdot, +)(+, *, *)/2 - \\ & - u^3\sigma r_\beta(\beta, \cdot, +)(\cdot, \cdot, +)/2 + u\sigma r_\beta(\beta, +, *)(\cdot, +, *)/2 + \\ & + (u^5-u^3)\sigma r_\beta(\beta, \cdot, \cdot)(\cdot, \cdot, \cdot)/12 - (u^3-u)\sigma r_\beta(\beta, \cdot, \cdot)(\cdot, +, +)/4 - \\ & - (u^4-3)(\cdot, \cdot, +)(\cdot, \cdot, +)/8 + (u^2-1)(\cdot, \cdot, +)(+, *, *)/4 + \\ & + (u^2-1)(\cdot, +, *)(\cdot, +, *)/4 + (u^2-1)(\cdot, +, +)^2/8 - \\ & - (u^4-3)(\cdot, \cdot, \cdot)(\cdot, +, +)/12 + (u^6-15)(\cdot, \cdot, \cdot)^2/72, \end{aligned}$$

where $u = \sigma^{-1}\big(x - \Sigma_{01}(P)\Sigma_{11}^{-1}(P)y\big)$ and $r = \Sigma_{11}^{-1}(P)y$.

Define

$$R_i(P, y, x) = \int_{-\infty}^{x} H_i(P, y, \xi)\, d\xi / H_0(P, y, x).$$

Then with u and r as above we obtain

(3.3) $$R_1(P, y, x) = -u^2(\cdot, \cdot, \cdot)/6 + (\cdot, +, +)/2 - (\cdot, \cdot, \cdot)/3 - u\sigma r_\beta(\beta, \cdot, \cdot)/2 - \\ - \sigma^2 r_\beta r_\gamma(\beta, \gamma, \cdot)/2,$$

(3.4)
$$\begin{aligned} R_2(P, y, x) = {} & (-u^2-3u)\big((\cdot, \cdot, \cdot, \cdot)-3\big)/24 + u\big((\cdot, \cdot, +, +)-3-q\big)/4 + \\ & + (-u^2-2)\sigma r_\beta(\beta, \cdot, \cdot, \cdot)/6 + \sigma r_\beta(\beta, \cdot, +, +)/2 - \\ & - u\sigma^2 r_\beta r_\gamma(\beta, \gamma, \cdot, \cdot)/4 - \sigma^3 r_\beta r_\gamma r_\delta(\beta, \gamma, \delta, \cdot)/6 + \\ & + u y_\beta r_\beta/4 - u\sigma^4 r_\beta r_\gamma r_\delta r_\varepsilon(\beta, \gamma, \cdot)(\delta, \varepsilon, \cdot)/72 + \\ & + \sigma^3 r_\beta r_\gamma r_\delta(\beta, \gamma, +)(\delta, \cdot, +)/2 + \\ & + (-u^2+3)\sigma^3 r_\beta r_\gamma r_\delta(\beta, \cdot, \cdot)(\gamma, \delta, \cdot)/4 + \end{aligned}$$

$$+u\sigma^2 r_\beta r_\gamma(\beta, \gamma, +)(+, \cdot, \cdot)/4-$$
$$-u\sigma^2 r_\beta r_\gamma(\beta, \cdot, +)(\gamma, \cdot, +)/2+$$
$$+(-u^3-3u)\sigma^2 r_\beta r_\gamma(\beta, \gamma, \cdot)(\cdot, \cdot, \cdot)/12+$$
$$+(-u^3-u)\sigma^2 r_\beta r_\gamma(\beta, \cdot, \cdot)(\gamma, \cdot, \cdot)/8+$$
$$+u\sigma^2 r_\beta r_\gamma(\beta, \gamma, \cdot)(\cdot, +, +)/4-\sigma r_\beta(\beta, \cdot, +)(+, *, *)/2-$$
$$-(-u^2-2)\sigma r_\beta(\beta, \cdot, +)(\cdot, \cdot, +)/2-$$
$$-\sigma r_\beta(\beta, +, *)(\cdot, +, *)/2+$$
$$+(-u^4-3u^2-6)\sigma r_\beta(\beta, \cdot, \cdot)(\cdot, \cdot, \cdot)/12-$$
$$-(-u^2-1)\sigma r_\beta(\beta, \cdot, \cdot)(\cdot, \cdot, \cdot)/12-$$
$$-(-u^2-1)\sigma r_\beta(\beta, \cdot, \cdot)(\cdot, +, +)/4-$$
$$-(-u^3-3u)(\cdot, \cdot, +)(\cdot, \cdot, +)/8-u(\cdot, \cdot, +)(+, *, *)/4-$$
$$-u(\cdot, +, *)(\cdot, +, *)/4-u(\cdot, +, +)^2/8-$$
$$-(-u^3-3u)(\cdot, \cdot, \cdot)(\cdot, +, +)/12+$$
$$+(-u^5-5u^3-15)(\cdot, \cdot, \cdot)^2/72.$$

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Received on 25. 6. 1981