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A CLASSIFICATION OF RANDOM MEASURES

BY

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Abstract. Modifying the definition of α-times ($0 < \alpha \leqslant \infty$) self-decomposable (selfdec.) distributions on linear spaces due to N. V. Thu, we define α-times selfdec. random measures (r.m.) on a Polish space. We prove representation theorems for such r.m. and study some related limit problems.

Throughout the paper we preserve the terminology and notation of [2]. Recall some of them. Let σ be a Polish space, $\mathscr{B}$ — the ring of all bounded Borel subsets of σ, F_c — the class of all continuous functions $f\colon \sigma \to R_+ = [0; \infty)$ with compact support and M — the class of all Radon measures on σ. We shall consider M as a Polish space with the vague topology. By a *random measure* (r.m.) on σ we mean a Borel probability measure on M. By M_0 we denote the class of all infinitely divisible random measures (i.d.r.m.) on σ (cf. [2]).

Let L_p denote the Laplace transform of an i.d.r.m. P on σ. By virtue of Theorem 6.1 in [2] we get the formula

$$(1) \qquad -\log L_p(f) = m(f) + \lambda(1 - e^{-\pi f}), \quad f \in \mathscr{F}_c,$$

where $m \in M$, λ is a measure on $M' = M \setminus \{0\}$ satisfying the condition

$$(2) \qquad \lambda(1 - e^{-\pi B}) < \infty, \quad B \in \mathscr{B}.$$

In what follows (m, λ) will be called *canonical measure* of P and we write $P = (m, \lambda)$. Further, by L_0 we denote the class of all measures λ on M' satisfying condition (2).

For every $\alpha > 0$ and $k = 0, 1, \dots$ we put

$$r_{\alpha,k} = \binom{\alpha+k-1}{k} = \begin{cases} 1 & \text{if } k = 0, \\ \dfrac{\alpha(\alpha+1)\dots(\alpha+k-1)}{k!} & \text{if } k = 1, 2, \dots \end{cases}$$

Given a number $c > 0$ and an r.m. P on σ, we define an r.m. $T_c P$ on σ by

$$T_c P(E) = P\{\mu\colon\ c\mu \in E\}$$

for every Borel subset E of M.

The concept of α-times selfdec. probability measures on linear spaces was introduced and studied by Thu [5, 6]. In the same way one can define α-times selfdec. r.m. Namely, an r.m. P on σ is said to be *α-times selfdec.* if for every $c \in (0, 1)$ there exists an i.d.r.m. $P_{\alpha,c}$ such that

$$P = \mathop{*}_{k=0}^{\infty} T_{c^k} A(P_{\alpha,c}; r_{\alpha,k}), \tag{3}$$

where for an i.d.r.m. Q and $t > 0$ the symbol $A(Q; t)$ denotes Q^{*t} and $*$ is the convolution operation.

Further, if (3) holds for some fixed $c \in (0, 1)$ and $P_{\alpha,c} \in M_0$, then we say that P is *α-times c-decomposable* (c-dec., cf. [4]).

By M_α (resp. $M_{\alpha,c}$), $0 < \alpha < \infty$, we denote the class of all α-times selfdec. (resp. c-dec.) r.m. on σ. Further, the r.m. in

$$M_\infty = \bigcap_{\alpha > 0} M_\alpha \quad (\text{resp. } M_{\infty,c} = \bigcap_{\alpha > 0} M_{\alpha,c})$$

are called *completely selfdec.* (resp. *completely c-dec.*).

An r.m. $P \in M_\alpha$ is said to be *α-differentiable* if the following limit exists in the weak sense:

$$D^{(\alpha)} P = \operatorname*{Lim}_{t \to 0} A(P_{\alpha,c}; t^{-\alpha});$$

$P_{\alpha,c}$ is determined in (3) with $c = e^{-t}$ (cf. [6]). For every $r > 0$ and $B \in \mathscr{B}$ we put $M_r(B) = \{\ \in M\colon \mu B > r\}$.

The following theorem is an analogon of Theorem 2.1 in [4] and its proof will be omitted:

THEOREM 1. *The following statements are equivalent:*

(i) *The infinite convolution* $\mathop{*}_{k=0}^{\infty} T_{c^k} A(P; r_{\alpha,k})$ *is weakly convergent.*

(ii) $\int_{M_1(B)} \log^\alpha \mu B P(d\mu) < \infty$, $B \in \mathscr{B}$.

(iii) $\int_{M_1(B)} \log^\alpha \mu B \lambda(d\mu) < \infty$, $B \in \mathscr{B}$.

Let $M_{0,\alpha}$ denote the class of all $P \in M_0$ satisfying condition (ii) of Theorem 1. Further, by $L_{0,\alpha}$ we denote the class of all $\lambda \in L_0$ such that $P = (0, \lambda) \in M_{0,\alpha}$.

THEOREM 2. *The following statements are equivalent*:

(i) $P \in M_\alpha$.

(ii) $P \in M_\alpha$ *and* $\{A(P_{\alpha,c}; t^{-\alpha}),\ t > 0,\ c = e^{-t}\}$ *is relatively compact in the weak sense.*

(iii) *There exist an* $m_\alpha \in M$ *and a* $\lambda_\alpha \in L_{0,\alpha}$ *such that*

$$-\log L_p(f) = m_\alpha(f) + \frac{1}{\Gamma(\alpha)} \int_0^\infty T_{e^{-t}} \lambda_\alpha (1 - e^{-\pi f}) t^{\alpha-1}\, dt, \qquad f \in \mathscr{F}_c.$$

(iv) *P is* α-*differentiable and* $D^{(\alpha)} P \in M_{0,\alpha}$.

Proof. Suppose first that (i) holds, i.e. $P \in M_\alpha$. By an elementary argument we get $1 - e^{-cy} \geqslant c(1 - e^{-y})$ for every $c \in (0, 1)$ and $y > 0$. Consequently,

$$L_P(f) \leqslant \{L_{P_{\alpha,c}}(f)\}^{(1-c)^{-\alpha}}, \qquad f \in \mathscr{F}_c.$$

By the last inequality and Lemma 4.5 in [2] we can show that

$$\{A(P_{\alpha,c}; t^{-\alpha}),\ t > 0,\ c = e^{-t}\}$$

is relatively compact, which proves (ii).

Now we assume that (ii) holds. Let $P_\alpha = (m_\alpha, \lambda_\alpha)$ be a limit point of $A(P_{\alpha,c}; t^{-\alpha})$ as $t \to 0$. By Theorem 2, X.9, in [1] and by the fact that

$$r_{\alpha,k} = \frac{1}{k!\,\Gamma(\alpha)} \int_0^\infty e^{-t} t^{\alpha+k-1}\, dt \tag{4}$$

it follows that $m_\alpha = m$ and

$$\lambda(1 - e^{-\pi f}) = \frac{1}{\Gamma(\alpha)} \int_0^\infty T_{e^{-t}} \lambda_\alpha (1 - e^{-\pi f}) t^{\alpha-1}\, dt, \qquad f \in \mathscr{F}_c, \tag{5}$$

which implies (iii).

Finally, if (iii) holds, then by (4), (5) and Theorem 2, X.9, in [1] it follows that

$$\lambda = \lim_{s \to 0} \sum_{k=0}^\infty r_{\alpha,k} T_{e^{-ks}} (s^\alpha \lambda_\alpha). \tag{6}$$

Putting, for $t > 0$, $t_n = t/2^n$, $c_n = e^{-t_n}$ and

$$\lambda_{t,n} = \sum_{k=0}^\infty r_{\alpha,k} T_{c_n}^k (t_n^\alpha \lambda_\alpha), \qquad n = 0, 1, 2, \ldots,$$

we get

(7) $$P_{t,n} = (m_\alpha, \lambda_{t,n}) \in M_{\alpha,c_n}.$$

Note that, for every $c \in (0, 1)$, $M_{\alpha,c}$ is closed in the weak topology and $M_{\alpha,c}$ is contained in M_{α,c^2}. Then (6) together with (7) imply that $P \in M_{\alpha,e^{-t}}$. Since $t > 0$ is arbitrary, we conclude that $P \in M_\alpha$. Hence (ii) holds.

It is easy to show that $P_\alpha = (m_\alpha, \lambda_\alpha)$ is uniquely limit point of $A(P_{\alpha,c}; t^{-\alpha})$ as $t \to 0$. Thus (iv) is proved.

It is clear that (iv) implies (i). Theorem 2 is thus proved.

Let S_∞ denote the class of all finite convolutions of stable m.s. on σ and their cluster points.

THEOREM 3. *The following statements are equivalent:*

(i) $P \in M_\infty$.

(ii) $P \in S_\infty$.

(ii) *There exist an* $m \in M$, *a subset* K *of* $(0, 1] \times M'$ *and a probability measure* λ_∞ *on* K *such that*

$$-\log L_P(f) = m(f) + \int_K [\mu(f)]^w \lambda_\infty(dw d\mu), \quad f \in \mathscr{F}_c.$$

Proof. By Theorem 1 in [3] one can show that (ii) implies (ii). It is clear that (ii) implies (i). We shall prove that (i) implies (iii). Suppose that $P = (m, \lambda) \in M_\infty$. Let L_∞ be the set of all measures $\lambda' \in L_0$ such that $P' = (0, \lambda') \in M_\infty$. By the arguments similar to those given in the proof of Proposition 11.5 in [7] one can show that L_∞ is the union of its caps (see [7], Section 11). Suppose that λ is in a cap C of L_∞. Note that if $R_+ l$ is an extreme ray of L_∞ (see [7], Section 11), then l is a canonical measure of a stable r.m. on σ. By Theorem 1 in [3] and Proposition 11.1 in [7] the extreme non-zero points of C are of the form $l_{w,\mu}$ with $w \in (0, 1]$, $\mu \in M'$, such that $l_{w,\mu}(1 - e^{-\pi f}) = [\mu(f)]^w$, $f \in F_c$. By Choquet's theorem ([7], Section 3) there exists a probability measure l_∞ on the set $e \times C$ of all extreme points of C such that

$$\lambda(1 - e^{-\pi f}) = \int_{e \times C} l(1 - e^{-\pi f}) l_\infty(dl), \quad f \in \mathscr{F}_c.$$

Let φ be the mapping from $(0, 1] \times M'$ into the set of all canonical measures of stable r.m. on σ, determined by the formula

$$\varphi(w, \mu)(1 - e^{-f}) = [\mu(f)]^w, \quad f \in \mathscr{F}_c.$$

Put $k = \varphi^{-1}(a \times C)$ and $\lambda_\infty = l_\infty \varphi^{-1}$. We get (iii). The proof of Theorem 3 is completed.

Now, by a minor changing the proof of Theorem 5.1 in [5], one can prove the following

THEOREM 4. (i) *Every* M_α $(0 < \alpha \leqslant \infty)$ *is closed under convolution operation shifts changes of scales and passages to weak limits.*

(ii) *For any* $0 \leqslant \alpha < \beta \leqslant \infty$,

$$M_\beta \subset M_\alpha, \quad M_\beta = \bigcap_{0<\gamma<\beta} M_\gamma, \quad M_\alpha = \overline{\bigcup_{\gamma>\alpha} M_\gamma},$$

where the bar denotes the closure in the weak topology.

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