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ON A SOLUTION OF THE DUGUÉ PROBLEM

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Abstract. The paper presents a general solution of the Dugué problem of finding the characteristic functions φ_1 and φ_2 such that

$$(1-c)\varphi_1+c\varphi_2 = \varphi_1\varphi_2, \quad 0 < c < 1.$$

1. Introduction. By D. Dugué ([1], [2]) the following problem was posed. For which couples (φ_1, φ_2) of characteristic functions the condition

$$\frac{\varphi_1(t)+\varphi_2(t)}{2} = \varphi_1(t)\varphi_2(t), \quad -\infty < t < \infty, \tag{1}$$

holds. He noticed that $\varphi_1(t) = 1/(1+it)$ and $\varphi_2(t) = 1/(1-it)$ are the characteristic functions satisfying (1).

Equation (1) with $\varphi_2(t) = \varphi_1(-t)$ was discussed in [7].

A more general setting of the Dugué problem is contained in the question on couples (φ_1, φ_2) of characteristic functions for which

$$(1-c)\varphi_1(t)+c\varphi_2(t) = \varphi_1(t)\varphi_2(t), \quad c\in(0, 1), \ -\infty < t < \infty, \tag{2}$$

is satisfied. Two couples of such characteristic functions were given in [3]. They are

$$\varphi_1(t) = c+(1-c)e^{-itb}, \quad \varphi_2(t) = 1-c+ce^{itb}, \quad b\in\mathbf{R};$$
$$\varphi_1(t) = a/(a+it), \quad \varphi_2(t) = (1-c)a/((1-c)a-cit), \quad a > 0.$$

Further examples of couples of characteristic functions satisfying (2) can be found in [6] and [8]. From the results of [5] one can conclude that if F_1 and F_2 are distribution functions such that $F_1(+0) = 0$, $F_2(+0) = 1$ and F_1 is no lattice, then the unique solution of (2) is given by the characteristic functions:

$$\varphi_1(t) = \frac{ac}{ac-(1-c)it}, \quad \varphi_2(t) = \frac{a}{a+it}, \quad a > 0.$$

This paper tends towards achieving a general solution of the Dugué problem.

At the begining we give some properties of distribution functions in that question, which will be useful throughout this paper.

LEMMA 1. *If φ_1 and φ_2 are two characteristic functions such that condition (2) is satisfied, then φ_1 and φ_2 are both characteristic functions of purely discrete distributions or φ_1 and φ_2 are both characteristic functions of continuous distributions.*

LEMMA 2. *If F_1 and F_2 are two purely discrete distribution functions such that $F_1(0) = 0$, $F_2(+0) = 1$ and the condition*

$$(1-c)F_1+cF_2 = F_1 * F_2, \quad 0 < c < 1$$

(equivalent to (2)) is satisfied, then F_1 and F_2 determined lattice distributions given on the same lattice with the origin as a lattice point.

2. Solution of the Dugué problem for distributions with supports on the different semi-axises.

THEOREM 1. *Let φ_1 and φ_2 be two characteristic functions of distribution functions F_1 and F_2, respectively. If $F_1(0) = 0$ and $F_2(+0) = 1$, then condition (2) is satisfied only by the following characteristic functions φ_1 and φ_2:*

(i) $$\varphi_1(t) = 1, \quad \varphi_2(t) = 1;$$

(ii) $$\varphi_1(t) = p+(1-p)e^{ith},$$

$$\varphi_2(t) = 1-c+c\frac{1-c}{(1-p)e^{ith}-(c-p)}, \quad 0 \leqslant p \leqslant c,\ h > 0;$$

(iii) $$\varphi_1(t) = c+(1-c)e^{ith}\frac{r-1}{r-e^{ith}},$$

$$\varphi_2(t) = 1-\frac{cr}{r-1}+\frac{cr}{r-1}e^{-ith}, \quad r \geqslant 1/(1-c),\ h > 0;$$

(iv) $$\varphi_1(t) = [p+(1-p)e^{ith}]\frac{r_1-1}{r_1-e^{ith}}, \quad \varphi_2(t) = [1-p+pe^{-ith}]\frac{1-r_2}{1-r_2e^{-ith}},$$

$1 < r_1 < 1/(1-c)$, $r_2 = 1-(1-c)(r_1-1)/[c+(1-p)(r_1-1)]$, $0 < p < 1$, $h > 0$;

(v) $$\varphi_1(t) = \frac{\alpha c}{\alpha c-(1-c)it}, \quad \varphi_2(t) = \frac{\alpha}{\alpha+it}, \quad \alpha > 0.$$

Proof. It can be easily verified that condition (2) is satisfied by the characteristic functions (i)-(v). We shall prove that they are the unique characteristic functions satisfying (2) and the assumptions of Theorem 1. Assume that φ_1 and φ_2 are characteristic functions of distribution functions F_1 and F_2 for which condition (2) and the assumptions of Theorem 1 are satisfied.

Let us consider the following functions of a complex variable $z = t+is$:

$$g_1(z) = \int_0^{\infty} e^{izx}\, dF_1(x), \qquad g_2(z) = \int_{-\infty}^{0} e^{izx}\, dF_2(x), \qquad z \in \boldsymbol{C}.$$

The function g_1 is analytic in the upper half-plane, while the function g_2 is analytic in the lower half-plane, and they are both continuous in those closed domains. On the real axis we have

(3) $$g_1(t) = \varphi_1(t), \qquad g_2(t) = \varphi_2(t), \qquad t \in \boldsymbol{R},$$

and $(1-c)g_1(z)+cg_2(z) = g_1(z)g_2(z)$, $\operatorname{Im} z = 0$.

Let

(4) $$\tilde{g}_1(z) = \begin{cases} g_1(z), & \operatorname{Im} z \geqslant 0, \\ \dfrac{cg_2(z)}{g_2(z)-(1-c)}, & \operatorname{Im} z < 0; \end{cases}$$

(5) $$\tilde{g}_2(z) = \begin{cases} \dfrac{(1-c)g_1(z)}{g_1(z)-c}, & \operatorname{Im} z > 0, \\ g_2(z), & \operatorname{Im} z \leqslant 0. \end{cases}$$

For the function $\tilde{g}_1$ and $\tilde{g}_2$ we have

(6) $$(1-c)\tilde{g}_1(z)+c\tilde{g}_2(z) = \tilde{g}_1(z)\tilde{g}_2(z), \qquad z \in \boldsymbol{C}.$$

This allows us to replace the search of the solution of (2) by the search of the solution of (6) on the real axis.

On the other hand, from Lemmas 1 and 2 we see that distributions F_1 and F_2 ought to be both continuous or both lattice. Then the proof of this theorem will be devided into two parts:

A. F_1 and F_2 are continuous distribution functions.

B. F_1 and F_2 are lattice distribution functions with a location of discontinuity points on the same lattice with the origin as a lattice point.

A. Let F_1 and F_2 be continuous distribution functions. First we consider the equation

(7) $$g_1(is)-c = 0, \qquad s \geqslant 0.$$

The function $g_1(is) = \int_0^{\infty} e^{-sx}\, dF_1(x)$ strictly decreases in the interval $[0, \infty)$, so equation (7) has at most one real solution.

Suppose that equation (7) has no solution. Then

$$\lim_{s\to\infty} g_1(is) = \lim_{s\to\infty} \int_0^{\infty} e^{-sx}\, dF_1(x) \geqslant c$$

and so F_1 has the saltus $p \geqslant c$ at the origin, which is impossible as F_1 is continuous.

Let $s = \alpha$, $\alpha > 0$, be the solution of equation (7). Then the function $\tilde{g}_2$ has a pole at the point $z = i\alpha$. It can be shown that it is the only pole of $\tilde{g}_2$. Indeed, since $\tilde{g}_2$ has the pole $z = i\alpha$, it may be analytic at most in the strip $|\mathrm{Im}\, z| < \alpha$. It is easy to state that $\tilde{g}_2$ is analytic in this strip and, moreover, it is analytic for $\mathrm{Im}\, z < -\alpha$. If $\mathrm{Im}\, z > \alpha$, then

$$|\tilde{g}_1(z)| \leqslant \int_0^\infty e^{-sx} dF_1(x) = g_1(is) < c$$

and so $\tilde{g}_2$ is also analytic in that domain. If $\mathrm{Im}\, z = \alpha$, then $g_1(z) \neq c$ for $z \neq i\alpha$. Indeed, if, for a certain $t \neq 0$, $g_1(t+i\alpha) = c$, then the distribution F_1 must be lattice as

$$h(t) = g_1(t+i\alpha) = \int_0^\infty e^{itx} d\Big(\int_0^x e^{-dy} dF_1(y)\Big)$$

is unnormalized characteristic function with $h(t) = h(0)$, which is impossible. Thus $z = i\alpha$ is the only pole of the function $\tilde{g}_2$.

Following the arguments of [7] it can be proved that the function $\tilde{g}_2$, given by (5), being the analytic extension of the characteristic function φ_2, is bounded outside a certain neighbourhood of its unique pole. Hence we conclude that $\tilde{g}_2$ is the rational function of the form

$$g_2(z) = \frac{P_k(z)}{(z-i\alpha)^k}, \quad k \in N, \tag{9}$$

where P_k is the polynomial of the k-th degree at most.

Let us consider the equation

$$g_2(is) - (1-c) = 0, \quad s \leqslant 0. \tag{10}$$

The function

$$g_2(is) = \int_{-\infty}^0 e^{-sx} dF_2(x)$$

strictly increases in the interval $(-\infty, 0]$ and so this equation has at most one real solution. Assume that $s = -\beta$, $\beta > 0$, is the solution of equation (10). Then the function $\tilde{g}_1$ has the pole at the point $z = -i\beta$. In the similar manner as above one can show that $z = -i\beta$ is the only pole of $\tilde{g}_1$ and that $|\tilde{g}_1|$ is bounded outside a neighbourhood of this pole. Thus the function $\tilde{g}_1$ has the form

$$\tilde{g}_1(z) = \frac{Q_l(z)}{(z+i\beta)^l}, \quad l \in N, \tag{11}$$

where Q_l is the polynomial of the l-th degree at most.

On the other hand, by (6) and (9), we get

$$\tilde{g}_1(z) = \frac{cP_k(z)}{P_k(z)-(1-c)(z-i\alpha)^k}, \quad k \in \mathbf{N}. \tag{12}$$

Let us observe that the numbers k and l in representations (11) and (12), respectively, are such that $\tilde{g}_1$ and $\tilde{g}_2$ are irreducible rational functions. Since $P_k(z)$ and $P_k(z)-(1-c)(z-i\alpha)^k$ are also relatively irreducible, it must be $k = l$ and

$$Q_l(z) = AcP_l(z), \quad (z+i\beta)^l = A[P_l(z)-(1-c)(z-i\alpha)^l], \quad A \in \mathbf{R}.$$

Hence

$$Q_l(z) = c(z+i\beta)^l + A(1-c)c(z-i\alpha)^l, \quad P_l(z) = \frac{1}{A}(z+i\beta)^l + (1-c)(z-i\alpha)^l,$$

and from $\tilde{g}_1(0) = \tilde{g}_2(0) = 1$ we get $A = (1/c)(-\beta/\alpha)^l$. Thus we obtain

$$\tilde{g}_1(z) = c+(1-c)\left(-\frac{\beta}{\alpha}\right)^l\left(\frac{z-i\alpha}{z+i\beta}\right)^l, \quad \tilde{g}_2(z) = 1-c+c\left(-\frac{\alpha}{\beta}\right)^l\left(\frac{z+i\beta}{z-i\alpha}\right)^l, \quad l \in \mathbf{N}.$$

Therefore

$$\begin{aligned} \varphi_1(t) &= \frac{c\alpha^l(t+i\beta)^l + (1-c)(-\beta)^l(t-i\alpha)^l}{\alpha^l(t+i\beta)^l}, \\ \varphi_2(t) &= \frac{(1-c)\beta^l(t-i\alpha)^l + c(-\alpha)^l(t+i\beta)^l}{\beta^l(t-i\alpha)^l}, \quad l \in \mathbf{N}. \end{aligned} \tag{13}$$

Since in this case $\lim_{s\to\infty} \tilde{g}_1(is) = 0$, we conclude that l should be odd and $(\alpha/\beta)^l = (1-c)/c$. We see then that

$$\lim_{|t|\to\infty} \varphi_1(t) = 0 \quad \text{and} \quad \lim_{|t|\to\infty} \varphi_2(t) = 0,$$

i.e. φ_1 and φ_2 are characteristic functions of absolutely continuous distributions. Then the polynomials from the numerators of (13) have the degree $l-1$ and the formulae

$$f_1(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\varphi_1(t)\,dt, \; x > 0, \quad f_2(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\varphi_2(t)\,dt, \; x < 0,$$

determine the density functions corresponding to the characteristic functions

φ_1 and φ_2, respectively. Applying the residue theorem we get

$$\int_{-\infty}^{\infty} \varphi_1(t)\, e^{-itx}\, dt = -2\pi i \operatorname{Res}_{-i\beta}[\tilde{g}_1(z)\, e^{-izx}]$$

$$= -2\pi i c \operatorname{Res}_{-i\beta}\left[\left(1-\left(\frac{z-i\alpha}{z+i\beta}\right)^l\right)e^{-izx}\right] = \frac{2\pi i c}{(l-1)!} \lim_{z\to -i\beta} \frac{d^{l-1}}{dz^{l-1}}[(z-i\alpha)^l e^{-izx}]$$

$$= 2\pi c(\alpha+\beta)\, e^{-\beta x} \sum_{k=0}^{l-1} \binom{l-1}{k} \frac{l}{(k+1)!} (-(\alpha+\beta)\, x)^k$$

$$= 2\pi c(\alpha+\beta)\, e^{-\beta x}\, W_l(-(\alpha+\beta)\, x)/(l-1)!,$$

where

$$W_l(u) = \sum_{k=0}^{l-1} \binom{l}{k+1} \frac{(l-1)!}{k!} u^k, \qquad u \in \boldsymbol{R}.$$

It was established in [7] that the polynomial $W_l(u)$ has $l-1$ different real negative roots. Consequently, if $l > 1$, then

$$f_1(x) = c(\alpha+\beta)\, e^{-\beta x}\, W_l(-(\alpha+\beta)\, x)/(l-1)!, \qquad x > 0,$$

could not be a density function, as W_l takes negative values.

If $l = 1$, then $\alpha = \beta(1-c)/c$ and

$$(14) \qquad f_1(x) = \begin{cases} 0 & \text{for } x \leqslant 0, \\ \beta e^{-\beta x} & \text{for } x > 0, \end{cases}$$

i.e. the density function of the exponential distribution, and so $\varphi_1(t) = \beta/(\beta - it)$. Furthermore, from (13) we get $\varphi_2(t) = \alpha/(\alpha + it)$, i.e. the characteristic function of the exponential distribution given on the left semi-axis by the density function

$$(15) \qquad f_2(x) = \begin{cases} \alpha e^{\alpha x} & \text{for } x < 0, \\ 0 & \text{for } x \geqslant 0. \end{cases}$$

Finally we obtain the following couple of characteristic functions

$$(16) \qquad \varphi_1(t) = \frac{\beta}{\beta - it}, \quad \varphi_2(t) = \frac{\alpha}{\alpha + it}, \quad \beta = \frac{c}{1-c}\alpha, \quad \alpha > 0,$$

for which condition (2) is satisfied.

At the end we note that if equation (10) has no solution, then

$$\lim_{s\to-\infty} \tilde{g}_2(is) = \lim_{s\to-\infty} \int_{-\infty}^{0} e^{-sx}\, dF_2(x) \geqslant 1-c$$

and F_2 has the saltus $q \geqslant 1-c$ at the origin, in contradiction to assumption A.

Let us remark that by the consideration of part A we obtain only one couple (16) of characteristic functions of continuous distributions satisfying condition (2) and the assumptions of Theorem 1.

B. Assume now that F_1 and F_2 are lattice distributions, defined on the same lattice with the step $h > 0$ and the origin as a lattice point. For the simplicity we can put $h = 1$. By the assumptions of Theorem 1, F_1 is a right-sided distribution, while F_2 is a left-sided distribution, and so the corresponding characteristic functions can be written as

$$(17) \qquad \varphi_1(t) = \sum_{k=0}^{\infty} p_k e^{itk}, \quad \varphi_2(t) = \sum_{k=0}^{\infty} q_k e^{-itk},$$

where $0 \leqslant p_k \leqslant 1$, $0 \leqslant q_k \leqslant 1$, $k = 0, 1, 2, \ldots$ and $\sum_{k=0}^{\infty} p_k = 1$, $\sum_{k=0}^{\infty} q_k = 1$.

Let us consider the following functions of a complex variable $z = re^{it}$ (r and t are real numbers):

$$(18) \qquad g_1(z) = \sum_{k=0}^{\infty} p_k z^k, \quad g_2(z) = \sum_{k=0}^{\infty} q_k z^{-k}, \quad z \in C.$$

The function g_1 is analytic inside the circle $K = \{z \in C: |z| = 1\}$, while the function g_2 is analytic outside K, and they are both continuous on K. Moreover, we have on K

$$g_1(z) = g_1(e^{it}) = \varphi_1(t), \quad g_2(z) = g_2(e^{it}) = \varphi_2(t), \quad |z| = 1,$$

and

$$(1-c)g_1(z) + cg_2(z) = g_1(z)g_2(z), \quad |z| = 1.$$

Hence we see that a meromorphic extension of g_1 to the outside of K can be given by

$$g_1(z) = \frac{cg_2(z)}{g_2(z)-(1-c)}, \quad |z| > 1,$$

and, similarly, the equation

$$g_2(z) = \frac{(1-c)g_1(z)}{g_1(z)-c}, \quad |z| \leqslant 1,$$

defines a meromorphic extension of g_2 to the interior of K.

Let us write

$$(19) \qquad \tilde{g}_1(z) = \begin{cases} g_1(z), & |z| \leqslant 1, \\ \dfrac{cg_2(z)}{g_2(z)-(1-c)}, & |z| > 1, \end{cases}$$

and

$$\tilde{g}_2(z) = \begin{cases} \dfrac{(1-c)g_1(z)}{g_1(z)-c}, & |z| < 1, \\ g_2(z), & |z| \geq 1. \end{cases} \tag{20}$$

Then

$$(1-c)\tilde{g}_1(z)+c\tilde{g}_2(z) = \tilde{g}_1(z)\tilde{g}_2(z), \quad z \in C. \tag{21}$$

Note that

$$\lim_{|z|\to\infty} g_2(z) = q_0. \tag{22}$$

In order to find the lattice solutions of (2) or, equivalently, the solutions of (21) on K, we consider the following cases: (I) $q_0 > 1-c$, (II) $q_0 = 1-c$, (III) $q_0 < 1-c$.

(I) Let $q_0 > 1-c$. From (2) for functions (17) we have

$$(1-c)p_0 + cq_0 = \sum_{k=0}^{\infty} p_k q_k, \tag{23}$$

$$(1-c-q_0)p_k = \sum_{n=1}^{\infty} p_{k+n} q_n, \quad k = 1, 2, \ldots, \tag{24}$$

$$cq_k = \sum_{n=0}^{\infty} p_n q_{k+n}, \quad k = 1, 2, \ldots \tag{25}$$

From (24) we get $p_k = 0$ $(k = 1, 2, \ldots)$ as $q_0 > 1-c$, and so $p_0 = 1$. Furthermore, from (23) and (25) we obtain $q_0 = 1$, $q_k = 0$ $(k = 1, 2, \ldots)$. Thus in this case we have only

$$\varphi_1(t) = 1, \quad \varphi_2(t) = 1, \tag{26}$$

as the solution of equation (2).

(II) Let $q_0 = 1-c$. Then φ_2 is the characteristic function of a distribution with at least two discontinuity points and $q_0 = 1-c$ is the saltus at the origin. Thus φ_2 can be represented in the form

$$\varphi_2(t) = 1-c+c\varphi(t), \tag{27}$$

where $\varphi(t)$ is the characteristic function of a left-sided lattice distribution without jump at the origin. From (2) we have

$$\varphi_2(t) = \varphi_1(t)\varphi(t). \tag{28}$$

If φ is the characteristic function of the degenerated distribution, i.e. if $\varphi(t) = e^{-imt}$, $m \in N$, then, by (27)-(28), we immediately get

$$\varphi_1(t) = c+(1-c)e^{imt}, \quad \varphi_2(t) = 1-c+ce^{-imt}. \tag{29}$$

If φ is the characteristic function of a distribution with at least two discontinuity points, then φ can be written as

(30) $$\varphi(t) = \frac{1}{c} \sum_{k=k_0}^{\infty} q_k e^{-itk}, \quad \text{where } 0 < q_{k_0} < c,\ k_0 \in N.$$

From (27)-(28) we get

(31) $$1 - c + c\varphi(t) = \varphi_1(t)\varphi(t), \quad 0 < c < 1.$$

Since the product of $\varphi_1(t)$ and $\varphi(t)$ is the characteristic function of the left-sided lattice distribution based on the same lattice as φ, we conclude that $\varphi_1(t)$ has the form

(32) $$\varphi_1(t) = \sum_{k=0}^{k_0} p_k e^{itk}.$$

As a consequence, we obtain the following system of equations:

(33) $$p_{k_0} q_{k_0} = c(1-c),$$

(34) $$\sum_{l=0}^{k} p_{l+k_0-l} q_{l+k_0} = 0, \quad k = 1, 2, \ldots, k_0 - 1,$$

(35) $$\sum_{l=0}^{k} p_l q_{l+k} = c q_k, \quad k = k_0, k_0+1, \ldots$$

From (33)-(34) we get

(36) $$p_1 = p_2 = \ldots = p_{k_0-1} = 0,$$
$$p_{k_0} \neq 0, \quad q_{k_0+1} = q_{k_0+2} = \ldots = q_{2k_0-1} = 0$$

and equations (35) reduce to

(35′) $$p_0 q_k + p_{k_0} q_{k+k_0} = c q_k, \quad k \geqslant k_0.$$

Note that $0 \leqslant p_0 < c$. Indeed, $q_{k_0} < c$ and, by (33), $p_{k_0} > 1-c$. From (32) and (36) we see that $p_0 + p_{k_0} = 1$, so $p_0 < c$. Hence we get

$$\varphi_1(t) = e^{itk_0},\ p_0 = 0, \quad \text{and} \quad \varphi_1(t) = p_0 + (1-p_0)e^{itk_0},\ p_0 < c.$$

We now can deduce from (35′) that $q_k \neq 0 \Leftrightarrow q_{k+k_0} \neq 0$ $(k \geqslant k_0)$ and

$$q_k = \begin{cases} q_{k_0}\left(\dfrac{c-p_0}{p_{k_0}}\right)^{n-1}, & k = nk_0,\ n \in N, \\ 0, & k \neq nk_0,\ n \in N. \end{cases}$$

If we put $r = (c-p_0)/(1-p_0)$, $r < 1$, then q_k can be written as

$$q_k = \begin{cases} c(1-r)r^{n-1} & \text{for } k = nk_0,\ n = 1, 2, \ldots, \\ 0 & \text{for } k \neq nk_0,\ n = 1, 2, \ldots \end{cases}$$

Hence, by (30), we get

$$\varphi(t) = \frac{r-1}{r-e^{itk_0}}, \quad k_0 \in N,$$

i.e. the characteristic function of the geometric distribution, and, by (27),

$$\varphi_2(t) = \begin{cases} \dfrac{1-c}{1-ce^{-itk_0}} & \text{if } p_0 = 0, \\ 1-c+c\dfrac{r-1}{r-e^{itk_0}} & \text{if } p_0 \neq 0,\ p_0 < c,\ k_0 \in N. \end{cases}$$

Finally, in case (II), we obtain the couples of the characteristic functions satisfying (2) which can be written jointly as

$$\text{(37)} \qquad \varphi_1(t) = p+(1-p)e^{imt}, \quad \varphi_2(t) = 1-c+c\frac{r-1}{r-e^{imt}}, \quad m \in N,$$

where $0 \leqslant p \leqslant c$, $r = (c-p)/(1-p)$.

(III) Let $q_0 = \lim_{|z| \to \infty} g_2(z) < 1-c$. We first show that the function $\tilde{g}_1$, given by (19), has exactly one pole on a certain circle $C_1 = \{z \in C\colon |z| = r_1\}$, where $r_1 > 1$.

Note that if $z = re^{it}$, $r \geqslant 1$, then for $t = 0$ we have

$$g_2(z) = g_2(r) = \sum_{k=0}^{\infty} q_k r^{-k}.$$

Since the function g_2 strictly decreases in the interval $[1, \infty)$ as $r \to \infty$ and takes values from 1 to q_0, where $q_0 < 1-c$, there exists an r_1 such that $g_2(r_1) = 1-c$. Moreover, for $t \neq 0$, we have

$$\text{(38)} \qquad |g_2(re^{it})| < g_2(r).$$

Hence and by the principle of maximum it follows that $g_2(z) = 1-c$ only for $z = r_1$, so the function $\tilde{g}_1$ has the only pole which is situated on the circle $C_1 = \{z \in C\colon |z| = r_1\}$ at the point $z = r_1$, $r_1 > 1$. Furthermore, the function $\tilde{g}_1$ is bounded outside a certain circle $K_1 = \{z \in C\colon |z| \leqslant R_1\}$, where $R_1 > r_1$. In consequence, we can deduce that $\tilde{g}_1$ is the rational function of the form

$$\text{(39)} \qquad \tilde{g}_1(z) = \frac{Q_l(z)}{(r_1-z)^l}, \quad l \in N,$$

where Q_l is the polynomial of the l-th degree at most. We see also that in this case

$$\text{(40)} \qquad \lim_{|z| \to \infty} \tilde{g}_1(z) = \lim_{|z| \to \infty} \frac{cg_2(z)}{g_2(z)-(1-c)} = \frac{cq_0}{q_0-(1-c)} < c.$$

We shall find the form of the function $\tilde{g}_2$ given by (20) when $\lim_{|z|\to\infty} g_2(z) < 1-c$. Since $p_0 = \lim_{|z|\to 0} g_1(z)$, we need to consider the following cases: 1° $p_0 < c$, 2° $p_0 = c$, 3° $p_0 > c$.

Assume first that $p_0 = \lim_{|z|\to 0} g_1(z) < c$. The function

$$g_1(r) = \sum_{k=0}^{\infty} p_k r^k$$

strictly increases in the interval [0, 1] and takes values from $p_0 < c$ to 1, so there exists r_2, $0 < r_2 < 1$, such that $g_1(r_2) = c$. By the similar reasoning as previously, we can state that the function $\tilde{g}_2$ has only one pole which is situated on the circle $C_2 = \{z \in \boldsymbol{C}: |z| = r_2\}$ at the point $z = r_2$, $0 < r_2 < 1$. Besides, the function $\tilde{g}_2$ is bounded outside a certain circle $K_2 = \{z \in \boldsymbol{C}: |z| \leqslant R_2\}$, where $R_2 > r_2$. Therefore, we conclude that $\tilde{g}_2$ is the rational function of the form

$$\tilde{g}_2(z) = \frac{P_k(z)}{(r_2-z)^k}, \quad k \in \boldsymbol{N}, \tag{41}$$

where P_k is the polynomial of the k-th degree at most.

We can assume that in representations (39) and (41) numbers l and k are such that $\tilde{g}_1$ and $\tilde{g}_2$ are irreducible rational functions. From condition (21) we have

$$\frac{P_k(z)}{(r_2-z)^k} = \frac{(1-c)\,Q_l(z)}{Q_l(z)-c(r_1-z)^l}.$$

Hence, by (40), $k = l$, and

$$Q_l(z) = A(r_2-z)^l + c(r_1-z)^l, \qquad P_l(z) = (1-c)(r_2-z)^l + c(1-c)(r_1-z)^l/A,$$

where A is a constant.

Since $g_1(1) = Q_l(1)/(r_1-1)^l = 1$, we get $A = (1-c)(r_1-1)^l/(r_2-1)^l$. Then

$$\begin{aligned} \tilde{g}_1(z) &= c+(1-c)\left[\frac{(r_1-1)(r_2-z)}{(r_2-1)(r_1-z)}\right]^l, \\ \tilde{g}_2(z) &= 1-c+c\left[\frac{(r_2-1)(r_1-z)}{(r_1-1)(r_2-z)}\right]^l, \quad r_1 > 1,\ 0 < r_2 < 1,\ l \in \boldsymbol{N}. \end{aligned} \tag{42}$$

Consequently,

$$\begin{aligned} \varphi_{1,l}(t) &= c+(1-c)\left(\frac{r_1-1}{r_2-1}\right)^l \left(\frac{r_2-e^{it}}{r_1-e^{it}}\right)^l, \\ \varphi_{2,l}(t) &= 1-c+c\left(\frac{r_2-1}{r_1-1}\right)^l \left(\frac{r_1-e^{it}}{r_2-e^{it}}\right)^l, \quad r_1 > 1,\ 0 < r_2 < 1,\ l \in \boldsymbol{N}. \end{aligned} \tag{43}$$

We show next that $\varphi_{1,l}$, given by formula (43), is the characteristic function if and only if $l=1$ and $r_2 \leqslant cr_1/(r_1-(1-c))$. Indeed, if $l=1$ and $r_2 \leqslant cr_1/(r_1-(1-c))$, then

$$\varphi_{1,1}(t) = \left[\frac{cr_1}{r_1-1}+\frac{(1-c)r_2}{r_2-1}\right]\frac{r_1-1}{r_1-e^{it}}+\left[1-\frac{cr_1}{r_1-1}-\frac{(1-c)r_2}{r_2-1}\right]\frac{r_1-1}{r_1-e^{it}}e^{it}$$

is the characteristic function.

Assume now that $\varphi_{1,l}$, given by (43), is the characteristic function. Then, for $m = 0, 1, 2, \ldots$,

$$p_m = \frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi_{1,l}(t)e^{-imt}dt = \frac{1}{2\pi i}\int_{|z|=1}\frac{\tilde{g}_1(z)}{z^{m+1}}dz = \frac{1}{m!}\frac{d^m}{dz^m}[\tilde{g}_1(z)]\Big|_{z=0}.$$

Thus

$$p_0 = c+(1-c)\left(\frac{r_1-1}{r_2-1}\right)^l\frac{r_2^l}{r_1^l},$$

and

$$p_m = \frac{1}{m!}\frac{(1-c)(r_1-1)^l}{(r_2-1)^l}\frac{d^m}{dz^m}\left[\left(1-\frac{r_1-r_2}{r_1-z}\right)^l\right]\Big|_{z=0}$$

$$= \frac{(1-c)(r_1-1)^l(-1)^l(r_1-r_2)}{m!(r_2-1)^l\, r_1^{m+1}}\sum_{k=0}^{l-1}\binom{l}{k}(-1)^k\frac{(l-k+m-1)!}{(l-k-1)!}\left(\frac{r_1-r_2}{r_1}\right)^{l-k-1},$$

$$m = 1, 2, \ldots$$

Putting now $u=(r_1-r_2)/r_1$, we see that $0<u<1$ ($r_0>1$, $0<r_1<1$), and studying the polynomials

$$W_m(u) = \sum_{k=0}^{l-1}(-1)^k\binom{l}{k}\frac{(l-k+m-1)!}{(l-k-1)!}u^{l-k-1},$$

it can be verified (see [7]) that for any $u \in (0, 1)$ there exists a number m such that $p_m < 0$ if $l > 1$, which is impossible. If $l = 1$, then

$$p_0 = c+(1-c)\left(\frac{r_1-1}{r_2-1}\right)\frac{r_2}{r_1} \quad \text{and} \quad p_m = \frac{(1-c)(r_1-r_2)(r_1-1)}{1-r_2}r_1^{-m-1},$$

$$m = 1, 2, \ldots,$$

while $p_0 \geqslant 0$ if $r_2 \leqslant cr_1/(r_1-(1-c))$. It can be easily verified that then $p_m > 0$ ($m = 1, 2, \ldots$), and $\sum_{m=0}^{\infty} p_m = 1$.

On the other hand, for $l = 1$ we get from (43)

$$\varphi_{2,1}(t) = 1-c+c\frac{r_2-1}{r_1-1}+c\frac{r_1-r_2}{r_1-1}\cdot\frac{r_2-1}{r_2-e^{it}}, \quad r_1 > 1,\ 0 < r_2 < 1.$$

Hence

$$q_0 = 1-c+c\frac{r_2-1}{r_1-1}, \quad q_m = c\frac{(1-r_2)(r_1-r_2)}{r_2(r_1-1)}r_2^m, \quad m = 1, 2, \ldots,$$

and they determine the discrete distribution if $r_2 \geqslant (1-(1-c)r_1)/c$. Besides, the fact that $r_2 > 0$ implies that $r_1 < 1/(1-c)$. Finally, we get the following restrictions to (43):

$$1 < r_1 < 1/(1-c), \quad \frac{1-(1-c)r_1}{c} \leqslant r_2 \leqslant \frac{cr_1}{r_1-(1-c)}, \quad l = 1.$$

Putting now $p = cr_1/(r_1-1)+(1-c)r_2/(r_2-1)$, we can rewrite (43) with these restrictions as follows:

$$(44) \quad \varphi_1(t) = [p+(1-p)e^{it}]\frac{r_1-1}{r_1-e^{it}}, \quad \varphi_2(t) = [1-p+pe^{-it}]\frac{1-r_2}{1-r_2e^{-it}},$$

where

$$r_2 = 1-\frac{(1-c)(r_1-1)}{c+(1-p)(r_1-1)}, \quad 1 < r_1 < 1/(1-c), \; 0 \leqslant p \leqslant 1.$$

In the particular case for $p = 0$, we obtain from (44)

$$\varphi_1(t) = \frac{r_1-1}{r_1-e^{it}}e^{it}, \quad \varphi_2(t) = \frac{1-r_2}{1-r_2e^{-it}},$$

where

$$r_2 = \frac{cr_1}{r_1-(1-c)}, \quad 1 < r_1 < 1/(1-c),$$

while, for $p = 1$, we get

$$\varphi_1(t) = \frac{r_1-1}{r_1-e^{it}}, \quad \varphi_2(t) = \frac{(1-r_2)e^{-it}}{1-r_2e^{-it}},$$

where

$$r_2 = \frac{1-(1-c)r_1}{c}, \quad 1 < r_1 < 1/(1-c).$$

In case 2°, when $p_0 = c$, we have $g_1(0) = c$ and, following the previous considerations, we can state that the function $\tilde{g}_2$ has only one pole at the point $z = 0$, and $\tilde{g}_2(z) = P_k(z)/z^k$ $(k \in N)$, where P_k is the polynomial of the k-th degree at most.

Now, by the similar reasoning as in the case 1°, we obtain $k = l$, and

$$\tilde{g}_1(z) = c+(1-c)\frac{(r_1-1)^l}{(r_1-z)^l}z^l, \quad \tilde{g}_2(z) = 1-c+c\frac{(r_1-z)^l}{(r_1-1)^l}z^{-l}, \quad r_1 > 1, \, l \in N.$$

Therefore

$$(45)\qquad \begin{aligned}\varphi_{1,l}(t) &= c+(1-c)\,e^{itl}\frac{(r_1-1)^l}{(r_1-e^{it})^l},\\ \varphi_{2,l}(t) &= 1-c+ce^{-itl}\frac{(r_1-e^{it})^l}{(r_1-1)^l}, \qquad r_1>1,\ l\in N.\end{aligned}$$

It is obvious that $\varphi_{1,l}$ is the characteristic function for any positive integer l and $r_1 > 1$. We show now that $\varphi_{2,l}$, given by (45), is the characteristic function if and only if $l = 1$ and $r_1 \geqslant 1/(1-c)$. Indeed, if $l = 1$ and $r_1 \geqslant 1/(1-c)$, then

$$\varphi_{2,l}(t) = 1-\frac{cr_1}{r_1-1}+\frac{cr_1}{r_1-1}e^{-it}$$

is the characteristic function of the discrete distribution with two discontinuity points. Assuming that $\varphi_{2,l}$, given by (45), is the characteristic function, we have

$$q_m = \frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi_{2,l}(t)\,e^{itm}\,dt = \frac{1}{2\pi i}\int_{|z|=1}\frac{\tilde{g}_2(z)}{z}z^m\,dz$$

$$= \frac{1}{l!\,(r_1-1)^l}\frac{d^l}{dz^l}\left[(1-c)(r_1-1)^l z^{m+l}+c\,(r_1-z)^l z^m\right]\Big|_{z=0}, \qquad m = 0, 1, 2, \ldots$$

Hence

$$q_m = \begin{cases}\dfrac{1}{l!}\left[1-c+c\left(\dfrac{1}{1-r_1}\right)^l\right], & m=0,\\[2mm] \dfrac{c}{m!\,(l-m)!}(-1)^{l-m}\dfrac{r_1^m}{(r_1-1)^l}, & m=1,2,\ldots,l,\\[2mm] 0, & m>l.\end{cases}$$

Note that $q_{l-1} < 0$ for any positive integer $l > 1$ and any $r_1 > 1$, which contradicts the assumption that $\varphi_{2,l}$ is a characteristic function. Now, if $l = 1$, then $q_0 = ((1-c)r_1-1)/(r_1-1)$, $q_1 = cr_1/(r_1-1)$, and we see that it should be $r_1 \geqslant 1/(1-c)$, as otherwise $q_0 < 0$. Finally, we obtain

$$(46)\qquad \begin{aligned}\varphi_1(t) &= c+(1-c)\,e^{it}\frac{r_1-1}{r_1-e^{it}},\\ \varphi_2(t) &= 1-\frac{cr_1}{r_1-1}+\frac{cr_1}{r_1-1}e^{-it}, \qquad r_1\geqslant 1/(1-c).\end{aligned}$$

In the special case, where $r_1 = 1/(1-c)$, we get

$$\varphi_1(t) = \frac{c}{1-(1-c)\,e^{it}}, \qquad \varphi_2(t) = e^{-it}.$$

In the last case, where $p_0 > c$, we shall prove that $\tilde{g}_2$ is a bounded entire function.

Note first that, by definition (20), $\tilde{g}_2$ is an analytic function for $|z| \geqslant 1$. Put $z = re^{it}$, $r \leqslant 1$. For $t = 0$ the function

$$g_1(z) = g_1(r) = \sum_{k=0}^{\infty} p_k r^k$$

strictly increases in the interval [0, 1] and takes the values from $p_0 > c$ to 1, so $g_1(r) > c$ in that interval and the equation $g_1(r) - c = 0$ has no solution in [0, 1]. Further, since $|g_1(re^{it})| \leqslant g_1(r)$, and by the assumption $\lim g_1(z) > c$ ($|z| \to 0$), it follows that the function $\tilde{g}_2$ does not have any poles inside the circle $K = \{z \in C: |z| = 1\}$, so it is analytic there. Thus the function $\tilde{g}_2$ is analytic in the entire complex plane, i.e. $\tilde{g}_2$ is an entire function. Moreover, $\tilde{g}_2$ is a bounded function as $|\tilde{g}_2(z)| \leqslant g_2(r) \leqslant g_2(1) = 1$ if $|z| \geqslant 1$, and $|\tilde{g}_2(z)| \leqslant g_2(r) = (1-c)g_1(r)/(g_1(r)-c) \leqslant (1-c)/(p_0-c)$ if $|z| < 1$. Then $\tilde{g}_2$ is a bounded entire function, so it is a constant function by the Liouville's theorem. Thus $\tilde{g}_2(z) = \tilde{g}_2(1) = 1$, $\varphi_2(t) = 1$, and $q_0 = 1$ in contradiction to assumption (III).

Remark that as a conclusion of the considerations in case (III) we obtain the couples (44) and (46) of the characteristic functions satisfying condition (2).

Putting in formulae (37), (44) and (46): $\varphi_{1,h}(t) = \varphi_1(th)$, $\varphi_{2,h}(t) = \varphi_2(th)$ ($h > 0$, $h \in R$), we obtain together with (26) all couples $(\varphi_{1,h}, \varphi_{2,h})$ of the characteristic functions of one-sided lattice distributions defined on the different semi-axises which fulfil (2).

The results of parts A and B complete the proof of Theorem.

3. Solution of the Dugué problem for distributions with supports on the same semi-axis.

THEOREM 2. *Let φ_1 and φ_2 be two characteristic functions of distributions F_1 and F_2, respectively. If $F_1(0) = 0$ and $F_2(0) = 0$, then condition* (2) *is satisfied only by the characteristic functions of the distributions degenerated at the origin, i.e.*

$$\varphi_1(t) = 1, \quad \varphi_2(t) = 1. \tag{47}$$

Proof. Assume first that (2) holds with F_1 and F_2 being right-sided distributions such that $F_1(+0) > 0$ or $F_2(+0) > 0$. The supports of F_1 and F_2 we denote by $\mathrm{supp}(F_1)$ and $\mathrm{supp}(F_2)$. From (2) we know that

$$\mathrm{supp}(F_1) \cup \mathrm{supp}(F_2) = \mathrm{supp}(F_1 * F_2) \tag{48}$$

and, therefore, under these assumptions F_1 has the saltus $p_0 > 0$ at the origin, while F_2 has the saltus $q_0 > 0$ at the origin. Furthermore, condition (2) implies that

$$(1-c)p_0 + cq_0 = p_0 q_0, \quad 0 < c < 1, \tag{49}$$

which holds only if $p_0 = q_0 = 1$.

Thus in this case we get only the trivial solution of (2).

Now assume that (2) holds with F_1 and F_2 being arbitrary right-sided distributions. Denote by $\operatorname{lext}[F_1]$ and $\operatorname{lext}[F_2]$ left extremities of distributions F_1 and F_2, respectively. We have

$$\operatorname{lext}[(1-c)F_1 + cF_2] = \min\{\operatorname{lext}[F_1], \operatorname{lext}[F_2]\},$$

$$\operatorname{lext}[F_1 * F_2] = \operatorname{lext}[F_1] + \operatorname{lext}[F_2].$$

From (2) it follows that $\operatorname{lext}[F_1] = \operatorname{lext}[F_2] = 0$ and

$$0 < (1-c)F_1(x) + cF_2(x) \leqslant F_1(x)F_2(x), \quad 0 < c < 1,\ x > 0,\ x \in \boldsymbol{R}. \tag{50}$$

We shall show that non-trivial distributions satisfying (50) do not exist.

Let x_0 be an arbitrary real positive number and write $p_1 = F_1(x_0)$, $p_2 = F_2(x_0)$. Inequality (50) implies that $0 < (1-c)p_1 + cp_2 \leqslant p_1 p_2$ $(0 < c < 1)$, but it holds true only if $p_1 = p_2 = 1$. Consequently, in this case the other solution of (2) does not exist and Theorem 2 is established.

The similar result can be obtain for left-sided distributions, namely we have

THEOREM 3. *Let φ_1 and φ_2 be two characteristic functions of distribution functions F_1 and F_2, respectively. If $F_1(+0) = 1$ and $F_2(+0) = 1$, then condition* (2) *holds only for the characteristic functions of the distributions degenerated at the origin.*

4. Remarks on the Dugué problem for couples of distributions when the support of one of them is on the whole real line. Theorems 1-3 give the direct solution of the discussed problems (1) and (2) for couples of characteristic functions of one-sided distributions defined only on semi-axises.

Suppose now that one of the distribution functions F_1 and F_2 in that question has the support on the whole real line. As an example we mention the couple of characteristic functions

$$\varphi_1(t) = \frac{1}{\left(1-\frac{it}{b}\right)^2}, \quad \varphi_2(t) = \frac{1-c}{1-c\left(1-\frac{it}{b}\right)^2}, \quad b > 0,$$

satisfying (2). In [8] it has been shown that starting with that couple (φ_1, φ_2) we can generate many couples of characteristic functions of this type for which (2) holds. Thus we see that in this case the family of couples of characteristic functions satisfying the Dugué condition is not finite. The above fact may elucidate the following theorem (see [8]):

THEOREM 4. *If* $(\varphi_{1,c}, \varphi_{2,c})$ *is a couple of characteristic functions satisfying* (2) *for each* $c \in (0, \frac{1}{2})$, *then condition* (2) *holds true also for the characteristic functions* $(\Phi_{n,c}, \Psi_{n,c})$, $n \geqslant 1$, *where*

$$\Phi_{1,c}(t) = \varphi_{1,c}(t), \qquad \Psi_{1,c}(t) = \varphi_{2,c}(t),$$

$$\Phi_{n,c}(t) = \left[\frac{\sqrt{c}\,\Phi_{n-1,(1-\sqrt{c})/2}(t)}{\Phi_{n-1,(1-\sqrt{c})/2}(t) - (1-\sqrt{c})}\right]^2,$$

$$\Psi_{n,c}(t) = \Phi_{n-1,(1-\sqrt{c})/2}(t)\,\Psi_{n-1,(1-\sqrt{c})/2}(t), \qquad n \geqslant 2.$$

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