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ASYMPTOTIC REPRESENTATIONS OF SELF-NORMALIZED SUMS *

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Abstract. For sequences of i.i.d. rv with the probability distributions belonging to the domain of attraction of a stable law with index $\alpha \in (0, 2)$, the asymptotical behaviour of partial sums divided by the $1/p$-th power of partial sums of the p-th absolute powers with $\alpha/p \in (0, 2)$ is considered.

1. Introduction. Let $X, X_1, X_2, \ldots$ be independent identically distributed random variables (i.i.d.rv) with distribution function F and quantile function

$$Q(s) = \inf\{x\colon F(x) \geqslant s\}, \quad 0 < s < 1,$$
$$Q(0) = Q(0+), \quad Q(1) = Q(1-). \tag{1.1}$$

Let $G(y) = \mathrm{P}\{|X| \leqslant y\}$, $y \geqslant 0$, and let $K(s)$, $0 \leqslant s \leqslant 1$, be the quantile function of G, similarly defined as Q is in (1.1). Define also

$$Q_1(s) = (-Q(1-s)) \vee 0, \quad 0 \leqslant s \leqslant 1,$$
$$Q_2(s) = Q(s) \vee 0, \quad 0 \leqslant s \leqslant 1,$$

where $\vee$ means maximum. We note that $Q(s) = -Q_1(1-s)+Q_2(s)$.

Throughout this paper we assume that X is in the domain of attraction of a stable law with index $\alpha \in (0, 2)$. The usual characterization of the latter law in terms of distribution functions can for example be found in Feller [5]. This characterization was formulated in terms of quantile functions by Csörgő et al. [1] as follows:

PROPOSITION. *The rv X is in the domain of attraction of a stable law with index $\alpha \in (0, 2)$ if and only if with some function L, slowly varying at zero, we have*

$$K(1-s) = s^{-1/\alpha} L(s), \tag{1.2}$$

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(1.3) $$\lim_{s\downarrow 0} Q_1(1-s)/K(1-s) = w_1,$$

and

(1.4) $$\lim_{s\downarrow 0} Q_2(1-s)/K(1-s) = w_2,$$

where $w_1,\ w_2 \geqslant 0$ *and* $w_1^\alpha + w_2^\alpha = 1$.

Define

$$S_\alpha(n) = \sum_{j=1}^{n} X_j \quad \text{and} \quad T_{\alpha,p}(n) = \sum_{j=1}^{n} |X_j|^p.$$

Introduce also $U_{1,n}, \ldots, U_{n,n}$, standing for order statistics of independent uniform-(0, 1) rv. Then for each n we have

(1.5) $$\big(S_\alpha(n),\ T_{\alpha,p}(n)\big) \overset{\mathcal{D}}{=} \Big(\sum_{j=1}^{n} Q(U_{j,n}),\ \sum_{j=1}^{n} |Q(U_{j,n})|^p\Big)$$
$$= \Big(\sum_{j=1}^{n} Q_2(U_{j,n}) - \sum_{j=1}^{n} Q_1(1-U_{j,n}),\ \sum_{j=1}^{n} \big(Q_2(U_{j,n})\big)^p + \sum_{j=1}^{n} \big(Q_1(1-U_{j,n})\big)^p\Big),$$

where $p > 0$ and $\overset{\mathcal{D}}{=}$ means equality in distribution.

Let $\{N^{(i)}(y);\ y \geqslant 0\}$ $(i = 1, 2)$ be two independent Poisson processes, and define the following rv:

if $0 < \alpha < 1$,

$$\Delta_{\alpha,i} = \alpha^{-1} \int_0^\infty N^{(i)}(y)\, y^{-1-1/\alpha}\, dy \qquad (i = 1, 2);$$

if $\alpha = 1$,

$$\Delta_{1,i} = \int_{Y_1^{(i)}}^\infty \big(N^{(i)}(y) - y\big) y^{-2}\, dy - (\log Y_1^{(i)} - 1) \qquad (i = 1, 2),$$

where $Y_1^{(i)}$ is the first jump point of $N^{(i)}(y)$ $(i = 1, 2)$;

if $1 < \alpha < 2$,

$$\Delta_{\alpha,i} = \alpha^{-1} \int_0^\infty \big(N^{(i)}(y) - y\big) y^{-1-1/\alpha}\, dy \qquad (i = 1, 2).$$

Throughout we will choose $p > 0$, so that the rv $|X|^p$ is in the domain of attraction of a stable law with index $\alpha/p \in (0, 2)$. Let $H(y) = \mathrm{P}\{|X|^p \leqslant y\}$, $y \geqslant 0$, and let $J(s)$, $0 \leqslant s \leqslant 1$, be its quantile function, similarly defined as Q is in (1.1). Then $H(y) = G(y^{1/p})$, $J(s) = \big(K(s)\big)^p$, and by (1.2) we get

(1.6) $$J(1-s) = s^{-p/\alpha} L^p(s).$$

As normalizing and centralizing factors we will use

$$a_\alpha(n) = n^{-1/\alpha}/L(1/n),$$

$$b_\alpha(n) = \begin{cases} nEX = n\int_0^1 Q(s)\,ds, & \text{if } 1 < \alpha < 2, \\ n(-\int_{1/n}^1 Q_1(1-s)\,ds + \int_{1/n}^1 Q_2(1-s)), & \text{if } \alpha = 1, \\ 0, & \text{if } 0 < \alpha < 1, \end{cases}$$

$$c_{\alpha,p}(n) = (a_\alpha(n))^p = n^{-p/\alpha}/L^p(1/n),$$

and

$$d_{\alpha,p}(n) = \begin{cases} nE|X|^p = n\int_0^1 |Q(s)|^p\,ds, & \text{if } 1 < \alpha/p < 2, \\ n(\int_{1/n}^1 (Q_1(1-s))^p\,ds + \int_{1/n}^1 (Q_2(1-s))^p\,ds, & \text{if } \alpha = p, \\ 0, & \text{if } 0 < \alpha/p < 1. \end{cases}$$

The representation (1.5) combined with the method of proof of Corollary 3.1 in Csörgő et al. [1] immediately implies the following result:

THEOREM 1. *Assume that X is in the domain of attraction of a stable law with index $\alpha \in (0, 2)$, and $p > 0$ is such that $0 < \alpha/p < 2$. Then, as $n \to \infty$,*

$$\{a_\alpha(n)(S_\alpha(n) - b_\alpha(n)),\ c_{\alpha,p}(n)(T_{\alpha,p}(n) - d_{\alpha,p}(n))\}$$
$$\xrightarrow{\mathscr{D}} \{-w_1 \Delta_{\alpha,1} + w_2 \Delta_{\alpha,2},\ w_1^p \Delta_{\alpha/p,1} + w_2^p \Delta_{\alpha/p,2}\},$$

where $\xrightarrow{\mathscr{D}}$ stands for convergence in distribution.

In this paper we are interested in the limiting distribution of the rv

$$r_{\alpha,p}(n)\left(\frac{S_\alpha(n)}{T_{\alpha,p}^{1/p}(n)} - e_{\alpha,p}(n)\right) \tag{1.7}$$

with appropriate normalizing and centralizing sequences of constants $r_{\alpha,p}(n)$ and $e_{\alpha,p}(n)$, respectively. In case of $\alpha = 2$ and $EX = 0$, we have Student's case of the classical central limit theorem with $p = 2$, $r_{2,2}(n) = n^{1/2}$ and $e_{2,2}(n) = 0$ in (1.7). If $0 < \alpha < 2$, then EX^2 does not exist, and seeking the appropriate normalizing and centralizing sequences $r_{\alpha,p}(n)$, $e_{\alpha,p}(n)$ for the by $T_{\alpha,p}^{1/p}(n)$ divided sequence of partial sums $S_\alpha(n)$ was considered in the literature by Darling [2], Efron [4] and Hotelling [6]. Logan et al. [8] studied the limiting behaviour of the rv in (1.7) directly when $r_{\alpha,p}(n) = 1$ and $e_{\alpha,p}(n) = 0$, and obtain descriptions of asymptotic densities in the latter cases. LePage et al. [7] obtained representations of the limiting rv in (4), again with $r_{\alpha,p}(n) = 1$ and $e_{\alpha,p}(n) = 0$. As compared to the way we treat the problem of the asymptotic behaviour of the rv of (4) here, Logan et al. [8], and also LePage et al. [7], did not have the easily handled forms of $a_\alpha(n)$, $b_\alpha(n)$, $c_{\alpha,p}(n)$ and $d_{\alpha,p}(n)$ as in our Theorem 1. On the basis of the latter theorem we will be able to give asymptotic representations of the rv of (1.7) in their full generality. The special cases of our asymptotic representations do not reduce algebraically to the same form as those of LePage et al. [7] in their cases.

THEOREM 2. *Assume that X is in the domain of attraction of a stable law with index $\alpha \in (0, 2)$, and $p > 0$ is such that $0 < \alpha/p < 2$. As $n \to \infty$, we have:*

(i) *if* $0 < \alpha < 1$ *and* $0 < \alpha/p < 1$, *then*

$$\frac{S_\alpha(n)}{T_{\alpha,p}^{1/p}(n)} \xrightarrow{\mathcal{D}} \frac{-w_1 \Delta_{\alpha,1} + w_2 \Delta_{\alpha,2}}{(w_1^p \Delta_{\alpha/p,1} + w_2^p \Delta_{\alpha/p,2})^{1/p}};$$

(ii) *if* $0 < \alpha < 1$ *and* $1 \leqslant \alpha/p < 2$, *then*

$$\left(c_{\alpha,p}(n)\, d_{\alpha,p}(n)\right)^{1/p} \frac{S_\alpha(n)}{T_{\alpha,p}^{1/p}(n)} \xrightarrow{\mathcal{D}} -w_1 \Delta_{\alpha,1} + w_2 \Delta_{\alpha,2};$$

(iii) *if* $\alpha = 1$, $0 < 1/p < 1$ *and*

(a) $w_1 \neq 1/2$ *(asymmetric), then*

$$\frac{1}{a_1(n)\, b_1(n)} \frac{S_1(n)}{T_{1,p}^{1/p}(n)} \xrightarrow{\mathcal{D}} \frac{1}{(w_1^p \Delta_{1/p,1} + w_2^p \Delta_{1/p,1})^{1/p}},$$

(b) $w_1 = w_2 = 1/2$ *(quasi symmetric),*

(b1) $\lim\limits_{n\to\infty} |a_1(n)\, b_1(n)| = \infty$, *then*

$$\frac{1}{a_1(n)\, b_1(n)} \frac{S_1(n)}{T_{1,p}^{1/p}(n)} \xrightarrow{\mathcal{D}} \frac{1}{((1/2)^p \Delta_{1/p,1} + (1/2)^p \Delta_{1/p,2})^{1/p}},$$

(b2) $\lim\limits_{n\to\infty} a_1(n)\, b_1(n) = c_0$, *then*

$$\frac{S_1(n)}{T_{1,p}^{1/p}(n)} \xrightarrow{\mathcal{D}} \frac{-(1/2)\Delta_{1,1} + (1/2)\Delta_{1,2} + c_0}{((1/2)^p \Delta_{1/p,1} + (1/2)^p \Delta_{1/p,2})^{1/p}};$$

(iv) *if* $\alpha = p = 1$, *then*

$$c_{1,1}(n)\, d_{1,1}(n) \left\{ \frac{S_1(n)}{T_{1,1}(n)} - \frac{b_1(n)}{d_{1,1}(n)} \right\} \xrightarrow{\mathcal{D}} -w_1 \Delta_{1,1} + w_2 \Delta_{1,2} - c_0 (w_1 \Delta_{1,1} + w_2 \Delta_{1,2}),$$

where

$$c_0 = \begin{cases} \mathrm{E}X/\mathrm{E}|X|, & \text{if } \mathrm{E}|X| < \infty, \\ -w_1 + w_2, & \text{if } \mathrm{E}|X| = \infty; \end{cases}$$

(v) *if* $\alpha = 1$ *and* $1 < 1/p < 2$, *then*

$$\left(c_{1,p}(n)\,d_{1,p}(n)\right)^{1/p}\left\{\frac{S_1(n)}{T_{1,p}^{1/p}(n)}-\frac{b_1(n)}{d_{1,p}^{1/p}(n)}\right\}\xrightarrow{\mathscr{D}} -w_1\,\Delta_{1,1}+w_2\,\Delta_{1,2};$$

(vi) *if* $1 < \alpha < 2$, $0 < \alpha/p < 1$ *and* $\mathrm{E}X = 0$, *then*

$$\frac{S_\alpha(n)}{T_{\alpha,p}^{1/p}(n)}\xrightarrow{\mathscr{D}}\frac{-w_1\,\Delta_{\alpha,1}+w_2\,\Delta_{\alpha,2}}{(w_1^p\,\Delta_{\alpha/p,1}+w_2^p\,\Delta_{\alpha/p,2})^{1/p}};$$

(vii) *if* $1 < \alpha < 2$, $1 \leqslant \alpha/p < 2$ *and* $\mathrm{E}X = 0$, *then*

$$\left(c_{\alpha,p}(n)\,d_{\alpha,p}(n)\right)^{1/p}\frac{S_\alpha(n)}{T_{\alpha,p}^{1/p}(n)}\xrightarrow{\mathscr{D}} -w_1\,\Delta_{\alpha,1}+w_2\,\Delta_{\alpha,2};$$

(viii) *if* $1 < \alpha < 2$, $0 < \alpha/p < 1$ *and* $\mathrm{E}X \neq 0$, *then*,

$$\frac{1}{a_\alpha(n)\,b_\alpha(n)}\frac{S_\alpha(n)}{T_{\alpha,p}^{1/p}(n)}\xrightarrow{\mathscr{D}}\frac{1}{(w_1^p\,\Delta_{\alpha/p,1}+w_2^p\,\Delta_{\alpha/p,2})^{1/p}};$$

(ix) *if* $1 < \alpha < 2$, $p = \alpha$ *and* $\mathrm{E}X \neq 0$

$$\frac{\left(c_{\alpha,\alpha}(n)\,d_{\alpha,\alpha}(n)\right)^{1+1/\alpha}}{a_\alpha(n)\,b_\alpha(n)}\left\{\frac{S_\alpha(n)}{T_{\alpha,\alpha}^{1/\alpha}(n)}-\frac{b_\alpha(n)}{d_{\alpha,\alpha}^{1/\alpha}(n)}\right\}\xrightarrow{\mathscr{D}} -\frac{1}{\alpha}(w_1^\alpha\,\Delta_{1,1}+w_2^\alpha\,\Delta_{1,2});$$

(x) *if* $1 < \alpha < 2$, $1 < \alpha/p < 2$, $\mathrm{E}X \neq 0$ *and*
(a) $p < 1$, *then*

$$\left(c_{\alpha,p}(n)\,d_{\alpha,p}(n)\right)^{1/p}\left\{\frac{S_\alpha(n)}{T_{\alpha,p}^{1/p}(n)}-n^{1-1/p}\frac{\mathrm{E}X}{(\mathrm{E}\,|X|^p)^{1/p}}\right\}\xrightarrow{\mathscr{D}} -w_1\,\Delta_{\alpha,1}+w_2\,\Delta_{\alpha,2},$$

(b) $p = 1$, *then*

$$c_{\alpha,1}(n)\,d_{\alpha,1}(n)\left\{\frac{S_\alpha(n)}{T_{\alpha,1}(n)}-\frac{\mathrm{E}X}{\mathrm{E}\,|X|}\right\}\to -w_1\,\Delta_{\alpha,1}+w_2\,\Delta_{\alpha,2}-\frac{\mathrm{E}X}{\mathrm{E}\,|X|}(w_1\,\Delta_{\alpha,1}+w_2\,\Delta_{\alpha,2}),$$

(c) $p > 1$, *then*

$$\frac{\left(c_{\alpha,p}(n)\,d_{\alpha,p}(n)\right)^{1+1/p}}{a_\alpha(n)\,b_\alpha(n)}\left\{\frac{S_\alpha(n)}{T_{\alpha,p}^{1/p}(n)}-n^{1-1/p}\frac{\mathrm{E}X}{(\mathrm{E}\,|X|^p)^{1/p}}\right\}$$

$$\to -\frac{1}{p}(w_1^p\,\Delta_{\alpha/p,1}+w_2^p\,\Delta_{\alpha/p,2}).$$

Remark 1. Logan et al. [8] studied the cases (i) and (vi) and obtained the Fourier transforms of the appropriate limit distributions. For a number of parameter values they have numerically inverted these Fourier transforms and produced graphs for the densities. LePage et al. [7] studied the same two cases and obtained asymptotic representations in terms of an infinite sequence of i.i.d. exponential rv and i.i.d. random signs. Our approach covers all possible cases and, an account of our asymptotic representations being in terms of two independent Poisson processes, in principle their distribution and density functions are also easily computable.

Remark 2. The case (iii) (b) has two subcases. We illustrate here that they are not empty. First for (b1) we consider

$$Q_1(1-u) = \frac{1}{u}\left(1-\frac{1}{\log u}\right), \quad u \geqslant u_0,$$

and

$$Q_2(1-u) = \frac{1}{u}\left(1+\frac{1}{\log u}\right), \quad u \geqslant u_0,$$

and conclude our claim. As to (b2), if F is symmetric around zero, then $Q_1 = Q_2$ and $b_1(n) = 0$ for all n, and $c_0 = 0$. Again, in case of (b2) with

$$Q_1(1-u) = \frac{1}{u}\left(1-\frac{1}{(\log u)^2}\right), \quad u \geqslant u_0,$$

and

$$Q_2(1-u) = \frac{1}{u}\left(1+\frac{1}{(\log u)^2}\right), \quad u \geqslant u_0,$$

we get a finite positive c_0.

2. Proofs. First a lemma for later use.

LEMMA. *Let $L(x) > 0$, slowly varying at zero. Then*

$$\lim_{x\to 0} \frac{1}{L(x)} \int_x^1 \frac{L(u)}{u}\, du = \infty.$$

Proof. Special case of Theorem 1.2.1 (a) of De Haan [3].

Proof of Theorem 2. (i) In this case we have

$$\frac{S_\alpha(n)}{T_{\alpha,p}^{1/p}(n)} = \frac{a_\alpha(n) S_\alpha(n)}{\left(c_{\alpha,p}(n)\, T_{\alpha,p}(n)\right)^{1/p}},$$

and Theorem 1 implies the statement.

(ii) In this case we have

$$\left(c_{\alpha,p}(n)\, d_{\alpha,p}(n)\right)^{1/p} \frac{S_\alpha(n)}{T_{\alpha,p}^{1/p}(n)}$$
$$= \frac{a_\alpha(n) S_\alpha(n)}{\left\{\frac{1}{c_{\alpha,p}(n)\, d_{\alpha,p}(n)} \left[c_{\alpha,p}(n)\left(T_{\alpha,p}(n) - d_{\alpha,p}(n)\right)\right] + 1\right\}^{1/p}}.$$

If $\alpha = p$, then

$$c_{\alpha,\alpha}(n)\, d_{\alpha,\alpha}(n) = \frac{1}{L^{\alpha}(1/n)} \int_{1/n}^{1} \left(Q_1^{\alpha}(1-u) + Q_2^{\alpha}(1-u)\right) du \to \infty$$

by (1.2), (1.3), (1.4) and Lemma. Otherwise, immediately

$$c_{\alpha,p}(n)\, d_{\alpha,p}(n) = \frac{n^{-p/\alpha}}{L^{p}(1/n)} n \mathrm{E}\,|X|^{p} = \frac{\mathrm{E}\,|X|^{p}\, n^{1-p/\alpha}}{L^{p}(1/n)} \to \infty .$$

The proof of this case is now complete.

(iii) For all the subcases we have

$$\frac{S_1(n)}{T_{1,p}^{1/p}(n)} = \frac{a_1(n)(S_1(n) - b_1(n))}{(c_{1,p}(n)\, T_{1,p}(n))^{1/p}} + \frac{a_1(n)\, b_1(n)}{(c_{1,p}(n)\, T_{1,p}(n))^{1/p}}. \tag{2.1}$$

In case of (a) we show that

$$\lim_{n\to\infty} |a_1(n)\, b_1(n)| = \infty . \tag{2.2}$$

By (1.2), (1.3), (1.4) we have

$$|a_1(n)\, b_1(n)| = \frac{1}{L(1/n)} \left| - \int_{1/n}^{1} Q_1(1-u) + \int_{1/n}^{1} Q_2(1-u)\, du \right| > K \frac{1}{L(1/n)} \int_{1/n}^{1} \frac{L(u)}{u} du$$

with a suitably chosen $K > 0$. Hence, by Lemma, we have (2.2) and (a) is proven. In case of (b1), (2.2) is assumed. Hence this follows by Theorem 1 combined with (2.1). As to (b2), it again follows by (2.1) and Theorem 1.

(iv) We have, upon keeping in mind that $c_{1,1}(n) = a_1(n)$ and using the latter instead,

$$\begin{aligned} &a_1(n)\, d_{1,1}(n) \left\{ \frac{a_1(n)(S_1(n) - b_1(n)) + a_1(n)\, b_1(n)}{a_1(n)(T_{1,1}(n) - d_{1,1}(n)) + a_1(n)\, d_{1,1}(n)} - \frac{a_1(n)\, b_1(n)}{a_1(n)\, d_{1,1}(n)} \right\} \\ &= \frac{a_1(n)(S_1(n) - b_1(n)) - a_1(n)(T_{1,1}(n) - d_{1,1}(n)) \dfrac{b_1(n)}{d_{1,1}(n)}}{1 + \left(a_1(n)(T_{1,1}(n) - d_{1,1}(n))\right) / \left(a_1(n)\, d_{1,1}(n)\right)} . \end{aligned} \tag{2.3}$$

By Lemma we have $a_1(n)\, d_{1,1}(n) \to \infty$. Hence and by Theorem 1 the denominator of the latter fraction tends to 1 in probability. If $\mathrm{E}\,|X| < \infty$, then

$$\frac{b_1(n)}{d_{1,1}(n)} \to \frac{-\mathrm{E}X^{-} + \mathrm{E}X^{+}}{\mathrm{E}\,|X|} = \frac{\mathrm{E}X}{\mathrm{E}\,|X|},$$

where $X^{-} = -(X \wedge 0)$, $X^{+} = X \vee 0$. If $\mathrm{E}\,|X| = \infty$, then at least one of $\mathrm{E}X^{-}$ and $\mathrm{E}X^{+}$ is also infinite. Hence by definition of $b_1(n)$ and that of $d_{1,1}(n)$, and applying L'Hospital's rule, we get

$$\frac{b_1(n)}{d_{1,1}(n)} \to \frac{-w_1+w_2}{w_1+w_2} = -w_1+w_2$$

by Proposition. Hence by (2.3) and Theorem 1 we obtain the result.

(v) We have

$$(c_{1,p}(n)\,d_{1,p}(n))^{1/p}\left(\frac{S_1(n)}{T_{1,p}^{1/p}(n)}-\frac{b_1(n)}{d_{1,p}^{1/p}(n)}\right)$$

$$=\frac{a_1(n)(S_1(n)-b_1(n))+a_1(n)\,b_1(n)\left\{1-\left(\dfrac{c_{1,p}(n)(T_{1,p}(n)-d_{1,p}(n))}{c_{1,p}(n)\,d_{1,p}(n)}+1\right)^{1/p}\right\}}{\left(\dfrac{c_{1,p}(n)(T_{1,p}(n)-d_{1,p}(n))}{c_{1,p}(n)\,d_{1,p}(n)}+1\right)^{1/p}}.$$

By definition again $c_{1,p}(n)\,d_{1,p}(n)\to\infty$, and hence by Theorem 1 the denominator of the latter fraction tends to 1 in probability. Now by the mean value theorem

$$1^{1/p}-(1+x)^{1/p} = -(1/p)\,x+o(x), \qquad x\to 0. \tag{2.4}$$

Consequently

$$\left|a_1(n)\,b_1(n)\left\{1-\left(\frac{c_{1,p}(n)(T_{1,p}(n)-d_{1,p}(n))}{c_{1,p}(n)\,d_{1,p}(n)}+1\right)^{1/p}\right\}\right|$$

$$= O_P\left(\frac{a_1(n)\,b_1(n)}{c_{1,p}(n)\,d_{1,p}(n)}\right) = o_P(1),$$

on applying also Theorem 1 and definitions of $a_1(n)$, $b_1(n)$, $c_{1,p}(n)$ and $d_{1,p}(n)$.

(vi) In this case $b_\alpha(n)=0$, and Theorem 1 immediately implies this result.

(vii) Here $c_{\alpha,p}(n)\,d_{\alpha,p}(n)\to\infty$ by Lemma if $\alpha=p$, and by definitions otherwise. In both cases

$$\left(\frac{c_{\alpha,p}(n)\,T_{\alpha,p}(n)}{c_{\alpha,p}(n)\,d_{\alpha,p}(n)}\right)^{1/p} \xrightarrow{P} 1$$

and

$$(c_{\alpha,p}(n))^{1/p}\,S_\alpha(n) = a_\alpha(n)\,S_\alpha(n) \xrightarrow{\mathscr{D}} -w_1\,\Delta_{\alpha,1}+w_2\,\Delta_{\alpha,2}$$

by Theorem 1.

(viii) We have

$$(2.5)\quad \frac{1}{a_\alpha(n)b_\alpha(n)}\frac{S_\alpha(n)}{T_{\alpha,p}^{1/p}(n)} = \frac{1}{a_\alpha(n)b_\alpha(n)}\frac{a_\alpha(n)(S_\alpha(n)-b_\alpha(n))}{(c_{\alpha,p}(n)\,T_{\alpha,p}(n))^{1/p}}+\frac{1}{(c_{\alpha,p}(n)\,T_{\alpha,p}(n))^{1/p}}.$$

Again, $|a_\alpha(n)b_\alpha(n)|\to\infty$. Hence by Theorem 1 the first right-hand member of (2.5) goes to 0 in probability while the second one yields the result.

As a preliminary step to the proofs of (ix) and (x) we need

$$(2.6)\quad \left\{\frac{S_\alpha(n)}{T_{\alpha,p}^{1/p}(n)}-\frac{a_\alpha(n)b_\alpha(n)}{(c_{\alpha,p}(n)d_{\alpha,p}(n))^{1/p}}\right\} = (c_{\alpha,p}(n)d_{\alpha,p}(n))^{-2/p}\left\{\frac{(c_{\alpha,p}(n)d_{\alpha,p}(n))^{1/p}[a_\alpha(n)(S_\alpha(n)-b_\alpha(n))]}{1+o_p(1)}+\frac{a_\alpha(n)b_\alpha(n)(c_{\alpha,p}(n)d_{\alpha,p}(n))^{1/p}\left[1-\left(\frac{c_{\alpha,p}(n)(T_{\alpha,p}(n)-d_{\alpha,p}(n))}{c_{\alpha,p}(n)d_{\alpha,p}(n)}+1\right)^{1/p}\right]}{1+o_P(1)}\right\}.$$

Also in these two cases we have

$$(2.7)\quad c_{\alpha,p}(n)d_{\alpha,p}(n)\to\infty.$$

(ix) We have

$$\frac{(c_{\alpha,\alpha}(n)d_{\alpha,\alpha}(n))^{1/\alpha}}{a_\alpha(n)|b_\alpha(n)|(c_{\alpha,\alpha}(n)d_{\alpha,\alpha}(n))^{1/\alpha-1}} = \frac{\left(\int_{1/n}^{1}((Q_1(1-u))^\alpha+(Q_2(1-u))^\alpha)du/L^\alpha(1/n)\right)^{1/\alpha}}{\frac{n^{1-1/\alpha}}{L(1/n)}|\mathrm{E}X|\left(\int_{1/n}^{1}((Q_1(1-u))^\alpha+(Q_2(1-u))^\alpha)du/L^\alpha(1/n)\right)^{1/\alpha-1}}\to 0.$$

Hence proof of (ix) is now complete by (2.6), (2.7) and Theorem 1 combined.

(x) In case of (a) elementary calculations show that

$$(2.8)\quad \frac{a_\alpha(n)b_\alpha(n)(c_{\alpha,p}(n)d_{\alpha,p}(n))^{1/p-1}}{((c_{\alpha,p}(n)d_{\alpha,p}(n))^{1/p}} = \frac{n^{1-1/\alpha}\mathrm{E}X/L(1/n)}{n^{1-p/\alpha}\mathrm{E}|X|^p/L^p(1/n)}\to 0,$$

and hence, by (2.6), (2.7) and Theorem 1 combined, we have proven this case. As to (b), we note that $a_\alpha(n)=c_{\alpha,1}(n)$, $b_\alpha(n)=n\mathrm{E}X$ and $d_{\alpha,1}(n)=n\mathrm{E}|X|$. Hence, by (2.6), (2.7) and Theorem 1, we have this case too. In case of (c) elementary calculations show that (2.8) this time tends to infinity. Hence our statement follows by combining (2.4), (2.6), (2.7) and Theorem 1.

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