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WIENER PROCESSES WITH VALUES IN p-HOMOGENEOUS FRÉCHET SPACES

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Abstract. It is known that Wiener processes taking values in separable Banach spaces can be expanded into series of independent real Brownian processes. This property is very useful in many instances, e.g., in the proof of the law of the iterated logarithm. Known proofs of this theorem are based on the usual convex technique of normed spaces and cannot be adapted for more general situations. In our paper we present a different approach, based on properties of unconditional convergence of double series in vector spaces. This technique allows to extend the theorem to p-homogeneous Fréchet spaces.

1. Preliminaries and notations. Let $(\mathscr{X}, |\cdot|)$ be a Fréchet space, i.e. a complete separable real metric vector space with topology generated by a subadditive F-norm $|\cdot|$ having the following properties: $|x| = 0$ if and only if $x = 0$; $|ax| = |x|$ for every $a \in \boldsymbol{R}$, $|a| = 1$, $x \in \mathscr{X}$; $|\lambda_n x_n - \lambda x| \to 0$ for $|x_n - x| \to 0$, $\lambda_n \to \lambda$, where $x_n, x \in \mathscr{X}$, $\lambda_n, \lambda \in \boldsymbol{R}$. We say, that F-norm $|\cdot|$ is *p-homogeneous*, $0 < p \leqslant 1$, if $|\lambda x| = |\lambda|^p |x|$ for every $x \in \mathscr{X}$, $\lambda \in \boldsymbol{R}$.

An $\mathscr{X}$-valued random vector X will be called *symmetric Gaussian* (in the sense of Fernique [3]) if for every pair X_1, X_2 of independent random vectors having the same distributions as X and, for every pair of real numbers a, b such that $a^2+b^2 = 1$, the random vectors aX_1+bX_2 and bX_1-aX_2 are independent and have the distribution of X.

We will say that a Gaussian random vector X has an *orthogonal expansion* in $\mathscr{X}$ if

$$X = \sum_{n=1}^{\infty} a_n \lambda_n,$$

where $\{\lambda_n\}$ is a sequence of independent real Gaussian r.v.'s with mean 0 and variance 1, a_n are suitable elements of $\mathscr{X}$ and the above series converges a.s. in $\mathscr{X}$. Up to now it has been proved that such an expansion holds for

Gaussian random vectors taking values in Banach spaces [6] or in Orlicz spaces L_{Φ} [1].

Consider now an $\mathscr{X}$-valued symmetric Gaussian vector X. A homogeneous stochastic process $\{W(t)\colon 0 \leqslant t \leqslant 1\}$ with independent increments and with continuous $\mathscr{X}$-valued sample paths will be called the *Wiener process generated by* X if $W(t)$ has the same distribution as $t^{1/2} X$ for $0 \leqslant t \leqslant 1$. Such a process exists by [2].

The purpose of this paper is to prove the following result:

THEOREM. *Let $\mathscr{X}$ be a separable Fréchet space with p-homogeneous F-norm $|\cdot|$, $0 < p \leqslant 1$. Let X be an $\mathscr{X}$-valued symmetric Gaussian random vector having an orthogonal expansion $\sum_{n=1}^{\infty} a_n \lambda_n$, where $a_n \in \mathscr{X}$, $n \in N$, and $\{\lambda_n\}$ is the standard Gaussian sequence. Then there exists a sequence $\{B_n(t)\colon 0 \leqslant t \leqslant 1\}$, $n \in N$, of independent real Brownian motions such that the series $\sum_{n=1}^{\infty} a_n B_n(t)$ converges with probability* 1 *uniformly with respect to t to a Wiener process generated by X.*

2. Unconditional convergence of double series. The main tool in the proof of our theorem is the application of some properties concerning the unconditional convergence of double series in Fréchet spaces. These properties seem to be also of independent interest.

Write $N_0 = \{H \subset N\colon \text{card } H < \infty\}$, $\mathscr{N} = N \times N$, $\mathscr{N}_0 = \{F \subset \mathscr{N}\colon \text{card } F < \infty\}$, where N denotes the set of natural numbers. It is known [10] that for single series $\sum_{n=1}^{\infty} x_n$ of elements of $\mathscr{X}$ the following conditions are equivalent:

(i) $\sum_{n=1}^{\infty} x_n$ is unconditionally convergent, i.e., for every permutation π of N, the series $\sum_{n=1}^{\infty} x_{\pi(n)}$ is convergent.

(ii) $\sum_{n=1}^{\infty} x_n$ is subseries convergent, i.e. for every sequence $\{\delta_n;\ n \in N\}$ such that $\delta_n = 0$ or $\delta_n = 1$ the series $\sum_{n=1}^{\infty} \delta_n x_n$ is convergent.

(iii) $\sum_{n=1}^{\infty} x_n$ is unordered convergent, i.e. there exists an element $x \in \mathscr{X}$ such that for every $\varepsilon > 0$ we can find such a set $H \in N_0$ that, for every $H_1 \in N_0$, $H_1 \supset H$, we have $|x - \sum_{n \in H_1} x_n| < \varepsilon$.

(iv) For every sequence $\{\varepsilon_n\colon n \in N\}$ such that $\varepsilon_n = 1$ or $\varepsilon_n = -1$ the series $\sum_{n=1}^{\infty} \varepsilon_n x_n$ is convergent.

Now we define similar type of convergence for double series. This is a particular case of usual summability of countable families.

Definition. A double series $\sum_{\mathcal{N}} x_{ij}$ of elements of $\mathcal{X}$ is called *unconditionally convergent* if, for every $\varepsilon > 0$, there exists a set $F \in \mathcal{N}_0$ such that for every $F_1 \in \mathcal{N}_0$, $F_1 \cap F = \emptyset$ we have $|\sum_{F_1} x_{ij}| < \varepsilon$.

By the completness of $\mathcal{X}$ it follows that if the series $\sum_{\mathcal{N}} x_{ij}$ converges unconditionally, then there exists an $x \in \mathcal{X}$ such that $\lim_{F \in \mathcal{N}_0} \sum_F x_{ij} = x$.

LEMMA. *Let $\{x_{ij}, i, j \in N\}$, be a double sequence of elements of $\mathcal{X}$. Then the following conditions are equivalent*:

(i) *for every sequence $\{\varepsilon_{ij}: i, j \in N\}$, $\varepsilon_{ij} = 1$ or $\varepsilon_{ij} = -1$, $i, j \in N$, there exists the iterated limit*:

$$\lim_{I \to \infty} \lim_{J \to \infty} \sum_{i=1}^{I} \sum_{j=1}^{J} \varepsilon_{ij} x_{ij};$$

(ii) *for every sequence $\{\delta_{ij}: i, j \in N\}$, $\delta_{ij} = 1$ or $\delta_{ij} = 0$, $i, j \in N$, there exists the iterated limit*:

$$\lim_{I \to \infty} \lim_{J \to \infty} \sum_{i=1}^{I} \sum_{j=1}^{J} \delta_{ij} x_{ij};$$

(iii) *the series $\sum_{\mathcal{N}} x_{ij}$ is unconditionally convergent*;

(iv) *for every sequence $\{\varepsilon_{ij}: i, j \in N\}$, $\varepsilon_{ij} = 1$ or $\varepsilon_{ij} = -1$, $i, j \in N$, the series $\sum_{\mathcal{N}} \varepsilon_{ij} x_{ij}$ is unconditionally convergent.*

Proof. Proofs of implications (i)$\Rightarrow$(ii), (iii)$\Rightarrow$(iv) and (iv)$\Rightarrow$(i) are standard (see [10], p. 458) and are omitted. We only prove that (ii) implies (iii). Suppose, to the contrary, that the series $\sum_{\mathcal{N}} x_{ij}$ is not unconditionally convergent. Write $N_k = \{1, \ldots, k\}$, $N_k^c = N \setminus N_k$. Note that it suffices to find an $\varepsilon > 0$ such that, for every k, there is a $F_k \subset (N_k^c \times N) \cap \mathcal{N}_0$ with $|\sum_{F_k} x_{ij}| > \varepsilon$. Then, choosing k_n such that $k_n > i$ whenever $(i, j) \in F_{k_{n-1}}$ and taking $\delta_{ij} = 1$ for $(i, j) \in \bigcup_{n=1}^{\infty} F_{k_n}$ and $\delta_{ij} = 0$ otherwise, we obtain a contradiction with (ii).

However, if (iii) fails, then there exists an $\varepsilon > 0$ such that, for every i, $j \in N$, we can find such a set $G_{ij} \in \mathcal{N}_0$, disjoint with $N_i \times N_j$, that $|\sum_{G_{ij}} x_{lk}| > 2\varepsilon$. Let k be arbitrary. By (ii) there exists a j_k such that, for every $F \subset (N_k \times N_{j_k}^c) \cap \mathcal{N}_0$, $|\sum_F x_{ij}| < \varepsilon$. Putting $F_k = G_{kj_k} \cap (N_k^c \times N)$, we obtain

$$\Big|\sum_{(i,j) \in F_k} x_{ij}\Big| \geqslant \Big|\sum_{(i,j) \in G_{kj_k}} x_{ij}\Big| - \Big|\sum_{(i,j) \in G_{kj_k} \cap (N_k \times N)} x_{ij}\Big| > \varepsilon,$$

which completes the proof.

As an easy consequence we obtain

COROLLARY 1. *If* $\lim_{F\in\mathcal{N}_0}\sum_F x_{ij} = x$, *then*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} x_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} x_{ij} = x.$$

3. Random series in vector spaces.

COROLLARY 2. *Let $\mathcal{X}$ be a separable Fréchet space and let $\{Y_{ij}: i, j\in N\}$ be a sequence of independent symmetric random vectors with values in $\mathcal{X}$. If the iterated series $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} Y_{ij}$ converges a.s. in $\mathcal{X}$ to some random vector Y, then the series $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} Y_{ij}$ also converges a.s. in $\mathcal{X}$ to Y.*

Proof. By $L_0^{\mathcal{X}}$ we denote the space of all measurable $\mathcal{X}$-valued functions defined on $(\Omega, \mathcal{B}_0, P)$, where $\mathcal{B}_0$ is the σ-field generated by Y_{ij}. $L_0^{\mathcal{X}}$ is a separable Fréchet space with convergence in probability [9]. By independence and symmetry of Y_{ij} it follows that for every scalar sequence $\{\varepsilon_{ij}\}$, $\varepsilon_{ij} = 1$ or $\varepsilon_{ij} = -1$, distributions of partial sums of sequences $\{\varepsilon_{ij} Y_{ij}\}$ and $\{Y_{ij}\}$ are identical. Hence the series $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \varepsilon_{ij} Y_{ij}$ converges in $L_0^{\mathcal{X}}$. Our conclusion follows now by Lemma and Corollary 1.

Let now $\{\varphi_j: j\in N\}$ denote the sequence of Schauder functions which form an orthonormal basis in the reproducing kernel Hilbert space of a real Brownian motion [4],

$$\varphi_j(t) = \int_0^t H_j(s)\,ds,$$

where $\{H_j: j\in N\}$ is the sequence of Haar functions [8]. Consider an orthogonal expansion $\sum_{n=1}^{\infty} a_n\lambda_n$ of X. On the product space $(\Omega^\infty, \Sigma^\infty, P^\infty)$ we define $\lambda_n^{(j)}(\omega) = \lambda_n(\omega_j)$, where $\omega = (\omega_1, \omega_2, \ldots)\in\Omega^\infty$. Then the sequences $\{\lambda_n^{(j)}: n\in N\}$ are standard normal and independent. Since $X = \sum_{n=1}^{\infty} a_n\lambda_n$ a.s., then $X_j = \sum_{n=1}^{\infty} a_n\lambda_n^{(j)}$ form the sequence of independent random vectors with distribution $\mathcal{L}(X)$.

PROPOSITION. *Let $\mathcal{X}$ be a separable Fréchet space with p-homogeneous F-norm, $0 < p \leqslant 1$. The series $\sum_{j=1}^{\infty} X_j\varphi_j(t)$ is convergent a.s. in $\mathcal{X}$ for all t, the convergence being uniform with respect to $t\in[0, 1]$.*

Proof. We use here the classical idea of Ciesielski's construction of real Brownian motion [8]. Since supports of the functions φ_j are disjoint for $2^n \leqslant j < 2^{n+1}$, we have

$$|\sum_{j=1}^{\infty} X_j(\omega)\varphi_j(t)| \leqslant \sum_{n=0}^{\infty} |\sum_{j=2^n}^{2^{n+1}} \frac{X_j(\omega)}{2^{n/2}\cdot 4} 2^{n/2}\cdot 4\varphi_j(t)| \leqslant \sum_{n=0}^{\infty} \max_{2^n \leqslant j < 2^{n+1}} |\frac{X_j(\omega)}{2^{n/2}}|.$$

Write $A_j = \{|X_j| > j^{p/4}\}$. Then, by a version of the exponential integrability of Gaussian pseudonorms [5], we obtain

$$\sum_{j=1}^{\infty} P^{\infty}(A_j) \leqslant \sum_{j=1}^{\infty} \frac{E\exp(\alpha |X|)}{\exp(\alpha j^{p/4})} = C\sum_{j=1}^{\infty} \exp(-\alpha j^{p/4}) < \infty.$$

By Borel–Cantelli lemma we get $P^{\infty}\{|X_j| < j^{p/4}$ for all j large enough$\} = 1$. Then, for almost all $\omega \in \Omega^{\infty}$, we can find $J(\omega) \in N$ such that, for $j > J(\omega)$, $|X_j(\omega)| \leqslant j^{p/4}$. Hence and by p-homogeneity of F-norm we obtain

$$|\sum_{j=1}^{\infty} X_j(\omega)\varphi_j(t)| \leqslant \sum_{n=1}^{J(\omega)} \max_{2^n \leqslant j < 2^{n+1}} |\frac{X_j(\omega)}{2^{n/2}}| + \sum_{n > J(\omega)} \frac{2^{(n+1)p/4}}{2^{pn/2}} < \infty$$

and this estimate is independent of t, which ends the proof.

Proof of the Theorem. Denote by $\mathscr{C}^{\mathfrak{X}}$ the space of $\mathfrak{X}$-valued continuous functions defined for $0 \leqslant t \leqslant 1$ and vanishing at zero. Let

$$\|f\| = \sup_{0 \leqslant t \leqslant 1} |f(t)| \quad \text{for } f \in \mathscr{C}^{\mathfrak{X}}.$$

Then $(\mathscr{C}^{\mathfrak{X}}, \|\cdot\|)$ is a separable Fréchet space. By Proposition and by the definition of X_j it follows that the iterated series $\sum_{j=1}^{\infty}\sum_{n=1}^{\infty} Y_{nj}$ is convergent a.s. in $\mathscr{C}^{\mathfrak{X}}$, where $Y_{nj} = a_n \varphi_j \lambda_n^{(j)}$. By Corollary 2, the series

$$\sum_{n=1}^{\infty} a_n \Big(\sum_{j=1}^{\infty} \lambda_n^{(j)} \varphi_j\Big)$$

converges a.s. in $\mathscr{C}^{\mathfrak{X}}$ to the same limit $W = W(t, \omega)$. Since, by [8], for every standard normal sequence $\{\lambda^{(j)}: j \in N\}$ the series $\sum_{j=1}^{\infty} \lambda^{(j)} \varphi_j(t)$ converges a.s. to a Brownian motion $\{B(t): 0 \leqslant t \leqslant 1\}$, we have

$$W(t) = \sum_{n=1}^{\infty} a_n B_n(t), \qquad 0 \leqslant t \leqslant 1,$$

where $\{B_n: n \in N\}$ is the sequence of independent Brownian motions and the series is convergent uniformly a.s. with respect to t. It is easy to see that the process $\{W(t): 0 \leqslant t \leqslant 1\}$ is homogeneous with independent increments and with continuous sample paths. Moreover,

$$\mathscr{L}(W(t)) = \mathscr{L}(\sum_{n=1}^{\infty} a_n B_n(t)) = \mathscr{L}(t^{1/2} \sum_{n=1}^{\infty} a_n \lambda_n) = \mathscr{L}(t^{1/2} X).$$

Thus $\{W(t)\colon 0 \leqslant t \leqslant 1\}$ is an $\mathscr{X}$-valued Wiener process generated by X. This completes the proof.

Remark. The p-homogeneity of the space $\mathscr{X}$ is needed only to prove our Proposition. It is easy to see that Proposition would be true in the whole generality, when it were possible to find a sequence $\{c_n\}$ satisfying $\sum 2^n P\{|X \cdot 2^{-n/2}| > c_n\} < \infty$ and $\sum c_n < \infty$. At present, we do not know in which spaces such sequences can be found.

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