PROBABILITY
AND
MATHEMATICAL STATISTICS

Vol. 10, Fasc. 1 (1989), p. 119–135

WEAK CONVERGENCE TO THE BROWNIAN MOTION OF THE PARTIAL SUMS OF INFIMA OF INDEPENDENT RANDOM VARIABLES

BY

H. HEBDA-GRABOWSKA (LUBLIN)

Abstract. Let $\{Y_n, n \geqslant 1\}$ be a sequence of independent, positive random variables, defined on a probability space $(\Omega, \mathscr{A}, P)$, with the common distribution function F.

Put $Y_m^* = \inf(Y_1, Y_2, \ldots, Y_m)$, $m \geqslant 1$, and

$$S_n = \sum_{m=1}^{n} Y_m^*, \quad n \geqslant 2, S_1 = 0.$$

The aim of this note is to give the rate of weak convergence of $\{S_n, n \geqslant 1\}$ to the Brownian motion. Moreover, the mixing limit theorem and the random functional limit theorem for the sums S_n, $n \geqslant 1$, are presented.

1. Introduction and results. Let $\{Y_n, n \geqslant 1\}$ be a sequence of independent, positive random variables with the common distribution function F, such that

$$(1) \qquad \int_0^1 \left|F(x) - \frac{x}{b}\right| x^{-2}\, dx < \infty \quad \text{for some } b\colon 0 < b < \infty.$$

Let us put $Y_m^* = \inf(Y_1, Y_2, \ldots, Y_m)$, $m \geqslant 1$ and write

$$S_n = \sum_{m=1}^{n} Y_m^*, \quad n \geqslant 2, S_1 = 0.$$

The convergence in probability, almost sure and in law, is established in [5]-[8] for sums S_n of infima of independent random variables uniformly distributed on [0, 1]. The almost sure invariance principle for them has been obtained in [9]. Weak convergence of sums and of random sums of infima of independent positive random variables with the common distribution function F was investigated in [11] and [10], respectively.

In this paper we examine the relation between the Wiener measure on the space $(C, \mathscr{B}_C)$ and the distribution of sums $\{S_n, n \geqslant 1\}$, where $C = C_{\langle 0,1 \rangle}$ is the space of continuous functions on [0, 1] with the metric

$$\varrho(x, y) = \sup_{t \in \langle 0,1 \rangle} |x(t) - y(t)|, \qquad x, y \in C,$$

$\mathscr{B}_C$ is the σ-field of Borel sets in C, and

$$S_n = \sum_{m=1}^{n} Y_m^*, \qquad n \geqslant 2, S_1 = 0.$$

Let $\mathscr{L}_C$ be the Lévy-Prohorov's distance defined as follows: Let, for $B \in \mathscr{B}_C$ and $\varepsilon > 0$,

$$G_\varepsilon(B) = \{x: \bigvee_{y \in B} \varrho(x, y) < \varepsilon\},$$

where ϱ is the metric on $C_{\langle 0,1 \rangle}$, and let P and Q be two measures on $(C, \mathscr{B}_C)$. Then we say that $\mathscr{L}_C(P, Q) < \varepsilon$ iff $P(B) \leqslant Q(G_\varepsilon(B)) + \varepsilon$ and $Q(B) \leqslant P(G_\varepsilon(B)) + \varepsilon$ for all $B \in \mathscr{B}_C$.

Now, let $\{Y_n, n \geqslant 1\}$ be a sequence of independent, positive random variables (i.p.r.vs.), with the common distribution function F, such that (1) holds. Let us define on $C_{\langle 0,1 \rangle}$ the random function $\{\tilde{Y}_n(t), t \in \langle 0, 1 \rangle\}$ as follows:

$$\tilde{Y}_n(0) = 0, \qquad n \geqslant 1, \tag{2}$$

$$\tilde{Y}_n(t) = \frac{S_k - b \log k}{b\sqrt{2 \log n}} + \frac{t - t_k}{t_{k+1} - t_k} \left(\frac{S_{k+1} - S_k - b \log \frac{k+1}{k}}{b\sqrt{2 \log n}} \right),$$

if $t \in \langle t_k, t_{k+1})$, where $t_k = \sigma_k / \sigma_{n-1}$, $1 \leqslant k \leqslant n-1$, $t_0 = 0$, and

$$\sigma_k = \sum_{m=1}^{k} \frac{1}{m}, k \geqslant 1, \qquad S_n = \sum_{m=1}^{n} Y_m^*, n \geqslant 2, S_1 = 0.$$

Now, we are going to prove the following

THEOREM 1. *Let P_n denote the distribution of $\{\tilde{Y}_n(t), t \in \langle 0, 1 \rangle\}$ in the space $(C, \mathscr{B}_C)$. Then*

$$\mathscr{L}_C(P_n, W) = O\big((\log n)^{-1/8}\big), \tag{3}$$

where W is the Wiener measure on $C_{\langle 0,1 \rangle}$.

From Theorem 1 we immediately obtain

COROLLARY 1. *$\tilde{Y}_n$ converges weakly to W: $\tilde{Y}_n \Rightarrow W$ as $n \to \infty$.*

Moreover, we can prove the following stronger

THEOREM 2. *Under the assumptions of Theorem* 1 *we have* $\tilde{Y}_n \Rightarrow W$ (*mixing*) *as* $n \to \infty$.

Now, let $\{N_n, n \geqslant 1\}$ be a sequence of positive integer-valued random variables, defined on the same probability space $(\Omega, \mathscr{A}, P)$. Let us suppose that

$$N_n/a_n \xrightarrow{P} \lambda \quad \text{as } n \to \infty, \tag{4}$$

where λ is a positive random variable which may depend only on finite number of Y_n, $n \geqslant 1$, and $\{a_n, n \geqslant 1\}$ is a sequence of positive numbers such that $a_n \to \infty$ as $n \to \infty$. Then we can obtain

THEOREM 3. *Under the assumptions of Theorem* 1 *we have* $\tilde{Y}_{N_n} \Rightarrow W$ *as* $n \to \infty$, *for every* $\{N_n, n \geqslant 1\}$ *satisfying* (4).

By Theorem 3 and corollaries 5.1 and 5.3 in [12] (p. 227 and 230), by putting

$$h_1(x) = \sup_{t \in \langle 0,1 \rangle} x(t), \qquad h_2(x) = \sup_{t \in \langle 0,1 \rangle} |x(t)|,$$

we get

COROLLARY 2. *Under the assumptions of Theorem* 3, *for each* $x > 0$,

$$\lim_{n \to \infty} P\left[\max_{1 \leqslant k \leqslant N_n} \frac{S_k - b\log k}{b\sqrt{2\log n}} < x\right] = \frac{2}{\sqrt{2\pi}} \int_0^x e^{-u^2/2}\, du$$

and

$$\lim_{n \to \infty} P\left[\max_{1 \leqslant k \leqslant N_n} \frac{|S_k - b\log k|}{b\sqrt{2\log n}} < x\right] = \frac{1}{\sqrt{2\pi}} \int_{-x}^{x} \sum_{k=-\infty}^{+\infty} (-1)^k e^{-(u-2kx)^2/2}\, du.$$

Let us observe that this paper gives a generalization of the results presented in [10].

2. Proofs. In the proof of Theorem 1 we apply some lemmas given by Dehéuvels [6] and Höglund [11]. For the sake of completeness we present them in section 3.

Proof of Theorem 1. Suppose that $\{X_n, n \geqslant 1\}$ is the sequence of independent random variables (i.r.vs.) uniformly distributed on [0, 1]. (In this case $b = 1$.)

Put

$$X_m^* = \inf(X_1, X_2, \ldots, X_m), \qquad m \geqslant 1,$$

$$\tilde{S}_n = \sum_{m=1}^{n} X_m^*, \; n \geqslant 2, \qquad \tilde{S}_1 = 0,$$

and define

$$\tilde{X}_n(0) = 0, \quad n \geqslant 1,$$

(5)

$$\tilde{X}_n(t) = \frac{\tilde{S}_k - \log k}{\sqrt{2\log n}} + \frac{t-t_k}{t_{k+1}-t_k}\left(\frac{\tilde{S}_{k+1} - \tilde{S}_k - \log\frac{k+1}{k}}{\sqrt{2\log n}}\right),$$

if $t \in \langle t_k, t_{k+1})$, $0 \leqslant k \leqslant n-1$, where $t_k = \sigma_k/\sigma_{n-1}$, $1 \leqslant k \leqslant n-1$, $t_0 = 0$, $\sigma_k = \sum_{m=1}^{k} (1/m)$, $n \geqslant 2$.

Let $\tilde{P}_n$ denote the distribution of the random function $\{\tilde{X}_n(t), t \in \langle 0, 1 \rangle\}$ in the space $(C, \mathscr{B}_C)$. We shall prove that

$$\mathscr{L}_C(\tilde{P}_n, W) = O\big((\log n)^{-1/8}\big). \tag{6}$$

Let us put $s_n = \sqrt{s_n^2} = \sqrt{\sigma^2 U_n}$, $n \geqslant 1$, and

$$V_m = [\tau_{m+1} - \tau_m - \mathrm{E}(\tau_{m+1} - \tau_m)]\,\varepsilon(m)/s_n, \qquad 1 \leqslant m \leqslant n-1,$$

where the random variables U_n and τ_n, n $\geqslant 1$, are given in section 3 by (3.4) and (3.1), respectively $(\varepsilon(n) = n^{-1})$.

Let $\{W_n^{(1)}(t), t \in \langle 0, 1 \rangle\}$ be the random function defined as follows:

$$W_n^{(1)}(0) = 0, \quad n \geqslant 1,$$

$$W_n^{(1)}(t) = \frac{U_k - \mathrm{E}U_k}{s_n} + \frac{t-t_k}{t_{k+1}-t_k}\left(\frac{U_{k+1} - U_k - \mathrm{E}(U_{k+1} - U_k)}{s_n}\right),$$

if $t \in \langle t_k, t_{k+1})$, where t_k are as in (5), $0 \leqslant k \leqslant n-1$.

First we show that

$$\mathscr{L}_C(P_n^{(1)}, W) = O\big((\log n)^{-1/8}\big), \tag{7}$$

where $P_n^{(1)}$ is the distribution of $\{W_n^{(1)}(t)\}$ in $(C, \mathscr{B}_C)$. To do this, it is enough to note that the sequence $\{V_n, n \geqslant 1\}$ satisfies the conditions of Theorem 1 ([3]). In fact, we have $\mathrm{E}V_m = 0$, $m \geqslant 1$,

$$\sigma^2\Big(\sum_{m=1}^{n} V_m\Big) = 1, \quad t_k = \sum_{m=1}^{k} \sigma^2 V_m.$$

Write $L_n^{(3)} = \sum_{m=1}^{n-1} \mathrm{E}|V_m|^3$. By (3.6) and (3.7),

$$L_n^{(3)} \leqslant \frac{1}{s_n^2} \sum_{m=1}^{n-1} 2^2 \{\mathrm{E}(\tau_{m+1} - \tau_m)^3 \varepsilon^3(m) + \mathrm{E}^3(\tau_{m+1} - \tau_m)\varepsilon^3(m)\} = O\big((\log n)^{-1/2}\big),$$

therefore, by Theorem 1 ([3]), we obtain (7).

Let us now define $\{W_n^{(2)}(t), t \in \langle 0, 1\rangle\}$ as follows:

$W_n^{(2)}(0) = 0, \quad n \geqslant 1,$

$$W_n^{(2)}(t) = \frac{U_k - \log k}{\sqrt{2\log n}} + \frac{t-t_k}{t_{k+1}-t_k}\left(\frac{U_{k+1}-U_k-\log\frac{k+1}{k}}{\sqrt{2\log n}}\right),$$

$$t \in \langle t_k, t_{k+1}), \; 0 \leqslant k \leqslant n-1.$$

If $a_n = s_n/(2\log n)^{1/2}$ and

$$b_{n,k}(t) = \frac{\mathrm{E}U_k - \log k}{\sqrt{2\log n}} + \frac{t-t_k}{t_{k+1}-t_k}\left(\frac{\mathrm{E}(U_{k+1}-U_k)-\log\frac{k+1}{k}}{\sqrt{2\log n}}\right)$$

for $t \in \langle t_k, t_{k+1})$, $0 \leqslant k \leqslant n-1$, $b_{n,k}(0) = 0$, $0 \leqslant k \leqslant n-1$, $n \geqslant 2$, then

$$W_n^{(2)}(t) = a_n W_n^{(1)}(t) + b_{n,k}(t) \quad \text{for } t \in \langle t_k, t_{k+1}).$$

Let $P_n^{(2)}$ denote the distribution of $\{W_n^{(2)}\}$ in $(C, \mathscr{B}_C)$. We are going to show that

$$\mathscr{L}_C(P_n^{(2)}, P_n^{(1)}) = O\big((\log n)^{-1/4}\big). \tag{8}$$

By simple evaluations we obtain

$P[\varrho(W_n^{(2)}, W_n^{(1)}) \geqslant C(\log n)^{-1/4}]$

$$\leqslant P\left[|1-a_n| \max_{1\leqslant k\leqslant n}\frac{|U_k - \mathrm{E}U_k|}{s_n} + \max_{1\leqslant k\leqslant n}\frac{|\mathrm{E}U_k - \log k|}{\sqrt{2\log n}} \geqslant C(\log n)^{-1/4}\right].$$

By (3.5) there exists a positive constant C_1 such that

$$C(\log n)^{-1/4} - \max_{1\leqslant k\leqslant n}\frac{|\mathrm{E}U_k - \log k|}{\sqrt{2\log n}} \geqslant C_1(\log n)^{-1/4}.$$

Thus, by Kolmogorov's inequality and (3.6), we get

$P[\max_{1\leqslant k\leqslant n}|U_k - \mathrm{E}U_k| \geqslant C_1(\log n)^{-1/4} s_n |1-a_n|^{-1}]$

$$\leqslant \frac{2\sigma^2 U_n (\log n)^{1/2}|1-a_n|^2}{C_1^2 s_n^2} = O\big((\log n)^{-3/2}\big).$$

Then, by Lemma 1.2 of [13], we get (8).

Now, let us define the random functions $\{Z_n(t),\ t\in\langle 0,1\rangle\}$:

$$Z_n(0)=0,\quad n\geqslant 1,$$

$$Z_n(t)=\frac{S(\tau_k)-\log k}{\sqrt{2\log n}}+\frac{t-t_k}{t_{k+1}-t_k}\left(\frac{S(\tau_{k+1})-S(\tau_k)-\log\frac{k+1}{k}}{\sqrt{2\log n}}\right),$$

$$n\geqslant 2,\ t\in\langle t_k,\ t_{k+1}),\ 0\leqslant k\leqslant n-1,$$

where $S(\tau_k)=X_1^*+X_2^*+\ldots+X_{\tau_k}^*,\ k\geqslant 1$.

By (3.4), (3.11), (3.8) and the fact that $\tau_1=1$ a.s. we obtain

$$P[\varrho(Z_n,\ W_n^{(2)})\geqslant C(\log n)^{-1/4}]\leqslant P\left[2\max_{1\leqslant k\leqslant n}\frac{|S(\tau_k)-U_k|}{\sqrt{2\log n}}\geqslant C(\log n)^{-1/4}\right]$$

$$\leqslant P\left[\max(S(\tau_1)),\ \max_{1\leqslant k\leqslant n}|U_k'-U_k|+|S(\tau_1)-2|\geqslant\frac{C\sqrt{2}}{2}(\log n)^{1/4}\right]$$

$$\leqslant P\left[(U_n-U_n')\geqslant\frac{C\sqrt{2}}{2}(\log n)^{1/4}-2\right]$$

$$\leqslant \mathrm{E}(U_n-U_n')^2\left(\frac{C\sqrt{2}}{2}(\log n)^{1/4}-2\right)^{-2}=O\left((\log n)^{-1/2}\right),$$

hence, by Lemma 1.2 ([13]), we have

$$(9)\qquad \mathscr{L}_C(P_n^{(3)},\ P_n^{(2)})=O\left((\log n)^{-1/4}\right),$$

where $P_n^{(3)}$ denotes the distribution of random function $\{Z_n(t),\ t\in\langle 0,1\rangle\}$ in $(C,\ \mathscr{B}_C)$.

Now, let $\{\tilde{X}_n(t),\ t\in\langle 0,1\rangle\}$ be the random function given by (5) and let $\tilde{P}_n$ be the distribution of $\{\tilde{X}_n(t)\}$ in $(C,\ \mathscr{B}_C)$. We observe that

$$(10)\ P[\varrho(\tilde{X}_n,\ Z_n)\geqslant C(\log n)^{-1/4}]\leqslant P\left[\max_{1\leqslant k\leqslant n}|\tilde{S}_k-S(\tau_k)|\geqslant\frac{C\sqrt{2}}{2}(\log n)^{1/4}\right]$$

$$\leqslant P\left[\max_{1\leqslant k\leqslant N(n)}|\tilde{S}_k-S(\tau_k)|\geqslant\frac{C\sqrt{2}}{4}(\log n)^{1/4}\right]+$$

$$+P\left[\max_{N(n)<k\leqslant n}|\tilde{S}_k-S(\tau_k)|\geqslant\frac{C\sqrt{2}}{4}(\log n)^{1/4}\right],$$

where $N(n)$ is a subsequence of integers.

It is easy to see, that if $N(n) = [\log n]$, then

$$(11)\quad P\left[\max_{1\leqslant k\leqslant N(n)} |\tilde{S}_k - S(\tau_k)| \geqslant \frac{C\sqrt{2}}{4}(\log n)^{1/4}\right]$$

$$\leqslant P\left[\tilde{S}_{N(n)} + S(\tau_{N(n)}) \geqslant \frac{C\sqrt{2}}{4}(\log n)^{1/4}\right]$$

$$\leqslant P\left[\tilde{S}_{N(n)} \geqslant \frac{C\sqrt{2}}{8}(\log n)^{1/4}\right] + P\left[S(\tau_{N(n)}) \geqslant \frac{C\sqrt{2}}{8}(\log n)^{1/4}\right]$$

$$\leqslant \frac{\mathrm{E}S_{N(n)} + \mathrm{E}S(\tau_{N(n)})}{\frac{C\sqrt{2}}{8}(\log n)^{1/4}} = O\left((\log_2 n)(\log n)^{-1/4}\right),$$

where $\log_2 n = \log(\log n)$, as $\mathrm{E}\tilde{S}_n = \sum_{m=1}^{n} 1/(m+1)$ and $\mathrm{E}S(\tau_n) \sim \log n$.

Now we are going to estimate

$$P\left[\max_{N(n)<k\leqslant n} |\tilde{S}_k - S(\tau_k)| \geqslant \frac{C\sqrt{2}}{4}(\log n)^{1/4}\right].$$

Note that for $k \geqslant \tau_k$ we have, by definition (3.1),

$$\inf(X_1, X_2, \ldots, X_{\tau_k+i}) \leqslant \varepsilon(k) \quad \text{for } i \geqslant 0.$$

In this case we get

$$\tilde{S}_k = S(\tau_k) + \sum_{m=\tau_k+1}^{k} X_m^* \quad \text{and} \quad |\tilde{S}_k - S(\tau_k)| \leqslant k\varepsilon(k) = 1.$$

If $k < \tau_k$, then, by Lemma 3.7,

$$|\tilde{S}_k - S(\tau_k)| = \sum_{m=k+1}^{\tau_k} X_m^* \leqslant (\tau_k - k) X_{k+1} \leqslant (\tau_k - k)\frac{(1+A)\log_2 k}{k} \quad \text{a.s.}$$

for sufficiently large k. Therefore, by Lemma 3.6, for sufficiently large n we have

$$P\left[\max_{N(n)<k\leqslant n} |\tilde{S}_k - S(\tau_k)| \geqslant \frac{C\sqrt{2}}{4}(\log n)^{1/4}\right]$$

$$\leqslant P\left[\max_{N(n)<k<\tau_k\leqslant n} (\tau_k - k)\frac{(1+A)\log_2 k}{k} \geqslant \frac{C\sqrt{2}}{4}(\log n)^{1/4}\right]$$

$$\leqslant P\left[\max_{N(n)<k<\tau_k\leqslant n} \left\{(\tau_k - \tau_{k-1})\frac{(1+A)\log_2 k}{k} + \tau_{k-1}\frac{(1+A)\log_2 k}{k} - \right.\right.$$

$$\left. -(1+A)\log_2 k \right\} \geqslant \frac{C\sqrt{2}}{4}(\log n)^{1/4} \Bigg]$$

$$\leqslant P\Bigg[\max_{N(n)<k<\tau_k\leqslant n} \Big\{(\tau_k-\tau_{k-1})\frac{(1+A)\log_2 k}{k}+$$

$$+\frac{\tau_{k-1}}{k\log_2 k}(1+A)(\log_2 k)^2\Big\} \geqslant \frac{C\sqrt{2}}{4}(\log n)^{1/4}+(1+A)\log_2 N(n)\Bigg]$$

$$\leqslant P\Bigg[\max_{N(n)<k<\tau_k\leqslant n} (\tau_k-\tau_{k-1})\frac{(1+A)\log_2 k}{k} \geqslant C_1(\log n)^{1/4}\Bigg],$$

where C_1 is a positive constant such that

$$\frac{C\sqrt{2}}{4}(\log n)^{1/4}+(1+A)\log_2 N(n)-(1+A)^2(\log_2 n)^2 \geqslant C_1(\log n)^{1/4}.$$

Hence, by a simple evaluation, we obtain

$$P\Bigg[\max_{N(n)<k<\tau_k\leqslant n} (\tau_k-\tau_{k-1})\frac{(1+A)\log_2 k}{k} \geqslant C_1(\log n)^{1/4}\Bigg]$$

$$\leqslant P\Bigg[\max_{N(n)<k<\tau_k\leqslant n} \frac{\tau_k-\tau_{k-1}}{(1+A)k\log_2 k} \geqslant \frac{C_1(\log n)^{1/4}}{(1+A)^2(\log_2 n)^2}\Bigg]$$

$$\leqslant \sum_{k=N(n)+1}^{n} P[\tau_k-\tau_{k-1} \geqslant (1+A)k(\log_2 k)A_n],$$

where $A_n = C_1(\log n)^{1/4}/(1+A)^2(\log_2 n)^2$.

Now, by (3.3) we have

$$P\Bigg[\max_{N(n)<k\leqslant n} |\tilde{S}_k - S(\tau_k)| \geqslant \frac{C\sqrt{2}}{4}(\log n)^{1/4}\Bigg] \tag{12}$$

$$\leqslant \sum_{k=N(n)+1}^{n} \frac{1}{k}\left(1-\frac{1}{k}\right)^{(1+A)k(\log_2 k)A_n-1}$$

$$\leqslant \left(1+\frac{1}{N(n)}\right)\sum_{k=N(n)+1}^{n} \frac{1}{k}e^{-(1+A)(\log_2 k)A_n}$$

$$= \left(1+\frac{1}{N(n)}\right)\sum_{k=N(n)+1}^{n} \frac{1}{k(\log k)^{(1+A)A_n}} = O\big((\log n)^{-1/4}\big).$$

Hence, by (10)-(12) and Lemma 1.2 of [13], we get

$$\mathscr{L}_C(\tilde{P}_n, P_n^{(3)}) = O\big((\log n)^{-1/4}\big). \tag{13}$$

Using (8), (9) and (13) we obtain (6).

Now, let $\{Y_n, n \geqslant 1\}$ be a sequence of i.p.r.vs. with the same distribution function F satisfying (1) and let, as previously, $\{X_n, n \geqslant 1\}$ be a sequence of i.r.vs. uniformly distributed on $[0, 1]$.

Put $G(t) = \inf\{x \geqslant 0\colon F(x) \geqslant t\}$. Then, by [7], the sequences $\{G(X_n), n \geqslant 1\}$ and $\{Y_n, n \geqslant 1\}$ are the same in law. Furthermore, the sums

$$S_n = \sum_{m=1}^{n} Y_m^*, \quad \text{where } Y_m^* = \inf(Y_1, Y_2, \ldots, Y_m)$$

may be represented as

$$\bar{S}_n = \sum_{m=1}^{n} G(X_m^*), \quad \text{where } X_m^* = \inf(X_1, X_2, \ldots, X_m).$$

Let us define the random functions $\{\bar{Y}_n(t), t \in \langle 0, 1\rangle\}$ as follows:

$$(14) \qquad \begin{aligned} &\bar{Y}_n(0) = 0, \quad n \geqslant 1, \\ &\bar{Y}_n(t) = \frac{\bar{S}_k - b\log k}{b\sqrt{2\log n}} + \frac{t - t_k}{t_{k+1} - t_k}\left(\frac{\bar{S}_{k+1} - \bar{S}_k - b\log\frac{k+1}{k}}{b\sqrt{2\log n}}\right), \end{aligned}$$

if $t \in \langle t_k, t_{k+1})$, $0 \leqslant k \leqslant n-1$, $n \geqslant 2$, where $t_k = \sigma_k/\sigma_{n-1}$, $1 \leqslant k \leqslant n-1$, $t_0 = 0$, $\bar{S}_1 = 0$.

We shall show that

$$(15) \qquad \mathcal{L}_C(\bar{P}_n, \tilde{P}_n) = O\left((\log n)^{-1/4}\right),$$

where $\bar{P}_n$ denotes the distribution of $\{\bar{Y}_n(t)\}$, in $(C, \mathcal{B}_C)$.

Indeed,

$$P[\sup_{t\in\langle 0,1\rangle} |\bar{Y}_n(t) - \tilde{X}_n(t)| \geqslant C(\log n)^{-1/4}]$$
$$\leqslant P\left[\max_{1\leqslant k\leqslant n} |\bar{S}_k - b\tilde{S}_k| \geqslant \frac{b\sqrt{2}}{2}(\log n)^{1/4}\right]$$
$$= P\left[\max_{1\leqslant k\leqslant n} \left|\sum_{m=1}^{k} \delta_m(G(X_m^*) - bX_m^*) + \sum_{m=1}^{k}(1-\delta_m)(G(X_m^*) - bX_m^*)\right| \geqslant \frac{b\sqrt{2}}{2}(\log n)^{1/4}\right]$$
$$\leqslant P\left[\sum_{m=1}^{n} \delta_m |G(X_m^*) - bX_m^*| + \sum_{m=1}^{n}(1-\delta_m)|G(X_m^*) - bX_m^*| \geqslant \frac{b\sqrt{2}}{2}(\log n)^{1/4}\right],$$

where

$$\delta_m = \begin{cases} 1 & \text{if } X_m^* \leqslant \delta, \\ 0 & \text{otherwise, } 0 < \delta < 1. \end{cases}$$

With probability 1 all but finitely many δ_m are equal to 1, so

$$P[\sup_{t \in \langle 0,1 \rangle} |\bar{Y}_n(t) - \tilde{X}_n(t)| \geqslant C(\log n)^{-1/4}]$$

$$\leqslant P[\sum_{m=1}^{n} \delta_m |G(X_m^*) - bX_m^*| \geqslant C_1 (\log n)^{1/4}],$$

where C_1 is a positive constant. Hence, by the Markoff inequality and Lemma 3.8, we get (15). Thus, taking into account (6) and (15) we immediately obtain $\mathscr{L}_C(\bar{P}_n, W) = O((\log n)^{-1/8})$ and the proof of Theorem 1 is completed.

Proof of Theorem 2. At first we assume that $\{X_n, n \geqslant 1\}$ is a sequence of i.r.vs. uniformly distributed on [0, 1]. We will show that

$$\tilde{X}_n \Rightarrow W \text{ (mixing)} \quad \text{as } n \to \infty, \tag{16}$$

where $\{\tilde{X}_n(t), t \in \langle 0, 1 \rangle\}$ is defined by (5). By Corollary 1 we have $\tilde{X}_n \Rightarrow W$ as $n \to \infty$. Putting

$$X_n^{(1)}(0) = 0, \quad n \geqslant 1,$$

$$X_n^{(1)}(t) = (\tilde{S}_k - \log k)/\sqrt{2 \log n} \quad \text{if } t \in \langle t_k, t_{k+1}),$$

$0 \leqslant k \leqslant n-1$, $n \geqslant 2$, we immediately obtain

$$X_n^{(1)} \Rightarrow W \quad \text{as } n \to \infty \tag{17}$$

and

$$\varrho(\tilde{X}_n, X_n^{(1)}) \leqslant \sup_{1 \leqslant k \leqslant n} \frac{\left| X_{k+1}^* - \log \frac{k+1}{k} \right|}{\sqrt{2 \log n}} \to 0 \text{ a.s.} \quad \text{as } n \to \infty. \tag{18}$$

Now, let

$$X_n^{(2)}(t) = (\tilde{S}_{[e^{t \log n}]} - t \log n)/\sqrt{2 \log n}, \quad t \in \langle 0, 1 \rangle, n > 1. \tag{19}$$

We shall estimate $\varrho(X_n^{(1)}, X_n^{(2)})$. Write $e_t^{(n)} = \exp(t \log n)$, $t \in \langle 0, 1 \rangle$, $n > 1$. We have

$$\varrho(X_n^{(1)}, X_n^{(2)}) \leqslant \frac{1}{\sqrt{2 \log n}} \max_{1 \leqslant k < n} \sup_{t \in \langle t_k, t_{k+1})} \{|\tilde{S}_{[e_t^n]} - \tilde{S}_k| + |t \log n - \log k|\}$$

$$\leqslant \frac{1}{\sqrt{2\log n}} \max_{1 \leqslant k < n} \sup_{t \in \langle t_k, t_{k+1})} \{\max(\tilde{S}_{[e_t^{(n)}]} - \tilde{S}_k, \tilde{S}_k - \tilde{S}_{[e_t^{(n)}]}) +$$

$$+ \max(t \log n - \log k, \log k - t \log n)\}$$

$$\leqslant \frac{1}{\sqrt{2\log n}} \max_{1 \leqslant k < n} \{\max(\tilde{S}_{[e_{t_{k+1}}^{(n)}]} - \tilde{S}_k, \tilde{S}_k - \tilde{S}_{[e_{t_k}^{(n)}]}) +$$

$$+ \max(t_{k+1} \log n - \log k, \log k - t_k \log n)\}$$

$$\leqslant \frac{1}{\sqrt{2\log n}} \max_{1 \leqslant k < n} \{ \sum_{m=[e_{t_k}^{(n)}]+1}^{[e_{t_{k+1}}^{(n)}]} X_m^* + \max(\sigma_{k+1}(\log n/\sigma_{n-1}) - \log k,$$

$$\log k - \sigma_k(\log n/\sigma_{n-1}))\}$$

$$\leqslant \frac{1}{\sqrt{2\log n}} \Big(\max_{1 \leqslant k \leqslant N(n)} \sum_{m=[e_{t_k}^{(n)}]+1}^{[e_{t_{k+1}}^{(n)}]} X_m^* + \max_{N(n) < k < n} \sum_{m=[e_{t_k}^{(n)}]+1}^{[e_{t_{k+1}}^{(n)}]} X_m^* \Big) +$$

$$+ \frac{1}{\sqrt{2\log n}} \max_{1 \leqslant k < n} \max[(\sigma_{k+1} - \log k)(\log n/\sigma_{n-1}) +$$

$$+ \log k(\log n/\sigma_{n-1} - 1), (\log k - \sigma_k)(\log n/\sigma_n) + \log k(1 - \log n/\sigma_{n-1})]$$

$$\leqslant \frac{1}{\sqrt{2\log n}} \{\tilde{S}_{[e_{t_{N(n)+1}}^{(n)}]} + \sum_{m=[e_{t_k}^{(n)}]+1}^{[e_{t_{k+1}}^{(n)}]} \frac{(1+A)\log_2 m}{m} + O(1)\} \text{ a.s.}$$

by Lemma 3.7.

We observe that

$$\frac{1}{\sqrt{2\log n}} \tilde{S}_{[e_{t_{N(n)+1}}^{(n)}]} = \frac{\tilde{S}_{[e_{t_{N(n)+1}}^{(n)}]}}{t_{N(n)+1} \log n} \frac{t_{N(n)+1} \log n}{\sqrt{2\log n}} \to 0 \text{ a.s.}$$

as $n \to \infty$, because, by Lemma 3.3,

$$\tilde{S}_{[e_{t_{N(n)+1}}^{(n)}]}/t_{N(n)+1} \log n \to 1, \quad n \to \infty,$$

and $t_{N(n)+1} \log n \sim \log N(n)$.

Moreover,

$$\frac{1}{\sqrt{2\log n}} \max_{N(n) < k < n} \sum_{m=[e_{t_k}^{(n)}]+1}^{[e_{t_{k+1}}^{(n)}]} \frac{(1+A)\log_2 m}{m}$$

$$\leqslant \frac{1}{\sqrt{2\log n}} \max_{N(n)<k<n} (1+A)(\log_2 e_{t_{k+1}}^{(n)})(\log e_{t_{k+1}}^{(n)} - \log e_{t_k}^{(n)})$$

$$\leqslant \frac{(1+A)\log_2 n}{\sqrt{2\log n}} \max_{N(n)<k<n} \left(\frac{\sigma_{k+1}}{\sigma_n}\log n - \frac{\sigma_k}{\sigma_n}\log n\right) = O\left(\left(\frac{\log_2 n}{(\log n)^{3/2}}\right)^{-1}\right) \to 0$$

as $n \to \infty$. Thus

$$(20) \qquad \varrho(X_n^{(1)}, X_n^{(2)}) \to 0 \text{ a.s.}, \quad n \to \infty,$$

and, by (18),

$$(21) \qquad X_n^{(2)} \Rightarrow W, \quad n \to \infty.$$

Now, let us put $X^*_{(l,m)} = \inf(X_{l+1}, X_{l+2}, \ldots, X_m)$ for $m > l$, and define $\{X_n^{(3)}(t), t \in \langle 0, 1\rangle\}$ by

$$(22) \qquad X_n^{(3)}(t) = \frac{1}{\sqrt{2\log n}}\Big(\sum_{m=N(n)+1}^{[e^{t\log n}]} X^*_{(N(n),m)} - t\log n\Big), \quad t \in \langle 0, 1\rangle,$$

where $N(n) = [\log n]$.

By Lemma 3 ([10]), $X^*_{(l,m)} \geqslant X_m$, $m > l$, and the sum $\sum (X^*_{(l,m)} - X^*_m)$, $m = l+1, l+2, \ldots$, converges almost surely. Moreover, one can note that the random variable $\sum X^*_{(l,m)}$, $m = l+1, l+2, \ldots, N$, is independent of $X_1, X_2, \ldots, X_l$ for all $l > 1$ and $N > l$. By definitions (19), (22), Lemma 3.3 and Lemma 3 ([10]) we obtain

(23)

$$\varrho(X_n^{(2)}, X_n^{(3)}) \leqslant \frac{1}{\sqrt{2\log n}}\Big(\tilde{S}_{N(n)} + \sum_{m=N(n)+1}^{n} (X^*_{(N(n),m)} - X^*_m)\Big) \to 0 \text{ a.s.}, \quad n \to \infty,$$

so, by (22),

$$(24) \qquad X_n^{(3)} \Rightarrow W, \quad n \to \infty.$$

Let $\mathscr{B}_0$ be the field of cylinders which consists of sets of the form $\{\omega\colon (X_1(\omega), X_2(\omega), \ldots, X_k(\omega)) \in H\}$, with $k \geqslant 1$ and $H \in \mathbf{R}^k$. Then, for any $E \in \mathscr{B}_0$, by the definition (22) and relation (24) we obtain that $P[(X_n^{(3)} \in A) \cap E] \to W(A)P(E)$, $n \to \infty$, for every W-continuity set A, so that $X_n^{(3)} \Rightarrow W$ (mixing) as $n \to \infty$, and, by (17), (18), (20) and (23), also

$$(25) \qquad \tilde{X}_n \Rightarrow W \text{ (mixing)} \quad \text{as } n \to \infty.$$

Now, let $\{Y_n, n \geqslant 1\}$ be a sequence of i.p.r.vs., with the common distribution function F, such that (1) holds for some b $(0 < b < \infty)$, and let $\{\bar{Y}_n(t), t \in \langle 0, 1\rangle\}$ be defined by (14). By (15) we see that $\varrho(\bar{Y}_n, \tilde{X}_n) \xrightarrow{P} 0$ as

$n \to \infty$, so, by (25), we immediately obtain that $\bar{Y}_n \Rightarrow W$ (mixing) as $n \to \infty$. Thus the proof of Theorem 2 is completed.

Proof of Theorem 3. Let $\{N_n, n \geqslant 1\}$ be a sequence of positive integer-valued random variables satisfying (3). To prove Theorem 3 it is enough to show that the random elements $\{\tilde{X}_n(t), t \in \langle 0, 1\rangle\}$, given by (5), satisfy the generalized Anscombe condition with the norming sequence $\{k_n = n, n \geqslant 1\}$, i.e.

$$\max_{i \in D_n(\delta)} \varrho(\tilde{X}_i, \tilde{X}_n) \xrightarrow{P} 0, \quad n \to \infty, \tag{26}$$

for some $\delta > 0$, where $D_n(\delta) = \{i: (1-\delta)n \leqslant i < (1+\delta)n\}$. (See Theorem 3 ([4]) and the relation (16).)

By (18), (20) and (23) we can only to estimate $\max_{i \in D_n(\delta)} \varrho(X_i^{(3)}, X_n^{(3)})$.

Observe that

$$\max_{i \in D_n(\delta)} \varrho(X_i^{(3)}, X_n^{(3)}) \leqslant \max_{i \in D_n(\delta)} \sup_{t \in \langle 0,1\rangle} \left| \frac{1}{\sqrt{2\log i}} \tilde{S}_{[e_t^{(i)}]} - \frac{1}{\sqrt{2\log n}} \tilde{S}_{[e_t^{(n)}]} \right| +$$

$$+ \max_{i \in D_n(\delta)} \frac{1}{\sqrt{2}} |\sqrt{\log i} - \sqrt{\log n}| \leqslant \max_{i \in D_n(\delta)} \sup_{t \in \langle 0,1\rangle} \frac{1}{\sqrt{2\log i}} |\tilde{S}_{[e_t^{(i)}]} - \tilde{S}_{[e_t^{(n)}]}| +$$

$$+ \max_{i \in D_n(\delta)} \sup_{t \in \langle 0,1\rangle} \tilde{S}_{[e_t^{(n)}]} \left| \frac{1}{\sqrt{2\log i}} - \frac{1}{\sqrt{2\log n}} \right| + \frac{1}{\sqrt{2}} \left(\sqrt{\log n(1+\delta)} - \sqrt{\log n(1-\delta)} \right)$$

$$\leqslant \frac{1}{\sqrt{2\log n(1-\delta)}} \max\Big\{ \max_{(1-\delta)n \leqslant i < n} \sup_{t \in \langle 0,1\rangle} (\tilde{S}_{[e_t^{(n)}]} - \tilde{S}_{[e_t^{(i)}]}),$$

$$\max_{n \leqslant i < (1+\delta)n} \sup_{t \in \langle 0,1\rangle} (\tilde{S}_{[e_t^{(i)}]} - \tilde{S}_{[e_t^{(n)}]}) \Big\} + \tilde{S}_n \left(\frac{1}{\sqrt{2\log n(1-\delta)}} - \frac{1}{\sqrt{2\log n(1+\delta)}} \right) +$$

$$+ \frac{\log[(1+\delta)/(1-\delta)]}{\sqrt{2}(\sqrt{\log n(1+\delta)} + \sqrt{\log n(1-\delta)})}.$$

By Lemma 3.3 we see that

$$\tilde{S}_n \left(\frac{1}{\sqrt{2\log n(1-\delta)}} - \frac{1}{\sqrt{2\log n(1+\delta)}} \right) \to 0 \text{ a.s.}, \quad n \to \infty.$$

Putting $t_{N(n)} = \log N(n)/\log n$, where $N(n) = [\log n]$, by Lemma 3.7 we get

$$\frac{1}{\sqrt{2\log n(1-\delta)}} \max\Big\{ \max_{(1-\delta)n \leqslant i < n} \sup_{t \in \langle 0,1\rangle} \sum_{m=[e_t^{(i)}]+1}^{[e_t^{(n)}]} X_m^*, \max_{n \leqslant i < n(1+\delta)} \sup_{t \in \langle 0,1\rangle} \sum_{m=[e_t^{(n)}]+1}^{[e_t^{(i)}]} X_m^* \Big\}$$

$$\leqslant \frac{1}{\sqrt{2\log n(1-\delta)}} \max\Big\{ \max_{(1-\delta)n\leqslant i<n} \Big(\sup_{t\in\langle 0,t_{N(n)}\rangle} \sum_{m=[e_t^{(i)}]+1}^{[e_t^{(n)}]} X_m^* + \sup_{t\in\langle t_{N(n)},1\rangle} \sum_{m=[e_t^{(i)}]+1}^{[e_t^{(n)}]} X_m^*\Big),$$

$$\max_{n\leqslant i<n(1+\delta)} \Big(\sup_{t\in\langle 0,t_{N(n)}\rangle} \sum_{m=[e_t^{(n)}]+1}^{[e_t^{(i)}]} X_m^* + \sup_{t\in\langle t_{N(n)},1\rangle} \sum_{m=[e_t^{(n)}]+1}^{[e_t^{(i)}]} X_m^*\Big)\Big\} \text{ a.s.}$$

$$\leqslant \frac{1}{\sqrt{2\log n(1-\delta)}} \max\Big\{ \tilde{S}_{[e_{t_{N(n)}}^{(n)}]} + \max_{(1-\delta)n\leqslant i<n} \sup_{t\in\langle t_{N(n)},1\rangle} \sum_{m=[e_t^{(i)}]+1}^{[e_t^{(n)}]} \frac{(1+A)\log_2 m}{m},$$

$$\max_{n\leqslant i<n(1+\delta)} \tilde{S}_{[e_{t_{N(n)}}^{(i)}]} + \max_{n\leqslant i<n(1+\delta)} \sup_{t\in\langle t_{N(n)},1\rangle} \sum_{m=[e_t^{(n)}]+1}^{[e_t^{(i)}]} \frac{(1+A)\log_2 m}{m}\Big\}$$

$$\leqslant \max\Big\{\big(\tilde{S}_{[e_{t_{N(n)}}^{(n)}]}/t_{N(n)}\log n\big)\big(t_{N(n)}\log n/\sqrt{2\log n(1-\delta)}\big),$$

$$\big(\tilde{S}_{[e_{t_{N(n)}}^{(n(1+\delta))}]}/t_{N(n)}\log n(1+\delta)\big)\big(t_{N(n)}\log n(1+\delta)/\sqrt{2\log n(1-\delta)}\big)+$$

$$+\frac{(1+A)\log_2 n}{\sqrt{2\log n(1-\delta)}} \max\Big\{ \max_{(1-\delta)n\leqslant i<n} \sup_{t\in\langle t_{N(n)},1\rangle} t(\log n-\log i),$$

$$\max_{n\leqslant i<n(1+\delta)} \sup_{t\in\langle t_{N(n)},1\rangle} t(\log i-\log n)\Big\} \to 0 \text{ a.s.}, \quad n\to\infty,$$

by Lemmas 3.3 and 3.7. Then

$$\max_{i\in D_n(\delta)} \varrho(X_n^{(3)}, X_i^{(3)}) \xrightarrow{P} 0, \quad n\to\infty.$$

Hence, taking into account the relation given above, we have (26), so that the generalized Anscombe condition holds, in this case.

Now, let $\{\bar{Y}_n(t), t\in\langle 0, 1\rangle\}$ be given by (14). By simple evaluation and Lemma 3.8 we get

$$\max_{i\in D_n(\delta)} \varrho(\bar{Y}_i, \tilde{X}_i) \leqslant \frac{1}{b\sqrt{2\log n(1-\delta)}} \sum_{m=1}^{[n(1+\delta)]} |G(X_m^*)-bX_m^*| \xrightarrow{P} 0,$$

as $n\to\infty$. Hence, by (26), we obtain

$$\max_{i\in D_n(\delta)} \varrho(\bar{Y}_i, \bar{Y}_n) \xrightarrow{P} 0 \quad \text{as } n\to\infty.$$

Thus by Theorem 3 of [4] the proof of Theorem 3 is completed.

3. Lemmas. In this section we present some lemmas we needed in the proofs of Theorems 1-3.

Let $\{\varepsilon(n), n\geqslant 1\}$ be a sequence of positive real numbers strictly decreasing to zero.

By $\{\tau_n = \tau(\varepsilon(n)),\ n \geqslant 1\}$ we denote the sequence of random variables such that

$$\tau_n = \inf\{m\colon \inf(X_1, X_2, \ldots, X_m) \leqslant \varepsilon(n)\}, \tag{3.1}$$

where $\{X_n,\ n \geqslant 1\}$ is a sequence of independent random variables uniformly distributed on $[0, 1]$.

LEMMA 3.1. *The sequence* $\{\tau_n,\ n \geqslant 1\}$ *increases with probability* 1, *and* $\tau_n \to \infty$ *a.s. as* $n \to \infty$.

LEMMA 3.2. *The random variables* $\tau_n - \tau_{n-1}$, $n \geqslant 2$, *are independent, and if* $\varepsilon(n) = n^{-1}$, *then*

$$\mathrm{E}(\tau_{n+1} - \tau_n) = 1, \quad \sigma^2(\tau_{n+1} - \tau_n) = 2n, \tag{3.2}$$

$$P[\tau_{n+1} - \tau_n \geqslant r] = \frac{1}{n+1}\left(1 - \frac{1}{n+1}\right)^{r-1} \quad \textit{for any } r > 0. \tag{3.3}$$

Let

$$U_n = \sum_{m=1}^{n-1} (\tau_{m+1} - \tau_m)\,\varepsilon(m), \quad U_n' = \sum_{m=1}^{n-1} (\tau_{m+1} - \tau_m)\,\varepsilon(m+1), \tag{3.4}$$

where $\varepsilon(n) = n^{-1}$. Then

$$\mathrm{E}U_n - \log n = O(1), \quad \mathrm{E}U_n' - \log n = O(1), \tag{3.5}$$

$$\sigma^2 U_n - 2\log n = O(1), \quad \sigma^2 U_n' - 2\log n = O(1), \tag{3.6}$$

$$\sum_{m=1}^{n} \mathrm{E}(\tau_{m+1} - \tau_m)^p \varepsilon^p(m) \sim \sum_{m=1} \mathrm{E}(\tau_{m+1} - \tau_m)^p \varepsilon^p(m+1) \sim p!\log n, \tag{3.7}$$

$$\mathrm{E}(U_n - U_n') = O(1), \quad \sigma^2(U_n - U_n') = O(1), \tag{3.8}$$

where $b_n = O(1)$ denotes that the sequence $\{b_n,\ n \geqslant 1\}$ is bounded as $n \to \infty$.

LEMMA 3.3. *We have*

$$\frac{U_n - \log n}{\sqrt{2\log n}} \xrightarrow{\mathscr{D}} N(0, 1), \quad \frac{\tilde{S}_n - \log n}{\sqrt{2\log n}} \xrightarrow{\mathscr{D}} N(0, 1), \quad n \to \infty, \tag{3.9}$$

$$\frac{S(\tau_n)}{\log n} \to 1 \text{ a.s.}, \quad \frac{\tilde{S}_n}{\log n} \to 1 \text{ a.s.}, \quad n \to \infty, \tag{3.10}$$

where

$$\tilde{S}_n = \sum_{m=1}^{n} X_m^*, \quad S(\tau_n) = X_1^* + X_2^* + \ldots + X_{\tau_n}^*, \quad n \geqslant 1,$$

$$X_m^* = \inf(X_1, X_2, \ldots, X_m), \quad m \geqslant 1.$$

LEMMA 3.4. *Let U_n, U'_n be given by* (3.4). *Then*

(3.11) $$-2+U'_n \leqslant S(\tau_n)-S(\tau_1) \leqslant U_n \text{ a.s.}, \quad n \geqslant 2,$$

(3.12) $$S(\tau_{n-1}) \leqslant \tilde{S}_m \leqslant S(\tau_n) \quad \text{for } m \in \langle \tau_{n-1}, \tau_n).$$

LEMMA 3.5. *For all $A > 0$*

$$\log n-(1+A)\log_2 n \leqslant \log \tau_n \leqslant \log n+(1+A)\log_3 n \text{ a.s.}$$

for sufficiently large n, where $\log_p x = \log(\log_{p-1} x)$, $p \geqslant 2$, $\log_1 x = \log x$.

LEMMA 3.6. *We have*

$$\limsup_{n\to\infty} \tau_n/n\log_2 n = 1 \text{ a.s.}.$$

LEMMA 3.7. *For all $A > 0$*

$$(n\log n\log_2 n \ldots (\log_p n)^{1+A})^{-1} \leqslant X_n^*$$
$$\leqslant (\log_2 n+\log_3 n+ \ldots +(1+A)\log_p n)/n \text{ a.s.}$$

for sufficiently large n.

LEMMA 3.8. *Under the assumptions of Theorem* 1

$$\frac{\sum_{m=1}^{n} \delta_m (G(X_m^*)-bX_m^*)}{b\sqrt{2\log n}} + \frac{\sum_{m=1}^{n} (1-\delta_m)(G(X_m^*)-bX_m^*)}{b\sqrt{2\log n}} \xrightarrow{P} 0$$

as $n \to \infty$, and

$$\mathrm{E}\big[\sum_{m=1} \delta_m |G(X_m^*)-bX_m^*|\big] < \infty,$$

where

$$\delta_m = \begin{cases} 1 \text{ if } X_m^* < \delta \\ 0 \text{ if } X_m^* \geqslant \delta \end{cases}, \quad 0 < \delta < 1, \qquad G(t) = \inf\{x \geqslant 0\colon F(x) \geqslant t\}.$$

REFERENCES

[1] D. J. Aldous, *Weak convergence of radomly indexed sequences of random variables*, Math. Proc. Cambridge Philos. Soc. 83 (1978), p. 117–126.

[2] P. Billingsley, *Convergence of Probability Measures*, John Wiley and Sons, Inc. New York, London, Sydney, Toronto 1968.

[3] A. A. Borovkov, *Ob ocenkach w principie invariantnosti*, DAN SSSR, 205, 5 (1972), p. 1037–1039.

[4] M. Csörgö and Z. Rychlik, *Weak convergence of sequences of random elements with random indices*, Math. Proc. Camb. Phil. Soc. 88 (1980), p. 171–174.

[5] P. Dehéuvels, *Sur la convergence de sommes de minima de variables aléatoires*, C. R. Acad. Sci. Paris 276, Série A (1973), p. 304–313.

[6] — *Valeurs extrémales d'échantillons croissants d'une variable aléatoire réelle*, Ann. Inst. Henri Poincaré, Sect. B, X (1974), p. 89–114.

[7] U. Grenander, *A limit theorem for sums of minima of stochastic variables*, Ann. Math. Stat. (1966), p. 1041–1042.

[8] H. Hebda-Grabowska and D. Szynal, *On the rate of convergence in law for the partial sums of infima of random variables*, Bull. Acad. Polon. Scienc. XXVII. 6 (1979), p. 503–509.

[9] — *An almost sure invariance principle for partial sums of infima of independent random variables*, Ann. Prob. 7.6 (1979), p. 1036–1045.

[10] H. Hebda-Grabowska, *Weak convergence of random sums of infima of independent random variables*, Prob. and Math. Stat. 8 (1987), p. 41–47.

[11] T. Höglund, *Asymptotic normality of sums of minima of random variables*, Ann. Math. Stat. 43 (1972), p. 351–353.

[12] K. R. Parthasarathy, *Probability measures on metric spaces*, Academic Press, New York and London, 1967.

[13] Yu. V. Prohorov, *Schodimost' slučajnych processov i predel'nye teoremy veroyatnosti*, Teor. Ver. i ee Prim. 1 (1956), p. 177–238.

Institute of Mathematics
Maria Curie-Skłodowska University
Plac Marii Curie-Skłodowskiej 1
20-031 Lublin, Poland.

Received on 7. 10. 1986;
revised version on 21. 4. 1987