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TOPOLOGY OF THE CONVERGENCE IN PROBABILITY ON A LINEAR SPAN OF A SEQUENCE OF INDEPENDENT RANDOM VARIABLES

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Abstract. Let $X_1, X_2, \ldots$ be a sequence of independent symmetric Hilbert space valued non-degenerated random variables and let L_X denote the closed linear span of $\{X_n\}$ in $L_0(\Omega, \mathscr{F}, P; H)$. If L_X is a locally convex subspace of L_0, then L_X is Banach iff L_X does not contain an isomorphic copy of R^∞ iff

$$\sup_n P(X_n = 0) < 1.$$

If, moreover, X_n are equidistributed and $P(X_n = 0) = 0$, then

$$\left\{Y \in L_X:\ P\left(\|Y\| > \frac{1}{201}\right) < \frac{1}{201}\right\}$$

is a bounded neighbourhood of zero.

In this note we will investigate the topology of the convergence in probability for random variables of the form $\sum a_n X_n$, $n = 1, 2, \ldots$, where a_n are real numbers, $\{X_n\}$ is a fixed sequence of independent symmetric non-degenerated Hilbert space valued random variables and the series converges in probability. We denote the linear space of random variables of this form by L_X. It is easy to see that L_X endowed with the topology τ_P of the convergence in probability is a complete separable linear-metric space.

THEOREM 1. *If (L_X, τ_P) is locally convex, then the following conditions are equivalent:*

(i) *(L_X, τ_P) is a Banach space;*

(ii) *L_X does not contain a subspace isomorphic to R^∞;*

(iii) $\sup P(X_n = 0) < 1$.

Before proving Theorem 1, we will introduce some notation and prove some lemmas. We use ":=" as "equal by definition".

For $n = 1, 2, \ldots$ and $t \in R$ we have $Q_n(t) := E\min(1, \|tX_n\|^2)$. It is easy to see that $Q_n(0) = 0$,

$$\lim_{t \to \infty} Q_n(t) = 1 - P(X_n = 0),$$

$Q_n(t) = Q_n(-t)$ and, for $t_1 \geqslant t_2 \geqslant 0$, $Q_n(t_1) \geqslant Q_n(t_2)$.

For $\varepsilon > 0$

$$U_\varepsilon := \{Y \in L_X\colon\ Y = \sum a_n X_n \text{ and } \sum Q_n(a_n) < \varepsilon\},$$

$$V_\varepsilon := \{Y \in L_X\colon\ P(\|Y\| > \varepsilon) < \varepsilon\}.$$

LEMMA 1. *$\varepsilon U_\varepsilon \subset V_{2\varepsilon} \subset U_{400\varepsilon}$ for $0 < \varepsilon < 1/400$.*

Proof. The inclusions follow directly from the following beautiful estimates [4]:

(1) if $0 < \varepsilon < 1/200$ and $P(\|\sum a_n X_n\| > \varepsilon) < \varepsilon$, then $\sum Q_n(a_n) < 200\varepsilon$;
(2) $P(\|\sum a_n X_n\| > \varepsilon) < 2\sum Q_n(a_n \varepsilon^{-1})$ for every $\varepsilon > 0$.

Remark. Propositions (1) and (2) are stated in [4] under the assumption that $X_1, X_2, \ldots$ are equidistributed real random variables. But those assumptions are not used in the proof, which can be rewritten (with obvious changes) in the Hilbert space case.

LEMMA 2. *If $\operatorname{conv} U_\varepsilon \subset U_\eta$ for some $0 < \varepsilon < 1 - \sup P(X_n = 0)$ and $\eta > 0$, then*

$$\forall_{\delta > 0} \exists_{r = r(\delta) > 0} \forall_{n \in N} \forall_{t \in R} \qquad Q_n(t) < \varepsilon \Rightarrow Q_n(rt) < \delta.$$

Proof. Let us assume that the implication is false. Then for some $\delta > 0$ there exist sequences (n_k) and (t_k) of positive integers such that

$$Q_{n_k}(t_k) < \varepsilon \quad \text{and} \quad Q_{n_k}\left(\frac{t_k}{k}\right) \geqslant \delta.$$

Since $\varepsilon < 1 - \sup P(X_n = 0)$, we have

$$\forall_{n \in N} \exists_{t_n > 0} \forall_{t > t_n} \qquad Q_n(t) > \varepsilon.$$

Thus the boundedness of (n_k) would entail the boundedness of (t_k). But for (n_k) and (t_k) bounded we would have

$$\lim_{k \to \infty} Q_{n_k}\left(\frac{t_k}{k}\right) = 0.$$

Hence we can assume that (n_k) is strictly increasing.

Consider the following sequence of elements of $\operatorname{conv} U_\varepsilon$:

$$Y_1 = t_1 X_{n_1},$$

$$Y_2 = \frac{1}{2} t_2 X_{n_2} + \frac{1}{2} t_3 X_{n_3},$$

$$\dots\dots\dots\dots\dots$$

$$Y_m = \sum_{k=m}^{2m-1} \frac{1}{m} t_k X_{n_k}.$$

$$\dots\dots\dots\dots\dots$$

It is clear that

$$\sum_{k=m}^{2m-1} Q_{n_k}\left(\frac{t_k}{m}\right) \geqslant \sum_{k=m}^{2m-1} Q_{n_k}\left(\frac{t_k}{k}\right) \geqslant m\delta.$$

This contradicts the assumption of the lemma that Y_m belongs to U_η.

LEMMA 3. *Let* $\varepsilon, \lambda > 0$ *and let* $Z = \sum b_n X_n$, $n = 1, 2, \dots$, *be an element of* U_ε. *If* $Q_n(b_n) < \lambda$ *for every* n, *then* $\lambda Z/(\lambda+\varepsilon)$ *is an element of* $\operatorname{conv} U_\lambda$.

Proof. Since $Q_n(b_n) < \lambda$, there exist positive integers M and $1 = n_0 < n_1 < n_2 < \dots < n_M$ such that

$$\sum_{n=1}^{n_1-1} Q_n(b_n) = \lambda_1 < \lambda \quad \text{and} \quad Q_{n_1}(b_{n_1}) \geqslant \lambda - \lambda_1,$$

$$\sum_{n=n_1}^{n_2-1} Q_n(b_n) = \lambda_2 < \lambda \quad \text{and} \quad Q_{n_2}(b_{n_2}) \geqslant \lambda - \lambda_2,$$

$$\dots\dots\dots\dots\dots$$

$$\sum_{n=n_{M-1}}^{n_M-1} Q_n(b_n) = \lambda_M < \lambda \quad \text{and} \quad Q_{n_M}(b_{n_M}) \geqslant \lambda - \lambda_M,$$

$$\sum_{n=n_M}^{\infty} Q_n(b_n) < \lambda.$$

Consequently, random variables

$$Z_k = \sum_{n=n_{k-1}}^{n_k-1} b_n X_n \ (k = 1, 2, \dots, M) \quad \text{and} \quad Z_{M+1} = \sum_{n=n_M}^{\infty} b_n X_n$$

are elements of U_λ such that $Z_1 + Z_2 + \dots + Z_M + Z_{M+1} = Z$.

Obviously $M+1 \leqslant \varepsilon/\lambda + 1$. Thus $\lambda Z/(\lambda+\varepsilon) \in U_\lambda$.

LEMMA 4. *If*

$$\sum_{n=1}^{\infty} P(X_n \neq 0) < \infty,$$

then (L_X, τ_P) *is isomorphic to* R^{∞}.

Proof. We have to prove that:

(a) for every sequence of real numbers (a_n) the series $\sum a_n X_n$, $n = 1, 2, \ldots$, converges in probability;

(b) the sequence

$$\Big(\sum_{n=1}^{\infty} a_{n,k} X_n\Big)_{k=1}^{\infty}$$

of elements of L_X converges to zero in probability iff

$$\lim_{k\to\infty} a_{n,k} = 0 \quad \text{for every } n.$$

Both (a) and (b) follow immediately from the Borel-Cantelli Lemma.

Proof of the Theorem 1. (i) $\Rightarrow$ (ii) is obvious.

$\sim$(iii) $\Rightarrow$ $\sim$(ii). Let (n_k) be an increasing sequence of positive integers such that $P(X_{n_k} = 0) > 1-1/2^k$. By Lemma 4, the closed linear span of (X_{n_k}) is ismorphic to R^{∞}.

(iii) $\Rightarrow$ (i). It is enough to prove the existence of a bounded neighborhood of zero. Thus, by Lemma 1, it is enough to show that

$$\exists_{\varepsilon>0} \forall_{\eta>0} \exists_{s>0} \quad sU_\varepsilon \subset U_\eta .$$

Let us take $\delta > 0$. Local convexity of (L_X, τ_P) and Lemma 1 imply the existence of an $\varepsilon > 0$ such that $\operatorname{conv} U_\varepsilon \subset U_\delta$. We can assume that $\varepsilon < 1-\sup P(X_n = 0)$.

Let us fix an $\eta > 0$ and let us take a $\lambda > 0$ such that $\operatorname{conv} U_\lambda \subset U_{\eta/2}$. By Lemma 2 there exists an $r = r(\eta\lambda/2\varepsilon)$ such that

$$\forall_{n\in N} \forall_{t\in R} \quad Q_n(t) < \varepsilon \Rightarrow Q_n(rt) < \frac{\eta\lambda}{2\varepsilon}.$$

We claim that

$$(*) \qquad sU_\varepsilon \subset U_\eta \quad \text{for } s = \min\left(\frac{1}{r}, \frac{\lambda}{\lambda+\varepsilon}\right).$$

Let $Y = \sum_{n=1}^{\infty} a_n X_n$ be an element of U_ε. Let $N_\lambda = \{n \in N\colon Q_n(a_n) \geqslant \lambda\}$. Since $Q_n(a_n) < \varepsilon$, we have $Q_n(ra_n) < \eta\lambda/2\varepsilon$. Obviously $\operatorname{card} N_\lambda \leqslant \varepsilon/\lambda$. Hence

$$\sum_{n\in N_\lambda} Q_n(ra_n) < \frac{\eta}{2}.$$

On the other hand, by Lemma 3, we have

$$\frac{\lambda}{\lambda+\varepsilon}\sum_{n\notin N_\lambda} a_n X_n \in \operatorname{conv} U_\lambda \subset U_{\eta/2}.$$

Thus

$$\sum_{n=1}^{\infty} Q_n(sa_n) < \eta,$$

q.e.d.

As a corollary we get

THEOREM 2. *If $X_1, X_2, \ldots$ are equidistributed and (L_X, τ_P) is locally convex, then*

(a) $\mathrm{E}\|X_1\|^p < \infty$, *for every* $0 < p < 1$

(b) *if, moreover,* $P(X_1 = 0) = 0$, *then*

$$\left\{Y \in L_X\colon\ P\left(\|Y\| > \frac{1}{201}\right) < \frac{1}{201}\right\}$$

is a bounded neighbourhood of zero in (L_X, τ_P).

Proof. (a) From Theorem 1 we know that (L_X, τ_P) is a Banach space. Thus, by a theorem of Nikishin ([5], Theorem 1)(¹) there exists an $A \in \mathscr{F}$, $P(A) \geqslant \frac{1}{2}$, such that $\mathrm{E}\|X_n\|^p \chi_A \leqslant c_p$. Since X_n are equidistributed and independent, it follows that $\mathrm{E}\|X_1\|^p < \infty$ for every $0 < p < 1$.

(b) In view of Lemma 1 it is enough to prove that

$$\forall_{\eta>0}\exists_{s>0} \quad sU_\varepsilon \subset U_\eta, \quad \text{where } \varepsilon = \frac{200}{201}.$$

Let us fix $\eta > 0$ and let us take $\lambda > 0$ such that $\operatorname{conv} U_\lambda \subset U_{\eta/2}$. Since $Q_1 = Q_2 = \ldots$ and $\lim_{t\to\infty} Q_1(t) = 1$, there exists an $r > 0$ such that

$$Q_n(t) < \varepsilon \Rightarrow Q_n(rt) < \frac{\eta\lambda}{2\varepsilon}.$$

Now we can rewrite the part of the previous proof starting from (*).

Remarks. The case of $H = R$ and $X_1, X_2, \ldots$ equidistributed symmetric random variables is better known.

1. It is proved in [1] that, for equidistributed real symmetric random variables, "locally convex" and "Banach" is the same for (L_X, τ_P) (see also [2] for a survey of results).

(¹) It is stated for $H = R$ and $\Omega = [0, 1]$ but, again, the proof can be just re-written to get what we want.

2. The case of $X_1, X_2, \ldots$ real symmetric equidistributed, with $P(X_1 = 1) = P(X_1 = -1) = \frac{1}{2}$, shows that $\{Y \in L_X\colon P(\|Y\| > \frac{1}{2}) < \frac{1}{2}\}$ is not, in general, a bounded neighbourhood for a locally convex τ_P. However, in this real case $\frac{1}{2}-\varepsilon$ works for every $\varepsilon > 0$. The last statement follows from the following estimate (obtained from Inequality II, p. 6, in [3] and from [6]): for every $0 < \lambda < \frac{1}{2}$, if $P(|\sum a_n X_n| > \varrho) < \lambda$, then

$$\sqrt{\sum a_n^2} < 4\frac{\varrho}{1-2\lambda}.$$

3. *For every $1 \leqslant p < 2$ there exists a sequence $X_1, X_2, \ldots$ of equidistributed symmetric independent real r.v.'s such that $\mathrm{E}|X_1|^p < \infty$, but (L_X, τ_P) is not locally convex*(²).

Indeed, let (l_i) be an increasing sequence of positive integers such that

$$(*) \qquad \sum_{i=1}^{\infty} l_i \left(\frac{l_{i-1}}{l_i}\right)^{2/p} i^{2/p} < \infty, \qquad l_0 = 1$$

(e.g. $l_i = 2^{c^{i+i_0}}$, $c > p/(2-p)$).

We put $a_i = (l_{i-1}/l_i)^2 i^2$, $i = 1, 2, \ldots$ Then

$$(**) \qquad \sum_{i=1}^{\infty} l_i a_i^{1/p} < \infty.$$

Let $g_1, g_2, \ldots$ be a sequence of independent symmetric random variables with distribution

$$P(g_i = l_i) = P(g_i = -l_i) = a_i = \frac{1}{2} - \frac{1}{2}P(g_i = 0)$$

and let $(g_{i1})_{i=1}^{\infty}, (g_{i2})_{i=1}^{\infty}, \ldots$ be independent copies of the sequence $(g_i)_{i=1}^{\infty}$. We put

$$X_j = \sum_{i=1}^{\infty} g_{ij}.$$

It follows from $(**)$ that $\mathrm{E}|X_j|^p < \infty$.

Let

$$A_i = \frac{i}{l_i}, \qquad k_i = \frac{1}{a_i i} = \left(\frac{l_{i-1}}{l_i}\right)^{-2} i^{-3}.$$

(²) We owe this remark to S. Kwapień.

For $0 < \delta < 1$ we have

$$P\left(\left|A_i \sum_{j=1}^{k_i} X_j\right| > \delta\right) \leqslant P\left(\left|A_i \sum_{j=1}^{k_i} \sum_{s=i}^{\infty} g_{sj}\right| > \frac{\delta}{2}\right) +$$

$$+ P\left(\left|A_i \sum_{j=1}^{k_i} \sum_{s=1}^{i-1} g_{sj}\right| > \frac{\delta}{2}\right) = \mathrm{I} + \mathrm{II},$$

$$\mathrm{I} \leqslant \sum_{j=1}^{k_i} \sum_{s=i}^{\infty} P(|g_{sj}| \neq 0) = k_i \sum_{s=i}^{\infty} a_s \leqslant 2Mk_i a_i = \frac{2M}{i} \to 0$$

((*) implies that $\sum_{s=i}^{\infty} a_s \leqslant Ma_i$ for some constant M),

$$\mathrm{II} \leqslant \left(\frac{2}{\delta}\right)^2 \mathrm{E}\left|A_i \sum_{j=1}^{k_i} \sum_{s=1}^{i-1} g_{sj}\right|^2 = \frac{4}{\delta^2} A_i^2 k_i \left(\sum_{s=1}^{i-1} l_s^2 a_s\right)$$

$$= \frac{4}{\delta^2} A_i^2 k_i \left(\sum_{s=1}^{i-1} l_{s-1}^2 s^2\right) \leqslant \frac{4}{\delta^2} M_1 A_i^2 k_i l_{i-1}^2 = \frac{4}{\delta^2} \frac{M_1}{i} \to 0$$

((*) implies that $\sum_{s=1}^{i-1} l_{s-1}^2 s^2 \leqslant M_1 l_{i-1}^2$ for some constant M_1).

Thus for every $0 < \delta < 1$ there exists an i such that

$$P\left(\left|A_i \sum_{j=1}^{k_i} X_j\right| > \delta\right) < \delta.$$

On the other hand, for every i we have

$$P\left(\frac{1}{i}\left|A_i \sum_{j=1}^{k_i} X_j + A_i \sum_{j=k_i+1}^{2k_i} X_j + \ldots + A_i \sum_{j=(i-1)k_i+1}^{ik_i} X_j\right| \geqslant \frac{1}{5}\right)$$

$$= P\left(\frac{A_i}{i}\left|\sum_{j=1}^{ik_i} X_j\right| \geqslant \frac{1}{5}\right) \geqslant \frac{1}{2} P\left(\frac{A_i}{i}\left|\sum_{j=1}^{ik_i} g_{ij}\right| \geqslant \frac{1}{5}\right)$$

$$\geqslant \frac{1}{4} P\left(\max_{1 \leqslant j \leqslant ik_i} \left|\frac{A_i}{i} g_{ij}\right| \geqslant \frac{1}{5}\right) = \frac{1}{4}\left(1 - (1 - 2a_i)^{ik_i}\right)$$

$$\geqslant \frac{1}{4}\left(1 - e^{-2ia_ik_i}\right) = \frac{1}{4}\left(1 - e^{-2}\right) \geqslant \frac{1}{5},$$

which shows that (L_X, τ_P) is not locally convex.

4. In this case we can give a simple sufficient condition to have (L_X, τ_P) locally convex, namely, for $t > t_0$, $tP(|X_1| > t)$ is decreasing.

It can be obtained by the calculating derivative of $Q(x)/x$. This condition is sufficient for $Q(x)/x$ to be decreasing in some small neighbourhood of zero, so that Q can be replaced by an equivalent convex function Q_1.

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