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CUMULANTS FOR STATIONARY MIXING RANDOM SEQUENCES AND APPLICATIONS TO EMPIRICAL SPECTRAL DENSITY

BY

P. DOUKHAN (ORSAY) AND J. R. LEÓN (CARACAS)

Abstract. We first give a central limit theorem for a stationary strongly mixing sequence without any mixing rate assumption following ideas of Rosenblatt [23]. We then study functional central limit convergence and law of the iterated logarithm for the empirical spectral density considered like a random element of some Sobolev space.

1. Introduction. In the first part of this work we use cumulant techniques derived from [23]. We first show moment sums inequalities for stationary random sequences with finite cumulant sums; we also show that those cumulant sums are finite under convenient mixing rate assumptions.

Recall that a discrete time process (X_n) is said to be *strongly mixing* with mixing coefficients α_t if $\alpha_t \to 0$ for $t \to \infty$ with

$$\alpha_t = \operatorname{Sup} |P(A \cap B) - P(A) P(B)|,$$

the supremum being taken over A, B, such that $A \in \sigma(\ldots, X_{p-1}, X_p)$, $B \in \sigma(X_{p+t}, X_{p+t+1}, \ldots)$, where p is any integer.

We derive a central limit theorem for a strongly mixing and stationary sequence under finiteness of cumulant sums and without any mixing rate assumption. We also obtain a law of iterated logarithm (LIL) assuming some mixing rate assumption weaker than usual (see [22]and a convergence rate in Lévy distance for central limit theorem.

In the second part we study the behaviour of empirical spectral density I_n of a centered stationary strongly mixing sequence $(X_n)_{n \geq 0}$:

$$I_n(g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\lambda) \{R_0(n) + 2 \sum_{k=1}^{n-1} R_k(n) \cos k\lambda\} \, d\lambda$$

with

$$R_k(n) = \frac{1}{n}\sum_{j=1}^{n-k} x_j x_{j+k}.$$

Here g belongs to a Sobolev space H_s defined below and I_n is considered like a random element of the dual space H_{-s} of H_s. We write I for the element of H_{-s} defined by the spectral density f of the sequence (x_n). Without mixing assumption we show that $\mathrm{E}\{n\|I_n - I\|^2_{-s}\}$ is bounded (using cumulant sums assumptions). Afterwards we show functional convergence of $\sqrt{n}(I_n - I)$ to a Gaussian random variable of H_{-s} under strongly mixing rate assumption. The problem is that $I_n - I$ is not a sum of mixing random elements of H_{-s} except if $I_n - I$ acts only on the finite-dimensional subspace of H_s of l-th degree trigonometric polynomials. We also prove a bounded law of iterated logarithm with a strong mixing rate assumption. To obtain it we use a decomposition of $I_n - I$ into an $l(n)$-dimensional sum of mixing elements and a little part. The mixing rate of those elements is $\mathrm{Inf}\{\alpha_{n-l(n)}, 1\}$. We give a Lévy speed of convergence for them and we conclude with usual techniques (see [9]). This leads to a result similar to those of [2] concerning almost sure behaviour of $\mathrm{Max}\{|R_k(n) - r_k|;\ 0 \leqslant k \leqslant w_n\}$. We also show a uniform law of iterated logarithm for $I_n - I$ on the Sobolev space H_s.

Classical results concerning Gaussian processes can be found in [3], like LIL for $I_n(g) - I(g)$ with $g \in L^2(f^2(\lambda)\,d\lambda)$. Rosenblatt [23] and Dalhaus [7] show a central limit theorem. Rosenblatt uses a kernel estimate and Dalhaus uses more general spectral estimates for a fixed function g. The only functional results that we know concerning $I_n - I$ use a martingale approach [4]; the author shows a uniform LIL for a class of functions with rapidly decreasing Fourier's coefficients.

We now investigate fields of applications for this work. First of all note that $R_k(n) = I_n(e_k)$ if $e_k(x) = \cos kx$, so results for empirical covariances are obtained. Then Whittle's approximate log-likehood of stationary Gaussian processes [25] can be studied with

$$\ln(f) = -\frac{1}{2}\left\{I_n\left(\frac{1}{f}\right) + n\log\sigma_0^2\right\}.$$

In chapter XIV of [3] the authors give results concerning identification of ARMA processes (see theorem 3.5); they use a uniform LIL for I_n relative to Gaussian processes proved by Bouaziz; our results seem to be tractable here. Now, in the convergence of approximate likehood used for estimation of parameters of spectral density, $I_n(f)$ is the main term [15]; for this likehood is maximized

$$D(f_\theta, I_X) = \int_{-\pi}^{\pi} \log f_\theta(\omega) + \frac{X^{(\omega)}}{f_\theta(\omega)}\,d\omega,$$

where I_X is the periodogram and f_θ the spectral density of the process; our results give a LIL for D. Finally, Bouaziz [5] gives other applications of our results.

2. Cumulants. Let $\{A_1, \dots, A_k\}$ be centered real-valued random variables. We write $A = (A_1, \dots, A_k)$ and $m_\nu = \mathrm{E} A_1^{\nu_1} \dots A_k^{\nu_k}$ if $\nu = (\nu_1, \dots, \nu_k) \in \boldsymbol{N}^k$.

If $\varphi(t) = \mathrm{E} e^{itA}$, $t \in \boldsymbol{R}^k$, is the characteristic function of A, we obtain Taylor expansions of $\varphi(t)$, and $\log \varphi(t)$ if A admits n-th order moments:

$$\varphi(t) = \sum_{|\nu| < n} \frac{i^{|\nu|}}{\nu!} m_\nu t^\nu + O(|t|^n),$$

$$\log \varphi(t) = \sum_{|\nu| < n} \frac{i^{|\nu|}}{\nu!} c_\nu t^\nu + O(|t|^n) \qquad \text{for } t \to 0.$$

Here $\nu = (\nu_1, \dots, \nu_k) \in \boldsymbol{N}^k$, $t = (t_1, \dots, t_k) \in \boldsymbol{R}^k$ and $|\nu| = \nu_1 + \dots + \nu_k$, $\nu! = \nu_1! \dots \nu_k!$, $t = t_1^{\nu_1} \dots t_k^{\nu_k}$.

The coefficients c_ν are called *cumulants* of A.

Taylor developments lead Leonov and Shyriaev [19] to write

$$m_\nu = \sum_{\lambda_1 + \dots + \lambda_q = \nu} \frac{1}{q!} \frac{\nu!}{\lambda_1! \dots \lambda_q!} \prod_{j=1}^{q} c_{\lambda_j}, \tag{1}$$

$$c_\nu = \sum_{\lambda_1 + \dots + \lambda_q = \nu} \frac{(-1)^{q-1}}{q} \frac{\nu!}{\lambda_1! \dots \lambda_q!} \prod_{j=1}^{q} m_{\lambda_j}. \tag{2}$$

(Sums are taken for every integer q and $\lambda_1, \dots, \lambda_q \in \boldsymbol{N}^k$ such that $\lambda_1 + \dots + \lambda_q = \nu$.)

In the following we also write $c(A_1, \dots, A_k) = c_\nu$ and $m(A_1, \dots, A_k) = m_\nu$ for $\nu = (1, \dots, 1)$, and, if (X_t) is the k-th order process, $c(t_2, \dots, t_k) = c(X_0, X_{t_2}, \dots, X_{t_k})$.

Proposition 2.1. *Let* $(X_t)_{t \in \boldsymbol{N}}$ *be a* $2p$-th order stationary centered process satisfying

$$C_k = \sum_{(s_1, \dots, s_{k-1}) \in \boldsymbol{N}^{k-1}} |c(s_1, \dots, s_{k-1})| < \infty \qquad \textit{for } k = 2, 3, \dots, 2p.$$

Then

$$\mathrm{E}(X_1 + \dots + X_n)^{2p} \leqslant \sum_{q=1}^{p} n^q \gamma_q,$$

where

$$\gamma_q = \sum_{|\pi| = 2p} \frac{(2p)!}{\pi!} C(\pi), \qquad C(\pi) = C_{p_1}, \dots, C_{p_q} \qquad \textit{for } \pi = (p_1, \dots, p_q) \in \boldsymbol{N}^q.$$

Remark. For $p = 1, 2$ more precise statements are classically obtained: $\mathrm{E}(X_1 + \ldots + X_n)^2 \leqslant nC_2$, $\mathrm{E}(X_1 + \ldots + X_n)^4 \leqslant nC_4 + 3n^2 C_2^2$.

Proof. Let $S_n = X_1 + \ldots + X_n$. We compute

$$\mathrm{E}\, S_n^{2p} = \sum_I \mathrm{E} X_{I(1)} \ldots X_{I(2p)},$$

where I runs over the set of maps from $\{1, \ldots, 2p\}$ to $\{1, \ldots, n\}$.

If $L = \{l_1, \ldots, l_r\} \subset \{1, \ldots, 2p\}$, we write $c(I, L) = c(X_{I(l_1)}, \ldots, X_{I(l_r)})$ and formula (1) implies

$$\mathrm{E}\, S_n^{2p} = \sum_I \sum_{q \geqslant 1} \frac{1}{q!} \sum_{p_1 + \ldots + p_q = 2p} T_{p_1 \ldots p_q} \quad \text{with } T_{p_1 \ldots p_q} = \sum_I \sum_{\{L_j\}} \prod_{j=1}^{q} c(I, L_j).$$

The number of partitions $\{L_1, \ldots, L_q\}$ is $q!(2p)!/p_1! \ldots p_q!$, so

$$|T_{p_1, \ldots, p_q}| \leqslant \prod_{j=1}^{q} \Big\{ \sum_{1 \leqslant t_1 \leqslant \ldots \leqslant t_{p_j} \leqslant n} |c(X_{t_1}, \ldots, X_{t_p})| \Big\} \frac{q!(2p)!}{p_1! \ldots p_q!}.$$

The result follows now from the stationarity of (X_t).

Proposition 2.2. *If $(X_t)_{t \in N}$ is a p-th order stationary centered and strongly mixing process such that there is a $\delta \in]0, 1]$ satisfying*

(i) $\exists_{C > 0}\, \forall_{n \in N} \quad \mathrm{E}|X_n|^{p+\delta} < C$,

(ii) $\displaystyle\sum_{r=0}^{\infty} (r+1)^{k-2} \alpha_r^{\delta/(k+\delta)} < \infty$ *for* $k = 1, \ldots, p$,

then

$$C_p = \sum_{t_1, \ldots, t_{p-1}} |c(t_1, \ldots, t_{p-1})| < \infty.$$

Remark 2.2.1. Propositions 2.1 and 2.2 imply a moment inequality (cf. [9]) already known, but Rosenblatt ([24], p. 1179) gives examples of processes with finite cumulants sums and with $\alpha_n \geqslant n^{-\varepsilon}$ for arbitrary $\varepsilon > 0$.

Proof. If $1 \leqslant t_1 \leqslant \ldots \leqslant t_p$ and $r = t_{l+1} - t_l = \mathrm{Max}\{t_{j+1} - t_j; j = 1, \ldots, p-1\}$, then by (2) we get

$$c(X_{t_1}, \ldots, X_{t_p}) \leqslant |\mathrm{E}\, X_{t_1} \ldots X_{t_p} - \mathrm{E}\, X_{t_1} \ldots X_{t_l} \mathrm{E}\, X_{t_{l+1}} \ldots X_{t_p}| + R,$$

$$R \leqslant \sum_q \frac{1}{q} \sum_{\lambda_1 + \ldots + \lambda_q = \nu, \lambda_j \neq \nu} \prod_{j=1}^{q} |m_{\lambda_j}| \text{ for } \nu = (1, \ldots, 1)$$

and $A = (X_{t_1}, \ldots, X_{t_p})$.

By (1) we have

$$R \leqslant \sum_q \frac{1}{q} R(p) \operatorname*{Max}_{\lambda_j \neq \nu} \prod_{j=1}^{q} |C_{\lambda_j}|$$

for some constant $R(p)$ only depending on p.

From another hand a result by Davydov [8] shows that the first term is majorized by $10\alpha_r^{\delta/(p+\delta)}C^{p/(p+\delta)}$ and we have

$$C_p \leqslant 10p!\,C^{p/(p+\delta)}\sum_{r=0}^{\infty}(r+1)^{p-2}\alpha_r^{\delta/(p+\delta)}+R(p)\sum_{q}\frac{1}{q}\sum_{\substack{p_1+\ldots+p_q=p\\ p_j\neq p}}C_{p_1}\ldots C_{p_q}.$$

The result follows by induction.

THEOREM 2.3. *Let $(X_t)_{t\in N}$ be a fourth order stationary centered and strongly mixing real process such that*

$$K=\sum_{k=0}^{\infty}k\,|c(k)|<\infty \quad\text{and}\quad C=\sum_{i,j,k}|c(i,j,k)|<\infty.$$

Then, if there are some positive constants A, $\gamma<\gamma_1$ with $\alpha_t<At^{-\gamma_1}$, and if

$$\sigma^2=c(0)+2\sum_{k=1}^{\infty}c(k)\neq 0,$$

we have

$$\sup_{z}|P(n^{-1/2}(X_1+\ldots+X_n)<z)-P(Y<z)|<\text{Const}\,n^{-\varepsilon},\qquad \varepsilon=3\gamma/(24+36\gamma),$$

where Y denotes a centered Gaussian random variable with variance σ^2.

Remarks. 2.3.1. Without any mixing rate assumption we also can get a CLT analogous to [23], chap. III, Pb. 4; it was shown by [24] that the assumptions are really weaker than usual condition

$$\sum_{n=0}^{\infty}\alpha_n^{(1+\delta)/(2+\delta)}<\infty$$

(see [14]). Moreover, in the example by Rosenblatt, $c(n)=n^{-1-\beta}$, so that the condition of Remark 2.3.3 is satisfied and the result is valid here.

2.3.2. Ibragimov [16] shows that if (X_n) is a Gaussian process, our cumulant assumptions imply mixing but no explicit estimation of the mixing rate in the case of CLT. To verify the assumptions of theorem 2.3 we have still some additional smoothness condition for spectral density to add ([16], Chap. 6, Théorème 8). Note that finiteness of K implies differentiability.

2.3.3. The assumption $K<\infty$ can be weakened by

$$C_2=\sum_{n=0}^{\infty}|c(n)|<\infty \quad\text{and}\quad \frac{1}{k}\sum_{n=0}^{k}n\,|c(n)|\xrightarrow[k\to\infty]{}0.$$

2.3.4. Peligrad [20] shows under ϱ-mixing assumption that

$$\sum_{i=0}^{\infty}\varrho^{2/k}(2^i)<\infty$$

implies $C_k<\infty$. This assumption is weaker than that usual for α-mixing.

Proof. Let f: $\boldsymbol{R} \to \boldsymbol{R}$ be a thrice continuously differentiable function with bounded derivatives. We estimate

$$\Delta = \left|\mathrm{E} f\left(\frac{X_1 + \dots + X_n}{\sqrt{n}}\right) - f(Y)\right|.$$

Condition $K < \infty$ implies $C_2 < \infty$ and σ^2 is finite. Set $p = p(n)$, $q = q(n)$, $l = [n/(p+q)]$ (integer part) and

$$\lim_{n \to \infty} \frac{p(n)}{q(n)} = \infty .$$

We group the variables as follows:

$$\underset{p}{I_1}\underset{q}{///}\underset{p}{I_2}\underset{q}{///} - \dots \underset{q}{///}\underset{p}{I_p}\underset{(n-l(p+q))\,\text{terms}}{///} .$$

So $\operatorname{Card}(I_1) = \dots = \operatorname{Card}(I_l) = p$, and $J = \{1, 2, \dots, n\} \setminus (I_1 \cup \dots \cup I_l)$ satisfies $\operatorname{Card} J = n - lp$. We now set

$$u_h = \frac{1}{\sqrt{n}} \sum_{i \in I_h} X_i \quad (h = 1, \dots, l) \qquad \text{and} \qquad v = \frac{1}{\sqrt{n}} \sum_{i \in J} X_i .$$

We have $\Delta \leqslant a + b + c$ and

$$a = |\mathrm{E} f(u_1 + \dots + u_l + v) - f(u_1 + \dots + u_l)|,$$

$$b = |E f(u_1 + \dots + u_l - f(y_1 + \dots + y_l)|,$$

$$c = |\mathrm{E} f(y_1 + \dots + y_l) - f(Y)|,$$

where $y_1, \dots, y_l$ are centered i.i.d. Gaussian variables with the variance σ_p^2 that u_1.

Write $M_j = \sup\{|f^{(j)}(t)|;\ t \in \boldsymbol{R}\}$, $j = 0, 1, 2, 3$. We have

(i) $$a^2 \leqslant M_1^2 n^{-1} \sum_{i,j \in J} \mathrm{E} X_i X_j \leqslant M_1^2 \frac{n - lp}{n} C_2 \leqslant \frac{q}{p+q} M_1^2 C_2 \xrightarrow[n \to \infty]{} 0,$$

(ii) $$b \leqslant \sum_{j=1}^{l-1} b_j \text{ with } b_j = |\mathrm{E} f(Z_j + u_j) - f(Z_j + y_j)|$$

$$\text{and } Z_j = u_1 + \dots + u_{j-1} + y_{j+1} + \dots + y_l .$$

Taylor's formula implies:

$$f(Z_j + u_j) = f(Z_j) + u_j f'(Z_j) + u_j^2 f''(Z_j)/2 + u_j^3 f'''(\xi_j)/6,$$

$$f(Z_j + y_j) = f(Z_j) + y_j f'(Z_j) + y_j^2 f''(Z_j)/2 + y_j^3 f'''(\eta_j)/6.$$

So

$$b_j \leqslant |\mathrm{E}\, u_j f'(Z_j)| + \frac{1}{2}|\mathrm{E}\, u_j^2 f''(Z_j) - \mathrm{E}u_j^2\, \mathrm{E}\, f''(Z_j)| + \frac{1}{3} M_3\, \mathrm{E}\, |u_j|^3.$$

But proposition 2.1 shows that

$$\mathrm{E}\, |u_1|^4 \leqslant \frac{1}{n^2}(pC_4 + 3p^2 C_2^2) \leqslant \mathrm{Const}\, p^2/n^2.$$

From another hand, mixing inequality shows [7] that, for $0 < \varrho < \frac{1}{2}$,

$$b_j \leqslant 10\alpha_q^{3/4-\varrho}(\mathrm{E}u_j^4)^{1/4} M_1 + 5\alpha_q^{1/2-\varrho}[\mathrm{E}(u_j^2 - \mathrm{E}u_j^2)^2]^{1/2} M_2 + (1/3M_3)(\mathrm{E}u_j^4)^{3/4},$$

$$b_j \leqslant 10\alpha_q^{3/4-\varrho}(\mathrm{E}u_j^4)^{1/4} M_1 + 10\alpha_q^{1/2-\varrho}(\mathrm{E}u_1^4)^{1/2} M_2 + M_3 (\mathrm{E}u_1^4)^{3/4}/3$$
$$\leqslant Ct\left\{\alpha_q^{3/4-\varrho}\sqrt{p/n} + \alpha_q^{1/2-\varrho}\frac{p}{n} + \left(\frac{p}{n}\right)^{3/2}\right\}.$$

Hence $b \leqslant Ct\,\{\sqrt{l}\,\alpha_q^{3/4-\varrho} + \alpha_q^{1/2-\varrho} + 1/\sqrt{l}\}$; we see that $b \to 0$ if

$$l(n)\,\alpha_{q(n)} \xrightarrow[n\to\infty]{} 0, \qquad l(n) \xrightarrow[n\to\infty]{} \infty \qquad \text{for } \varrho = 1/4.$$

We have

(iii) $c \leqslant |l\sigma_p^2 - \sigma^2|\, M_2$ and $\sigma_p^2 = \frac{1}{n}\{\mathrm{E}X_0^2 + 2\sum_{k=1}^{p}(p-k+1)\,\mathrm{E}X_0 X_k\}$,

so (see 2.3.3)

$$|l\sigma_p^2 - \sigma^2| \leqslant \frac{l}{n}\sum_{k=1}^{p} k\mathrm{E}X_0 X_k + \frac{q}{p+q}\sigma^2.$$

The assumption $K < \infty$ implies $c \to 0$ (see 2.3.3) if $\lim q/p = 0$.

The CLT follows from a choice of $p(n)$ and $q(n)$ satisfying

$$\lim_{n\to\infty}\frac{p(n)}{q(n)} = \lim_{n\to\infty} q(n) = \infty, \qquad \lim_{n\to\infty}\frac{q(n)}{p(n)}\alpha_{q(n)} = 0.$$

To prove 2.3 we can use functions f such that $M_j \leqslant C_1 \varepsilon^{-j}$ (like in [17]) and the left member of the inequality is majorized by $C_2\varepsilon + \Delta(\varepsilon)$, where $\Delta(\varepsilon)$ is estimated before.

THEOREM 2.4. *If* $(X_t)_{t\in N}$ *is a strongly mixing process satisfying assumptions of theorem* 2.3 *with* $\gamma = 1$, *then*

$$\overline{\lim_{n\to\infty}}\frac{|X_1 + \ldots + X_n|}{2\sigma_n^2 \log\log\sigma_n^2} = 1 \ \textit{a.s.}, \qquad \textit{where } \sigma_n^2 = \mathrm{E}(X_1 + \ldots + X_n)^2.$$

Sketch of the proof. We follow the lines of the proof of [22]. Thus we show that

(a) $$|P(X_1+\ldots+X_n<z\sigma_n)-\varphi(z)|\leqslant Cn^{-\varepsilon}.$$

Indeed, it follows from 2.4 and the fact that Levy me distance of two Gaussian variables $N(0,\sigma_1^2)$, $N(0,\sigma_2^2)$ is of order $|\sigma_1-\sigma_2|^{1/3}$, and $|\sigma_n^2/n-\sigma^2|<4K/n$.

(b) If $\alpha_n<Cn^{-\gamma}$, $\gamma>1$,

$$P\big(\operatorname*{Max}_{1\leqslant k\leqslant n}(X_1+\ldots+X_k)\geqslant x\big)\leqslant 2P(X_1+\ldots+X_n\geqslant x-2\sigma_n)+Cn^{-\varepsilon}$$

for $\varepsilon<\gamma-1$.

(c) If $t=b\sqrt{\log\log\sigma_n^2}$, then

$$\exists_{C_1,C_2}\, C_1(\log\sigma_n^2)^{-b^2}\Big/\Big(t+\frac{1}{t}\Big)\leqslant P(X_1+\ldots+X_n\geqslant b\sqrt{2\sigma_n}\log\log\sigma_n^2)$$
$$\leqslant C_2(\log\sigma_n^2)^{-b^2}/t.$$

The proofs of (b) and (c) are analogous to those of [22]; proof of theorem then follows as in [21].

Remark 2.5.1. Kuelbs and Philipp [18] show a strong invariance principle which implies LIL under stationarity assumption and $\alpha_n\leqslant Cn^{-(1+\varepsilon)(1+2/\delta)}$ for some $\varepsilon,\delta>0$ and $E|X_1|^{2+\delta}<\infty$.

For $\delta=2$ it is Reznik's [22] assumption. We replace here a strong mixing rate assumption by mixing rate and cumulants assumptions.

3. Empirical spectral density.

In the following $(X_t)_{t\in N}$ is a fourth order stationary centered and strongly mixing real random process. We define empirical covariances $R_k(n)$ and periodogram I_n of (X_t):

$$R_k(n)=\frac{1}{n}\sum_{j=1}^{n-k}X_jX_{j+k},\qquad k=0,1,\ldots,n-1;$$

$$I_n(\lambda)=\frac{1}{2\pi}\{R_0(n)+2\sum_{k=1}^{n-1}R_k(n)\cos k\lambda\},\qquad -\pi\leqslant\lambda\leqslant\pi.$$

Let $r_k=\mathrm{E}X_0X_{|k|}$, $k=0,\pm1,\pm2,\ldots$ We make the following assumption:

(A.1) $$\varrho=\sum_{k=1}^{\infty}r_k^2<\infty.$$

The spectral density f of (X_t) is the even L^2- function defined by

$$f(\lambda)=\sum_{k=-\infty}^{+\infty}r_ke^{ik\lambda}=r_0+2\sum_{k=1}^{\infty}r_k\cos k\lambda\in L^2[-\pi,\pi].$$

We define

$$I_n(g) = \int_{-\pi}^{\pi} g(\lambda) I_n(\lambda)\, d\lambda, \qquad g \in L^2[-\pi, \pi].$$

Note that $R_k(n)$ estimates r_k, and $I_n(g)$ estimates

$$I(g) = \int_{-\pi}^{\pi} g(\lambda) f(\lambda) \frac{d\lambda}{2\pi}.$$

If g is odd, then $I(g) = I_n(g) = 0$ and else for

$$\bar{g}(x) = \tfrac{1}{2}\big(g(x) + g(-x)\big), \qquad I(\bar{g}) = I(g),\ I_n(\bar{g}) = I_n(g);$$

so we only consider even functions $g \in L^2[-\pi, \pi]$.

Our aim is to study the functional asymptotic behaviour of $I_n(g)$. We consider the Sobolev space H_s for $s > 1$:

$$H_s = \{g \in L^2[-\pi, \pi];\ g(-x) = g(x),\ \sum_{k=1}^{\infty} k^s |\hat{g}(k)|^2 < \infty\}$$

with

$$\hat{g}(k) = \int_{-\pi}^{\pi} \cos kx g(x) \frac{dx}{2\pi}, \qquad \hat{g}(0) = \int_{-\pi}^{\pi} g(x) \frac{dx}{2\pi}.$$

This space is Hilbert with the norm

$$\|g\|_s = \{|\hat{g}(0)|^2 + 2 \sum_{k=1}^{\infty} k^s |\hat{g}(k)|^2\}^{1/2}.$$

The dual space H_{-s} of H_s relatively to the duality

$$(g_1, g_2) = \int_{-\pi}^{\pi} g_1(x) g_2(x) \frac{dx}{2\pi}$$

has norm $\|\cdot\|_{-s}$,

$$\|T\|_{-s} = \sup\{|T(g)|;\ \|g\|_s \leqslant 1\} = \{|T(e_0)|^2 + 2 \sum_{k=1}^{\infty} k^{-s}\, T(e_k)^2\}^{1/2}$$

with $e_k(x) = \cos kx$, $k = 0, 1, \ldots$

We write $B_s = \{g \in H_s;\ \|g\|_s \leqslant 1\}$. Note that $I_n,\ I \in H_{-s}$. Let $Y_{j,k} = X_j X_{j+k} - r_k$. We have

PROPOSITION 3.1. *If* $(X_t)_{t \in N}$ *is a fourth-order stationary process satisfying Assumptions* (A.1) and (A.2),

$$\text{(A.2)} \qquad d(k) = \sum_{j=1}^{\infty} |c(Y_{0,k}, Y_{j,k})| \leqslant \gamma < \infty$$

for some constant $C > 0$, $\quad k = 0, 1, \ldots,$

then

$$\mathrm{E}\,\|I_n - I\|_{-s}^2 = \mathrm{E}\sup\{|I_n(g) - I(g)|^2;\ g \in B_s\} \leqslant C_1/n$$

for some constant C_1, *if* $s > 1$.

Proof. $I_n(g) - I(g) = T_1 + T_2 + T_3$ with $g \in B_s$,

$$T_1 = \hat{g}(0)\,A_{0,n} + 2\sum_{k=1}^{n-1}\hat{g}(k)\,A_{k,n}, \qquad \text{where } A_{k,n} = R_k(n) - \mathrm{E}R_k(n),$$

$$T_2 = 2\sum_{k=1}^{n-1}(\mathrm{E}R_k(n) - r_k)\,\hat{g}(k), \qquad T_3 = 2\sum_{k=n}^{\infty}\hat{g}(k)\,r_k.$$

So

$$T_2 = 2\sum_{k=1}^{n-1}\frac{k}{n}r_k\,\hat{g}(k) \quad \text{and} \quad |T_2| \leqslant 2\varrho/\sqrt{n}, \quad |T_3| \leqslant 2\varrho/\sqrt{n}.$$

From another hand,

$$\mathrm{E}\sup_{g} T_1^2 \leqslant \mathrm{E}A_{0,n}^2 + 2\sum_{k=1}^{n-1}k^{-s}\,\mathrm{E}A_{k,n}^2 \leqslant \Big(1 + 2\sum_{k=1}^{\infty}k^{-s}\Big)\gamma/n.$$

Remarks. 3.1.1. Leonov and Shyriaev's [19] formula for products implies $c(Y_{0,k}, Y_{j,k}) = c(k, j, j+k) + r_j^2 + r_{j+k}r_{j-k}$, hence.

$$d(k) \leqslant \sum_{k=0}^{\infty}|c(k, j, j+k)| + 3\sum_{j=1}^{\infty}r_j^2.$$

Assumptions (A.1) and (A.2) are satisfied if (A.1) is realized and $C_4 < \infty$.

3.1.2. If $(X_t)_{t\in N}$ is strongly mixing with

$$A = \sum_{n=0}^{\infty}\alpha_n^{\delta/(2+\delta)} < \infty \quad \text{and} \quad \mathrm{E}\,|X_0|^{4+2\delta} < \infty,$$

assumptions (A.1) and (A.2) are satisfied with

$$\varrho \leqslant 100A^2(\mathrm{E}\,|X_0|^{2+\delta})^{4/(2+\delta)}, \qquad \gamma = 40A(\mathrm{E}\,|X_0|^{4+2\delta})^{4/(4+2\delta)}.$$

PROPOSITION 3.2. *If* $(X_t)_{t\in N}$ *is fourth order stationary strognly mixing and satisfies assumptions* (A.1) *and* "$C_4 < \infty$" *and one of the following*:

(a) $\sum_{i,j,k} i\,|c(i, j, k)| < \infty$, $C_8 < \infty$ *and* (X_t) *is stationary to order* 8,

(b) $\sum_{n=0}^{\infty}\alpha_n^{\delta/(4+\delta)} < \infty$ *for some* $\delta \in]0, 2]$ *with* $\mathrm{E}\,|X_0|^{4+\delta} < \infty$,

then $\sqrt{n}(I_n(g) - I(g))$ *converges weakly to a centered Gaussian random variable with variance* $\sigma^2(g)$ *for* $g \in H_s$ *(see formula* $(*)$*) if* $\sigma^2(g) > 0$.

Proof. $n(I_n(g)-I(g)) = A+B+C$ with

$$A = \sum_{j=1}^{n} \{\hat{g}(0)\, Y_{j,0} + 2\sum_{k=1}^{l} Y_{j,k}\,\hat{g}(k)\},$$

$$B = -2\sum_{k=1}^{l} \hat{g}(k) \sum_{j=n-k}^{n} Y_{j,k},$$

$$C = 2\sum_{k=l}^{n-1} \hat{g}(k) \sum_{j=1}^{n-k} Y_{j,k} \quad \text{for any } l \leqslant \frac{n}{2}-1.$$

We note that $\mathrm{E}C^2 \leqslant 4n\gamma \|g\|_s^2\, l^{1-s}(s-1)^{-1}$, $\mathrm{E}B^2 \leqslant 4l^2 \|g\|_s^2\, \mathrm{E}X_0^4 (s-1)^{-1}$ using cumulants. Moreover $A/\sqrt{n}$ satisfies assumptions of the central limit result 2.3 under hypothesis (a), with the help of transformation formula for products of cumulants of [19], and assumptions of the result from [10] under hypothesis (b). The limit Gaussian variable of $A/\sqrt{n}$ has variance σ_l^2 converging to $\sigma^2(g)$ for $l \to \infty$, where

$$(*) \qquad \sigma^2(g) = 2\pi\Big[\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} g(x)g(y) f_4(x,\, -y,\, y)\,dx\,dy + 2\int_{-\pi}^{\pi} g^2(x) f^2(x)\,dx\Big]$$

and

$$f_4(x,\, y,\, z) = \frac{1}{(2\pi)^3} \sum_{\alpha,\beta,\gamma=-\infty}^{+\infty} c(\alpha,\, \beta,\, \gamma) \exp(-i(\alpha x + \beta x + \gamma z))$$

are the cumulant spectra of fourth order [6].

Dalhaus's method [7] completes the proof.

THEOREM 3.3. *If the assumptions of proposition* 3.2 *are satisfied and* $s > 1$, *then the random sequence* $\sqrt{n}(I_n - I) \in H_{-s}$ *converges weakly in the quotient space* H_{-s}/N_0 *to a Gaussian random variable* Y *with covariance* Γ:

$$\Gamma(g,\, g) = \mathrm{E}(Y(g))^2 = 2\pi\Big[\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} g(x)g(y) f_4(x,\, -y,\, y)\,dx\,dy + 2\int_{-\pi}^{\pi} g^2(x) f^2(x)\,dx\Big], \qquad g \in H.$$

Here N_0 *denotes the subspace* $\{g \in H_{-s},\ \Gamma(g,\, g) = 0\}$.

Remark 3.3.1. In the Gaussian case $f_4 = 0$, so that $N_0 = \{0\}$ if f is not zero a.s.

Proof. The assumptions $C_4 < \infty$ and $\sum r_k^2 < \infty$, $k = 0, 1, 2, \ldots$, show that

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |f_4(x,\, -y,\, y)|\,dx\,dy < \infty, \quad \int_{-\pi}^{\pi} f^2(x)\,dx < \infty.$$

The covariance Γ satisfies $|\Gamma(g, g)| \leqslant G^2(s)\|g\|_s^2$ for some constant $G(s)$ only depending on $s > 1$. We can consider the completion H of H_{-s} with norm $\|g\|_\Gamma = \sqrt{\Gamma(g, g)}$ by the injection i: $H \to H_{-s}$, $i(h)(g) = \Gamma(h, g)$. Let $(h_n)_{n \geqslant 0}$ be a complete orthonormal system for H. We see that

$$\sum_{n=0}^{\infty} \Gamma(h_n, g)^2 \leqslant G^2(s)\|g\|_s^2,$$

so Giné ([13], proof of lemma 3.2) shows that $Y \in H_{-s}$ (with the help of Kolmogorov's inequality).

Using de Acosta's method [1] we have to show that $\sqrt{n}(I_n - I)$ is flattly concentrated because of the convergence of finite dimensional repartitions by proposition 3.2, because of linearity (lemma 3.3 of [13] makes use of it).

Let p_l be the orthogonal projection of H_{-s} on F_l, the closed-subspace spanned by $\{e_k;\, k > l\}$. We see that

$$\mathrm{E}\,\|p_l(\sqrt{n}(I_n - I))\|_{-s}^2 \leqslant 2l^{1-s} C_4 (s-1)^{-1},$$

which completes the proof.

Let now

$$Z_{j,l} = \sum_{k=0}^{l} Y_{j,k}\, e_k, \qquad A_{n,l} = \sum_{j=i}^{n} Z_{j,l}.$$

We follow lines of the proof of [11], which is analogous to that of theorem 2.3 in the l-dimensional subspace E_l of H_{-s}, to show the following

Lemma 3.4. *If* $\sum k\,|c(k)| < \infty$, $C_4 < \infty$, $C_8 < \infty$ *and* $\alpha_n \leqslant Cn^{-\tau}$, *then*

$$L_{n,l} = \operatorname*{Sup}_{t>0} \left| P\left(\frac{1}{\sqrt{n}}\|A_{n,l}\|_{-s} > t\right) - P(\|Y_l\|_{-s} > t)\right| \leqslant \mathrm{Const}\, l^{3/4}\, n^{(b-1)/12}$$

with Y_l *the projection of* Y *on* E_l, *uniformly for* $l \leqslant n^\beta$ *for* $\beta < b \leqslant 1/10$ *such that* $\tau \geqslant 1/12b$.

Proof. For any $l \leqslant n^\beta$ the random sequence $(Z_{1,l}, \ldots, Z_{n,l}, \ldots)$ is strongly mixing with mixing coefficient $\tilde{\alpha}_n = 1$ for $n \leqslant l$ and $\tilde{\alpha}_n = \alpha_{n-l}$, else. For suitable functions f such that $\|D^j f\|_\infty \leqslant C t \varepsilon^{-j}$ $(j = 0, 1, 2, 3)$, we estimate $\Delta = |\mathrm{E} f(A_{n,l}/\sqrt{n}) - f(Y_l)|$ so that $L_{n,l} \leqslant Ct\,\{\varepsilon + \Delta\}$. This computation is based on estimates like

$$\mathrm{E}\,\|Z_{1,l} + \ldots + Z_{p,l}\|_{-s}^4 \leqslant \mathrm{Const}\, C_8\, l^3\, p^2.$$

It leads to

$$\Delta \leqslant \mathrm{Const}\left\{\varepsilon^{-1}\left(\left(\frac{lq}{p}\right)^{1/2}+\alpha_{q-l}^{4}\left(\frac{nl}{p}\right)^{1/2}\right)+\varepsilon^{-2}\left(\alpha_{q-l}^{4}\, l^{3/2}+l^{2}\frac{q}{p}\right)+\right.$$

$$\left.+\varepsilon^{-3}\, l^{9/4}\left(\frac{p}{n}\right)^{1/2}\right\}.$$

Put $\varepsilon = l^{3/4}\, n^{(b-1)/12}$. We get

LEMMA 3.5 [9]. *If* $\mathrm{E}\,|X_0|^{4+2\delta} < \infty$ *for some* $0 < \delta \leqslant 2/3$ *and* $\alpha_n \leqslant \mathrm{Const}\, n^{-\tau}$ *with* $\tau > 2(1+2/\delta)$, *then, for any* β, $\varrho > 0$ *such that* $\tau\beta > 1+\varrho$, $\beta+\varrho < \delta/(2(2+\delta))$, *we have*

$$\boldsymbol{P}\big(\operatorname*{Max}_{k\leqslant n}\|A_{k,l}\|_{-s} > x\big) \leqslant 2\boldsymbol{P}\big(\|A_{n,l}\|_{-s} > x-20\sqrt{n}\,\sigma_l\big)+n^{-\varrho}$$

with $\sigma_l = \operatorname*{Sup}_{n>0} n^{-1/2}\,\|A_{n,l}\|_{-s}$.

Note that $\sigma_l \leqslant \Sigma = (4/(s-1))\, C_4$.

Write now $n(I_n-I) = A_{n,l}+B_{n,l}+C_{n,l}$ like in the proof of proposition 3.2; we see that $\mathrm{E}\,\|B_{n,l}\|_{-s}^{4} \leqslant \mathrm{Const}\,\mathrm{E}X_0^8\, l^4$, $\mathrm{E}\,\|C_{n,l}\|_{-s}^{2} \leqslant \mathrm{Const}\, nl^{1-s}$. So

$$P_k = \boldsymbol{P}\left(\operatorname*{Max}_{n_k<n\leqslant n_{k+1}} \frac{n}{a_n}\|I_n-I\|_{-s} > C\right) \quad \text{for } n_k = [e^k];$$

$a_n = \sqrt{n\log\log n}$ can be estimated by $P_k \leqslant P_k^1+P_k^2+P_k^3$ for $C = C_1+$ $+C_2+C_3$,

$$P_k^1 = \boldsymbol{P}\left\{\operatorname*{Max}_{n\leqslant n_{k+1}} \|A_{n,l}\|_{-s} \geqslant C_1\, a_{n_k}\right\},$$

$$P_k^2 \leqslant \sum_{n=n_k+1}^{n_{k+1}} \boldsymbol{P}(\|B_{n,l}\|_{-s} \geqslant C_2\, a_n) \leqslant \sum_{n=n_k+1}^{n_{k-1}} \mathrm{E}\,\|B_{n,l}\|_{-s}^{4}\,(C_2\, a_n)^{-4}$$

$$\leqslant Ctl^4\, n_k^{-1}(\log\log n_k)^{-2} \leqslant Ctl^4\, e^{-k}(\log k)^{-2},$$

$$P_k^3 \sum_{n=n_k+1}^{n_{k+1}} \boldsymbol{P}(\|C_{n,l}\|_{-s} > C_3\, a_n) \leqslant Cte^k\, l^{1-s}(\log k)^{-1}.$$

From another hand, we see like [9] that, for $l \leqslant n_{k+1}^{\beta}$,

$$P_k^1 \leqslant 2\boldsymbol{P}(\|A_{n_{k+1},l}\|_{-s} \geqslant C_1\, a_{n_k}-20\textstyle\sum\sqrt{n_{k+1}})+n_{k+1}^{-\varrho} \qquad \text{(by 3.5)}$$

$$\leqslant 2\boldsymbol{P}\left(\frac{1}{\sqrt{n_{k+1}}}\|A_{n_{k+1},l}\|_{-s} \geqslant C_1/2\sqrt{2\log k}\right)+n_{k+1}^{-\varrho} \qquad \text{for } k > k_0,$$

$$P_k^1 \leqslant 2P(\|Y_l\|_{-s} \geqslant C_1/2\sqrt{2\log k}) + n_{k+1}^{-\varrho} + Cn_{k+1}^{-3/40}\, l^{3/4}$$

(for $b = 1/10$ in 3.4),

$$P_k^1 \leqslant 2P(\|Y\|_{-s} \geqslant C_1/2\sqrt{2\log k}) + n_{k+1}^{-\varrho} + Cn_{k+1}^{-3/40}\, l^{3/4}.$$

Choose C_1 such that $\mathrm{E}\exp\{\|Y\|_{-s}^2/C_1^2\} < \infty$, $l = l(k) = e^{\beta k}$ with $\beta < 1/10$ and $\beta > 1/(s-1)$. We get, by Borel–Cantelli's lemma,

THEOREM 3.6. *If* (X_n) *is an* 8^{th} *order stationary mixing sequence with* $C_8 < \infty$, $\alpha_n \leqslant C_0 n^{-\tau}$, $\tau > 10$, $s > 11$, $\exists_{C>0}$

$$\overline{\lim} \sqrt{\frac{n}{\log\log n}}\, \|I_n - I\|_{-s} \leqslant C \text{ a.s.}$$

Note that condition $\tau > 10$ implies $C_4 < \infty$ and $\sum k|c(k)| < \infty$.
Let $1 \leqslant w \leqslant n$. We see that

$$\|I_n - I\|_{-s} \geqslant w^{-s/2} \operatorname*{Max}_{0 \leqslant k \leqslant w} |R_k(n) - r_k|$$

and

COROLLARY 3.7. *Let* (X_n) *be an* 8^{th} *order stationary mixing sequence with* $\alpha_n \leqslant C_0 n^{-\tau}$ *for some* $C_0 > 0$, $\tau > 10$ and $C_8 < \infty$. *Then for* $s > 11$ *there is a constant* C *such that, for* $1 \leqslant w_n \leqslant n$,

$$\overline{\lim_{n\to\infty}}\, n^{1/2}(\log\log n)^{-1/2} w_n^{-s/2} \operatorname*{Max}_{0 \leqslant k \leqslant w_n} |R_k(n) - r_k| \leqslant C \text{ a.s.}$$

Remark 3.7.1. The normalization goes to infinity if $w_n = n^a$ with $a < 1$. Paper [2] gives an analogous of this result for linear sequences, where $w_n = n^{2/3}$ with normalization $w_n^{-1/4}$ in place of $w_n^{-s/2}$.

COROLLARY 3.8. *Let* (X_n) *be an* 8^{th} *order stationary mixing sequence with* $C_8 < \infty$ *and* $\alpha_n \leqslant C_0 n^{-\tau}$, $\tau > 10$. *Then, if* $s > 12$, $\exists_{\Omega_0}$ $P(\Omega_0) = 1$, *and* $\omega \in \Omega_0$ *implies*

$$\overline{\lim_{n\to\infty}} \sqrt{\frac{n}{\log\log n}}\,(I_n(g) - I(g)) = \Gamma^{1/2}(g, g),\ \forall_{g \in H_s}.$$

Proof. We see that B_s is a compact of H_t if $t+1 < s$. So there is a family $g_1, \ldots, g_{N(\varepsilon)} \in E_{l(\varepsilon)}$ such that

$$B_s \subset \bigcup_{i=1}^{N(\varepsilon)} (g_i + \varepsilon B_t).$$

We show LIL for $I_n(g_i)-I(g_i)$. If $g_i = \sum_{k=0}^{l} a_k e_k$, we write

$$n\big(I_n(g_i)I(g_i)\big) = \sum_{j=1}^{n} \Big\{a_0(X_j^2-r_0^2)+2\sum_{k=1}^{l}(X_j X_{j+k}-r_k)\,a_k\Big\}+A_n$$

with $EA_n^4 \leqslant \text{Const}$, so that $A_n(n\log\log n)^{-1/2} \xrightarrow[n\to\infty]{} 0$ a.s.

Theorem 2.5 shows that

$$\overline{\lim_{n\to\infty}} \sqrt{\frac{n}{2\log\log n}}\big(I_n(g_i)-I(g_i)\big) = \Gamma^{1/2}(g_i, g_i), \qquad \omega\in\Omega_1,\ i = 1, \ldots, N(\varepsilon).$$

Now $g \to \Gamma^{1/2}(g, g)$ is continuous on H_t and we have

$$\Omega_2 = \Big\{\omega;\ \overline{\lim} \sqrt{\frac{n}{\log\log n}}\,\|I_n-I\|_{-t} \leqslant C\Big\}.$$

If $\omega\in\Omega_1\cap\Omega_2$, $\|g\|_s \leqslant 1$, and $h\in\{g_1, \ldots, g_{N(\varepsilon)}\}$ is an ε-neighbourhood of g, then we set

$$J_n = \sqrt{\frac{n}{2\log\log n}}(I_n-I)$$

and get

$$|J_n(g)-\Gamma^{1/2}(g, g)| \leqslant \|J_n\|_{-t}\,\varepsilon+|J_n(h)-\Gamma^{1/2}(h, h)|+|\Gamma^{1/2}(h, h)-\Gamma^{1/2}(g, g)|.$$

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Unité Associée 743 CNRS
Bâtiment 425
Statistique Appliquée
Université Paris XI
91405 ORSAY Cedex
France

Departamento de Matematicas
Fac. de Ciencias U. C. V.
Apartado Postal 21291
Caracas 1020-A
Venezuela

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