PROBABILITY
AND
MATHEMATICAL STATISTICS

Vol. 10, Fasc. 1 (1989), p. 107–118

SHIFTED MOMENT PROBLEM FOR GAUSSIAN MEASURES IN SOME ORLICZ SPACES

BY

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Abstract. Suppose that two Gaussian measures μ_1 and μ_2 on Orlicz space $(L_M(T, F, m), \| \|_M)$ fulfill the condition

$$(*) \qquad \int \|x+y\|_M^q (\mu_1 - \mu_2)(dy) = 0$$

for each x from L_M.

It is proved that, under some assumptions on modular M, measure m and q, condition(*) implies $\mu_1 = \mu_2$.

Introduction. Suppose that μ_1 and μ_2 are two probability measures on a separable Banach space $(E, \|\cdot\|)$ satisfying the condition

$$(+) \qquad \int \|x+y\|^q \mu_1(dy) = \int \|x+y\|^q \mu_2(dy)$$

for every $x \in E$ and some fixed $q > 0$.

Linde [7] has recently proved that if $E = L^r$ and $q \neq kr$ for positive integers k, then $\mu_1 = \mu_2$ (cf. also [4]). The purpose of this paper is to prove a similar theorem for some Orlicz spaces L_M, under the additional assumption that μ_i are Gaussian.

The technique employed here differs somewhat from that used in [7]; in particular, we do not rely on Hoffman-Jørgensen's result [5] and, even in the case of $L^r[0, 1]$ spaces, our theorem is a little stronger, namely, $\mu_1 = \mu_2$ if we only assume that $q \neq r$.

Preliminaries. We recall here briefly basic notions concerning Orlicz spaces and, after restricting our attention to the class of spaces we will work with, we state and prove some simple facts implied by our axioms.

Throughout the paper (T, F, m) will stand for a finite separable measure space. By M we denote a fixed Young function, that is a convex, strictly increasing function such that $M(t) = 0$ iff $t = 0$ and $M(-t) = M(t)$. We assume further that M satisfies (Δ_2) condition,

$$M(2t) \leqslant M(t)k \quad \text{for } t \geqslant t_0$$

for some $k > 0$ and $t_0 \geqslant 0$. Observe that convexity of M implies $k \geqslant 2$.

Now, by L_M we denote the space of all real-valued measurable functions f such that $\int M(f)\,dm < \infty$.

It is easy to see that L_M is a linear space. Moreover, if

$$\|f\|_M = \inf\{u > 0;\ \int M(f/u)\,dm \leqslant 1\}, \qquad f \in L_M,$$

then $\|\cdot\|_M$ is a norm on L_M and $(L_M, \|\cdot\|_M)$ becomes a (separable) Banach space [6].

Now, we restrict our attention to a rather special class of L_M-spaces which are a generalization of L^r-spaces $(r \geqslant 2)$.

First of all, we impose some smoothness properties on M. Namely, we assume that

(I) $M \in C^2$, that is, M has continuous derivatives up to the order two.

For the sake of convenience we write: $p = dM/dt$ and $p' = d^2M/dt^2$.

Next condition means that p', roughly speaking, cannot decrease "too fast":

(II) For all t, s such that $0 \leqslant t \leqslant s$, we have $p'(t) \leqslant p'(s) + A$ for some $A \geqslant 0$.

The third condition,

(III) $M(t) \geqslant Bt^2$ for $t \geqslant t_1$ and a $B > 0$, says that $L_M \subset L^2$ and that the natural embedding of L_M into L^2 is continuous. The L^2-norm and the corresponding inner product on L_M will be denoted by $\|\cdot\|_2$ and $\langle\cdot,\cdot\rangle$, respectively.

In the sequel we assume that our L_M-space satisfies properties (I)–(III). We now draw some simple conclusions from these properties.

PROPOSITION 1. *Under the assumptions as above the following properties hold*:

(i) $$M(t) \leqslant tp(t) \leqslant M(2t) \quad \textit{for } t \geqslant 0,$$

(ii) $$tp'(t) \leqslant p(2t) + At \quad \textit{for } t \geqslant 0,$$

(iii) $$t^2 p'(t) \leqslant (k^2/2 + A/B)\,M(t) \quad \textit{for } t \geqslant \max(t_0, t),$$

(iv) $$\sup_{\|f\|_M \leqslant c} \int M(f)\,dm < \infty$$

for every positive constant c.

Proof. (i) By convexity of M the function p is nondecreasing, so

$$M(t) = \int_0^t p(s)\,ds \geqslant \frac{t}{2}\,p\left(\frac{t}{2}\right),$$

whereas $M(t)/t \leqslant p(t)$, again by convexity of M.

(ii) Since $p' \geqslant 0$, this , together with (II), yields

$$p(t) = \int_0^t p'(s)\,ds \geqslant \min_{t/2 \leqslant s \leqslant t} p'(s)(t/2) \geqslant (t/2)\big(p'(t/2) - A\big).$$

(iii) This follows easily by (i), (ii), and (Δ_2)-condition.

(iv) Observe that the (Δ_2)-condition implies, for $t \geqslant t_0$ and all $s \geqslant 0$, $M(st) \leqslant K(s)\,M(t)$, where $K(s) = k(s \vee 1)^{\log_2 k}$. Since $k \geqslant 2$, $K(s)$ is non-decreasing, so

$$\begin{aligned}\int M(f)\,dm &= \int_{\{f/\|f\|_M \leqslant t_0\}} M(f)\,dm + \int_{\{f/\|f\|_M > t_0\}} M(f)\,dm \\ &\leqslant M(t_0\|f\|_M)\,m(T) + K(\|f\|_M)\int M(f/\|f\|_M)\,dm \\ &= M(t_0\|f\|_M)\,m(T) + K(\|f\|_M),\end{aligned}$$

because $\int M(f/\|f\|_M)\,dm = 1$.

Now, for fixed u, $h \in L_M$, let $G(t) = \|u+th\|_M$, $t \in [-1, 1]$. Further, let $u_t = u+th$ and $w_t = u_t/\|u_t\|_M$. The next proposition contains basic facts concerning smoothness of G as well as some estimates essential in the sequel.

Proposition 2. *Let* $u \notin \operatorname{lin}\{h\}$. *Then* $G \in C^2$ *and* $|G'(t)| \leqslant C\|h\|_M$, $|G''(t)| \leqslant C\|h\|_M^2/\|u_t\|_M$, *where* C *does not depend on* h, u, t. *Moreover*,

(i) $$G'(t) = I_1/I_2,$$

(ii) $$G''(t) = \frac{I_3 I_2 - I_1 I_4}{I_2^2},$$

where

$$I_1 = \int p(w_t)\,h\,dm, \qquad I_2 = \int p(w_t)\,w_t\,dm, \qquad I_3 = \int p'(w_t)\,w_t'\,h\,dm,$$

$$I_4 = \int \big(p'(w_t)\,w_t + p(w_t)\big)\,w_t'\,dm \quad \textit{with } w_t' = \frac{h\|u_t\|_M - u_t\,G'(t)}{\|u\|_M^2}.$$

Proof. The existence of G' is standard and easily follows from our assumptions [6]. Formula (i) can be obtained by differentiating the equation $\int M(w_t)\,dm = 1$.

We show that $|G'(t)| \leqslant C\|h\|_M$. To do this assume that $\|h\|_M \leqslant 1$. Then

(i) and (iv) of Proposition 1 yield that

$$|I_1| = \left|\int p(w_t)\, h\, dm\right| \leqslant \int p(|w_t|+|h|)(|w_t|+|h|)\, dm$$

$$\leqslant \int M(2|w_t|+2|h|)\, dm \leqslant \sup_{\|f\|_M \leqslant 4} \int M(f)\, dm < \infty,$$

so $|I_1| \leqslant C\|h\|_M$.

To complete this part of the proof, observe that property (i) of Proposition 1 gives

$$I_2 = \int p(w_t)\, w_t\, dm = \int p(|w_t|)(|w_t|)\, dm \geqslant \int M(w_t)\, dm \geqslant 1.$$

Next, observe that the formula for G'' is obtained by formal differentiation of expressions under the integral sign in I_1 and I_2, respectively. To justify this procedure it is enough, by virtue of the Mean Value Theorem, to show that

(a) $$\int \sup_{-1 \leqslant t \leqslant 1} |p'(w_t)\, w_t'\, h|\, dm < \infty$$

and

(b) $$\int \sup_{-1 \leqslant t \leqslant 1} \left(|p'(w_t)\, w_t'\, w_t| + |p(w_t)\, w_t'|\right) dm < \infty.$$

This can be done in a similar way as in estimating I_1 and is left to the reader. The above arguments also show that G'' is continuous.

Finally, to show that $|G''(t)| \leqslant C\|h\|_M^2/\|u_t\|_M$, we have to estimate I_2, I_3 and I_4. It is easy to see that

$$|I_2| \leqslant \int M(2w_t)\, dm \leqslant \sup_{\|f\|_M \leqslant 2} \int M(f)\, dm < \infty.$$

Next, using the formula for w_t' and the fact that $|G'(t)| \leqslant C\|h\|_M$, writing $z_t = |h|+|w_t|$, $C_1 = C \vee 1$, we get, for $\|h\|_M \leqslant 1$,

$$|I_3| \leqslant \frac{C_1}{\|u_t\|_M} \int p'(|w_t|)(|h|+|w_t|)\,|h|\, dm \leqslant \frac{C_1}{\|u_t\|_M} \int \left(A + p'(z_t)\right) z_t^2\, dm$$

$$\leqslant \frac{C_1}{\|u_t\|_M} \left[At_1^2\, m(T) + (2A/B + k^2/2) \sup_{\|f\|_M \leqslant 2} \int M(f)\, dm\right] \leqslant \frac{C_2}{\|u_t\|_M}$$

for a constant C_2 independent from u, h, t. Similarly, we get $|I_4| \leqslant C_3\|h\|_M/\|u_t\|_M$, which completes the proof.

COROLLARY 1. *Let u be a function on T with values in $\{-1, 0, 1\}$ such that $\|u\|_M \neq 0$ and let q be a fixed positive number. Write $S_u = \operatorname{supp} u$. Then*

(i) $$\frac{dG^q}{dt}(0) = q\frac{\|u\|_M^q}{m(S_u)}\langle u, h\rangle,$$

(ii) $$\frac{d^2 G^q}{dt^2}(0) = q\,\|u\|_M^{q-1}\,[\Phi_1(u)\,\|1_{S_u^c} h\|_2^2 + \Phi_2(u)\,\|1_{S_u} h\|_2^2 + \Phi_3(u)\,\langle u, h\rangle^2],$$

where

$$\Phi_1(u) = \frac{p'(0)}{p(1/\|u\|_M)\,m(S_u)}, \qquad \Phi_2(u) = \frac{p'(1/\|u\|_M)}{p(1/\|u\|_M)\,m(S_u)},$$

$$\Phi_3(u) = \frac{\|u\|_M}{m^2(S_u)}\left[(q-1) - \frac{p'(1/\|u\|_M)}{\|u\|_M\,p(1/\|u\|_M)}\right],$$

and $G^q(t) = \|u+th\|_M^q$.

The main result. We begin with one technical lemma.

LEMMA 1. *Let (Y, F, v) be a finite measure space and let h be a real-valued function defined on $(-1, 1)\times Y$ with the properties:*

(i) $$t \to h(t, y) \text{ belongs to } C^1\ v\text{-a.e.},$$

(ii) $$y \to h(t, y) \text{ belongs to } L^1(v) \text{ for all } t \in (-1, 1),$$

(iii) $$\sup_{-1<t<1} \int (h'(t, y))^2\, v(dy) < \infty, \text{ where } h' = dh/dt.$$

Put $g(t) = \int h(t, y)\, v(dy)$.
Then for every ε, $0 < \varepsilon < 1$, *we have*

$$\frac{dg}{dt}(t) = \int h'(t, y)\, v(dy) \quad \text{for } t \in (-1+\varepsilon, 1-\varepsilon).$$

Proof. Let

$$A_n = \{y \in Y;\ \sup_{-1+\varepsilon \leqslant t \leqslant 1-\varepsilon} |h'(t, y)| < n\}.$$

Then the functions $g_n(t) = \int 1_{A_n}(y)\, h(t, y)\, v(dy)$ are differentiable for $t \in (-1+\varepsilon, 1-\varepsilon)$. By the Mean Value Theorem we get, for t and t_0 belonging to $(-1+\varepsilon, 1-\varepsilon)$,

$$\frac{g_n(t) - g_n(t_0)}{t - t_0} = \int 1_{A_n}(y)\, h'(t_n, y)\, v(dy).$$

Choosing subsequence $t_{n_k} \to t^*$ and applying (i) and (ii) we get

$$\frac{g(t) - g(t_0)}{t - t_0} = \int h'(t^*, y)\, v(dy).$$

Since $t^* \to t_0$ if $t \to t_0$, applying once more (i) and (ii) we finally obtain

$$\frac{dg}{dt}(t_0) = \int h'(t_0, y)\, v(dy).$$

Now we recall some standard facts about Gaussian measures on Banach spaces [2].

Suppose that $(E, \|\cdot\|)$ is a separable Banach space and E' is its dual space. A Borel probability measure μ on E is called *centered Gaussian* if every $\xi \in E'$ is a real (symmetric) Gaussian random variable on the probability space $(E, \mathscr{B}_E, \mu)$, with $\mathscr{B}_E$ being the Borel σ-field of E. It is well-known that if μ is Gaussian, then there exists a (unique) $x \in E$ (called the *barycenter* of μ) such that $\mu_0(\cdot) = \mu(\cdot + x)$ is centered.

Now, let $E'(\mu)$ be the $L^2(\mu_0)$-closure of E', endowed with the usual $L^2(\mu_0)$ inner product. For every $\xi \in E'(\mu)$ there exists a unique $\Lambda\xi \in E$ such that, for every $\eta \in E'$, we have

$$\eta(\Lambda\xi) = \int \eta\xi\, d\mu_0 = \langle \eta, \xi \rangle_{H(\mu)}.$$

Λ: $E'(\mu) \to E$ is a linear injective mapping.

The image of Λ, endowed with the inner product induced from $E'(\mu)$, is denoted by $H(\mu)$ and is called the *reproducing kernel Hilbert space* (RKHS) of (E, μ). It is well-known that $H(\mu)$ is the space of all admissible translates of μ.

The following theorem describes the density of the measure μ translated by some $x = \Lambda\xi \in H(\mu)$, $\xi \in E'(\mu)$:

CAMERON-MARTIN'S FORMULA. *Let $x = \Lambda\xi \in H(\mu)$, where $\xi \in E'(\mu)$, with μ centered Gaussian. Then*

$$d\mu(\cdot - x) = \exp\left(\xi - \frac{1}{2}\|\xi\|^2_{H(\mu)}\right) d\mu.$$

The next theorem generalizes an immediate observation that one-dimensional symmetric Gaussian measure of translates of a fixed interval takes on the greatest value when this (translated) interval is symmetric with respect to the origin.

ANDERSON'S INEQUALITY. *Let μ be a centered Gaussian measure and V a Borel symmetric with respect to the origin convex set.*

Then, for every $x \in E$, $\mu(x+V) \leqslant \mu(V)$.

LEMMA 2. *Let μ be a Gaussian measure on E with* $\dim(\operatorname{supp}\mu) = \infty$. *Assume that the barycenter of μ belongs to $H(\mu)$. Then, for every $x \neq 0$, $x \in H(\mu)$ and $r \in R$,*

$$\sup_{-1 \leqslant t \leqslant 1} \int \|x + ty\|^r \mu(dy) < \infty. \tag{1}$$

Proof. For $r \geqslant 0$ the conclusion easily follows from the finiteness of all moments of any Gaussian measure.

It remains to prove our lemma for $r < 0$. Write $V = \{y;\ \|y\| \leqslant 1\}$. *Then*

(2) $$\int \|x+ty\|^r \mu(dy) = (-r) \int_0^\infty u^{r-1} \mu_0 \left(\frac{x}{t} + x_0 + \frac{u}{t} V\right) du,$$

where $\mu_0(\cdot + x_0) = \mu(\cdot)$.

Put $a = x/t + x_0$, $u/t = s$. Take a fixed $\xi \in E'$ and let $b = \Lambda\xi$, $\eta = \Lambda^{-1} a$. By Cameron-Martin's Formula we get (see also [3])

$$\mu_0(a+sV) = \exp\left(-\frac{1}{2}\|a\|^2_{H(\mu)}\right) \int_{sV} \exp(-\eta)\, d\mu_0$$
$$\leqslant \exp\left(-\frac{1}{2}\|a\|^2_{H(\mu)}\right) \int_{sV} \exp(\xi - \inf_{sV} \xi - \eta)\, d\mu_0$$
$$= \exp\left(-\frac{1}{2}\|a\|^2_{H(\mu)} - \inf_{sV} \xi\right) \int_{sV} \exp(\xi - \eta)\, d\mu_0.$$

Since

$$\int_{sV} \exp(\xi - \eta)\, d\mu_0$$
$$= \exp\left\{\frac{1}{2}\|a-b\|^2_{H(\mu)}\right\} \int_{sV} \exp\left(\xi - \eta - \frac{1}{2}\|a-b\|^2_{H(\mu)}\right) d\mu_0$$
$$= \exp\left\{\frac{1}{2}\|a-b\|^2_{H(\mu)}\right\} \mu_0(a-b+sV) \leqslant \exp\left\{\frac{1}{2}\|a-b\|^2_{H(\mu)}\right\} \mu_0(sV),$$

we get

(3) $$\mu_0(a+sV) \leqslant \exp\left(\frac{1}{2}\|a-b\|^2_{H(\mu)} - \frac{1}{2}\|a\|^2_{H(\mu)} - \inf_{sV} \xi\right) \mu_0(sV).$$

We now choose a $w \in H(\mu)$ such that $\Lambda^{-1} w \in E'$ and

(4) $$\|w-x\|^2_{H(\mu)} < \frac{1}{16}\|x\|^2_{H(\mu)}.$$

For $t_0 > 0$ which is small enough we have

(5) $$\frac{1}{2}\left\|\frac{x}{t} + x_0\right\|^2_{H(\mu)} > \frac{1}{4t^2}\|x\|^2_{H(\mu)} \quad \text{for } |t| \leqslant t_0$$

and

(6) $$\left\|\frac{w-x}{t} + x_0\right\|^2_{H(\mu)} \leqslant \frac{2}{t^2}\|w-x\|^2_{H(\mu)}.$$

Moreover, if $\xi = \Lambda^{-1}w/t$, then, for some $u_0 > 0$, we get

$$|\inf_{(u/t)V} \xi| \leqslant \frac{1}{16t^2}\|x\|^2_{H(\mu)} \quad \text{for } 0 < u \leqslant u_0. \tag{7}$$

Combining (3)-(7) we have, for $|t| \leqslant t_0$ and $0 < u \leqslant u_0$,

$$\mu_0\left(\frac{x}{t}+x_0+\frac{u}{t}V\right) < C\mu_0\left(\frac{u}{t}V\right)\exp\left(-\frac{D}{t^2}\right) \tag{8}$$

for some positive constants C and D. Now, (8) yields the following estimate for (2):

$$\begin{aligned}&\int_0^\infty u^{r-1}\mu_0\left(\frac{x}{t}+x_0+\frac{u}{t}V\right)du\\ &\leqslant C\exp\left(-\frac{D}{t^2}\right)\int_0^{u_0} u^{r-1}\mu_0\left(\frac{u}{t}V\right)du+\int_{u_0}^\infty u^{r-1}\,du\\ &\leqslant \frac{C}{t^r}\exp\left(-\frac{D}{t^2}\right)\int_0^\infty u^{r-1}\mu_0(uV)\,du-\frac{1}{r}u_0^r \quad \text{for } |t| \leqslant t_0.\end{aligned}$$

Since for every positive integer n there exists a constant C_n such that $\mu(uV) \leqslant C_n u^n$ for $0 < u \leqslant 1$ [1], we obtain

$$\int_0^\infty u^{r-1}\mu_0(uV)\,du < \infty.$$

Finally, if $|t| > t_0$, then

$$\int_0^\infty u^{r-1}\mu_0\left(\frac{x}{t}+x_0+\frac{u}{t}V\right)du \leqslant \int_0^\infty u^{r-1}\mu_0\left(\frac{u}{t}V\right)du = |t|^r\int_0^\infty u^{r-1}\mu_0(uV)\,du,$$

which completes the proof of the lemma.

The last lemma and Corollary 1 give the following

COROLLARY 2. *Let ν be a Gaussian measure on $E = L_M$. Assume that the barycenter of ν belongs to $H(\nu)$ and let $u \neq 0$, $u \in H(\nu)$.*

If $\dim(\operatorname{supp}\nu) = \infty$, then the functions $G^q(t) = \|u+th\|_M^q$ and dG^q/dt satisfy all assumptions of Lemma 1 for every $q > 0$.

Before formulating our theorem, we again introduce some notation. Write $T_a = \{x;\, m\{x\} > 0\}$ and $T_c = T\setminus T_a$. Let

$$R_c = \left\{\frac{tp'(t)}{p(t)}+1;\; M(t) \in \left[\frac{1}{m(T)},\, \infty\right)\right\},$$

$$R_d = \left\{\frac{tp'(0)}{p(t)}+1;\ M(t) = \frac{1}{m(\{x\})} \text{ with } x \in T_d\right\}.$$

THEOREM. *Let $L_M \equiv L_M(T, F, m)$ be a separable Orlicz space, where M satisifies* (I)-(III). *Assume that* $\dim L_M = \infty$, *m is not purely atomic and that* $q \notin R_c \cup R_d$, $q > 0$.

If ν_1 and ν_2 are Gaussian measures on L_M such that, for every $u \in L_M$,

$$(+) \qquad \int \|u+h\|_M^q \nu_1(dh) = \int \|u+h\|_M^q \nu_2(dh),$$

then $\nu_1 = \nu_2$.

Proof. Denote $\nu_1 - \nu_2$ by ν. We show that for disjoint sets $A, B \in F$ the following conditions hold:

$$\text{(a)} \qquad \int \langle 1_A, h\rangle \nu(dh) = 0,$$

$$\text{(b)} \qquad \int \langle 1_A, h\rangle \langle 1_B, h\rangle \nu(dh) = 0,$$

$$\text{(c)} \qquad \int \langle 1_A, h\rangle^2 \nu(dh) = 0.$$

This clearly implies that, for every continuous linear functional ψ on L_M, the one-dimensional Gaussian measures $\psi(\nu_1)$ and $\psi(\nu_2)$ are identical, which, of course, yields the equality $\nu_1 = \nu_2$.

Let g_i be the barycenters of $\nu_i (i = 1, 2)$ and let u be a fixed element of L_M with values in $\{-1, 0, 1\}$. In order to apply Corollary 2 we construct a centered Gaussian random vector Z with infinite-dimensional support satisfying $g_1, g_2, u \in H(\gamma)$, where γ is the distribution of Z. Furthermore, if γ_s is the distribution of sZ, $s > 0$, then γ_s as well as $\nu_i * \gamma_s$ $(i = 1, 2)$ have all the above-mentioned properties.

Moreover, the property $(+)$ still holds with ν_i replaced by $\nu_i * \gamma_s$.

Applying Lemma 1 to G^q and dG^q/dt with ν replaced by $\nu * \gamma_s$, we get

$$\int \frac{dG^q}{dt}(0)\, \nu * \gamma_s(dh) = 0 \quad \text{and} \quad \int \frac{d^2 G^q}{dt^2}(0)\, \nu * \gamma_s(dh) = 0.$$

It is easy to see that these equations yield, as $s \to 0$,

$$(9) \qquad \int \frac{dG^q}{dt}(0)\, \nu(dh) = 0,$$

$$(10) \qquad \int \frac{d^2 G^q}{dt^2}(0)\, \nu(dh) = 0.$$

To conclude this part of the proof we have to construct a random vector Z with the above listed properties. To do this, let $\{f_i\}_{i=1}^{\infty}$ be a sequence of linearly independent functions belonging to the unit sphere of L_M

and let $\xi_{-2}, \xi_{-1}, \xi_0, \xi_1, \ldots$ be a sequence of standard Gaussian random variables. It is easy to see that

$$Z = g_2 \xi_{-2} + g_1 \xi_{-1} + u\xi_0 + \sum_{i=1}^{\infty} f_i \xi_i \frac{1}{2^i}$$

satisfies all the requirements.

Next, applying Corollary 1 to formulas (9) and (10) we get

$$\int \langle u, h \rangle \nu(dh) = 0 \tag{11}$$

and

$$\Phi_1(u) \int \|(1-|u|)\cdot h\|_2^2 \nu(dh) + \\ + \Phi_2(u) \int \| |u| \cdot h\|_2^2 \nu(dh) + \Phi_3(u) \int \langle u, h \rangle^2 \nu(dh) = 0. \tag{12}$$

Property (a) now follows from (11) if we put $u = \mathbf{1}_A$, $A \in F$. Further, if A and B are disjoint and of positive measure, then writing (12) for $u_1 = \mathbf{1}_A + \mathbf{1}_B$ and $u_2 = \mathbf{1}_A - \mathbf{1}_B$, and taking into account that $\Phi_k(u_1) = \Phi_k(u_2)$ $(k = 1, 2, 3)$ and that $\Phi_3(u_2) \neq 0$, we get (b).

Now, if $A \subset T_c$, $A \in F$, $m(A) > 0$, then we construct a sequence $\{r_n\}$ such that $r_0 = \mathbf{1}_A$, $\operatorname{supp} r_n = A$, $r_n = \pm 1$ with equal measure $= \frac{1}{2} m(A)$, and $\{r_n\}$ is an orthogonal sequence of $L^2(T, F, m)$. Substituting r_n in place of u in (12) for $n = 1, 2, \ldots$, we get the following system of linear equations:

$$D + \Phi_3(\mathbf{1}_A) \int \langle r_n, h \rangle^2 \nu(dh) = 0, \qquad n = 1, 2, \ldots \tag{13}$$

Adding first n equations and dividing by n we get

$$D + \frac{1}{n} \Phi_3(\mathbf{1}_A) \sum_{k=0}^{n} \int \langle r_k, h \rangle^2 \nu(dh) = 0.$$

When $n \to \infty$, we get $D = 0$. Substituting this value into (13) and taking into account that $\Phi_3(\mathbf{1}_A) \neq 0$ we obtain (c) for all $A \subset T_c$, $A \in F$.

Next, we show that (c) yields

$$\int \|\mathbf{1}_A h\|_2^2 \nu(dh) = 0 \tag{13}$$

for all $A \subseteq T_c$. Indeed, let now $\{f_k\}$ denote an orthonormal basis of $L^2(T_c, F \cap T_c, m)$. Then

$$\|\mathbf{1}_A h\|_2^2 = \sum_{k=1}^{\infty} \langle f_k, \mathbf{1}_A h \rangle^2,$$

which, by virtue of (c), proves (13).

Observe that, in particular, (13) implies

$$p'(0) \int \|h\|_2^2 \nu(dh) = 0. \tag{14}$$

For, putting $A = T_c$ in (c) and in (13) and using (12) we immediately get that $p'(0)\|\mathbf{1}_{T_d} h\|_2^2 \nu(dh) = 0$, which, together with (13), gives (14).

To complete the proof, take $\{x_0\} \subseteq T_d$ and put $u = \mathbf{1}_{x_0}$ into (12). Applying (14) we obtain

$$(\Phi_2(u) - \Phi_1(u) + \Phi_3(u)\, m(\{x_0\}))\int \|uh\|_2^2\, \nu(dh) = 0$$

or, equivalently,

$$\left(\|u\|_M (q-1) - \frac{p'(0)}{p(1/\|u\|_M)}\right)\int \|uh\|_2^2\, d\nu(h) = 0. \tag{15}$$

Since $q \notin R_d$, (15) implies that (c) holds for $u = \mathbf{1}_{x_0}$, which clearly completes the proof.

Corollary 3. *If the function $t \to p(t)/t$ is bounded or $p'(0) = 0$, then Theorem holds even if m is purely atomic.*

Proof. Boundedness of $t \to p(t)/t$ or condition $p'(0) = 0$ give that

$$p'(0)\int \|\mathbf{1}_{T_d} h\|_2^2\, \nu(dh) = 0.$$

In fact, if $p'(0) \neq 0$, then, putting in (12) $u_n = \mathbf{1}_{x_n}$ ($\{x_n\} = T_d$), we have

$$\int \|h\|_2^2\, \nu(dh) + C_n \int \|u_n h\|_2^2\, \nu(dh) = 0, \tag{16}$$

where

$$C_n = \frac{\|u_n\|_M\, p(1/\|u_n\|_M)}{p'(0)} - 1$$

and C_n ($n = 1, 2, \ldots$) is a bounded sequence. But $\sum_n \int \|u_n h\|_2^2\, \nu(dh) < \infty$, so

$$\lim_{n\to\infty} C_n \int \|u_n h\|_2^2\, \nu(dh) = 0 \qquad (\dim L_M = \infty),$$

whence $\int \|h\|_2^2\, \nu(dh) = 0$, which completes the proof.

We say that $q > 0$ is *admissible* for L_M if (+), satisfied for this particular q, implies that $\nu_1 = \nu_2$.

Corollary 4. *For $L_M = L^r$ with $r > 2$ the exponent q is admissible if $q \neq 1$, $q \neq r$. When m is non-atomic or $r = 2$, then q is admissible if $q \neq r$.*

For any I_M with non-purely atomic measure m, every $q < 1$ is admissible.

Remark. The following example shows that if $q = r$, then there exist two different Gaussian measures on L^r such that (+) is satisfied.

Example. Let h_1, h_2, h be such functions on T that $h_1 = \mathbf{1}_A$, $h_2 = \mathbf{1}_A c$, $h = h_1 + h_2$, $A \in F$, $m(A) < m(T)$ and let θ_1, θ_2 be two independent and standard Gaussian variables. If $X_1 = h_1\theta_1 + h_2\theta_2$, $X_2 = h\theta_1$, then it is clear that X_1 and X_2 generate different measures on L^r.

We have, for any $f \in L^r$,

$$\begin{aligned}
\mathrm{E}\|f+h_1\theta_1+h_2\theta_2\|_r^r &= \mathrm{E}\Big(\int_T |f+h_1\theta_1+h_2\theta_2|^r dm\Big) \\
&= \mathrm{E}\Big(\int_A |f+h_1\theta_1|^r dm\Big)+\mathrm{E}\Big(\int_{A^c} |f+h_2\theta_2|^r dm \\
&= \mathrm{E}\Big(\int_A |f+h_1\theta_1|^r dm\Big)+\mathrm{E}\Big(\int_{A^c} |f+h_2\theta_1|^r dm\Big) \\
&= \mathrm{E}\|f+h\theta_1\|_r^r
\end{aligned}$$

Acknowledgment. The author is indebted to Tomasz Byczkowski for his remarks and suggestions during the preparation of this paper.

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Received on 30. 3. 1987